EconS 501 - Microeconomic Theory I
Final exam - Answer key

1. **Substitution and output effects.** Consider a firm employing labor l and renting capital k at prices w and r, respectively, where $w, r > 0$, to produce output q under the following production technology

$$f(l,k) = l^{\frac{1}{3}} k^{\frac{1}{3}}$$

   (a) Find the conditional demand for labor $l^c(r,w,q)$ and capital $k^c(r,w,q)$, and the firm's cost function, $C(r,w,q)$.

   - *Finding conditional factor demands.* The above production function is a Cobb-Douglas type, so we focus on interior solutions. Setting the marginal rate of technical substitution, $\frac{MP_l}{MP_k}$, equal to the input price ratio, $\frac{w}{r}$, yields

$$MRTS_{lk} = \frac{MP_l}{MP_k} = \frac{\frac{1}{3} l^{-\frac{2}{3}} k^{\frac{1}{3}}}{\frac{1}{3} l^{\frac{1}{3}} k^{-\frac{2}{3}}} = \frac{k}{l} = \frac{w}{r}$$

Rearranging, we have

$$k = \frac{wl}{r}$$

Substituting the above expressions into the production function, we obtain

$$q^3 = lk = l \cdot \overbrace{\frac{wl}{r}}^{k} = \frac{w}{r} l^2$$

Therefore, solving for l in $q^3 = \frac{w}{r} l^2$, we find the conditional demand for labor

$$l^c(r,w,q) = \sqrt{\frac{r}{w} q^3} = \sqrt{\frac{r}{w}} q^{\frac{3}{2}}.$$

Similarly, from the tangency condition above, $\frac{k}{l} = \frac{w}{r}$, we have that $l = \frac{kr}{w}$, which we can insert into the production function to obtain

$$q^3 = lk = k \cdot \underbrace{\frac{kr}{w}}_{l} = \frac{r}{w} k^2$$

Solving for k in $q^3 = \frac{r}{w} k^2$, we find the conditional demand for capital

$$k^c(r,w,q) = \sqrt{\frac{w}{r} q^3} = \sqrt{\frac{w}{r}} q^{\frac{3}{2}}.$$

- *Finding the cost function.* The cost function, which the firm utilizes input in the cost-minimizing way, is

$$\begin{aligned} C(r,w,q) &= wl^c(r,w,q) + rk^c(r,w,q) \\ &= w\underbrace{\sqrt{\frac{r}{w}}q^{\frac{3}{2}}}_{l^c} + r\underbrace{\sqrt{\frac{w}{r}}q^{\frac{3}{2}}}_{k^c} \\ &= 2\sqrt{rw}q^{\frac{3}{2}} \end{aligned}$$

(b) Let p be output price, where $p > 0$. Find the supply function $q(p,r,w)$, and show that the law of supply holds.

- The firm's profit-maximization problem (PMP) is

$$\begin{aligned} \max_{q\geq 0} \pi(q) &= pq - C(r,w,q) \\ &= pq - 2\sqrt{rw}q^{\frac{3}{2}} \end{aligned}$$

where the first term denotes total revenue and the second represents total costs, which we found in part (a). Differentiating with respect to q, yields

$$\frac{\partial \pi(q)}{\partial q} = p - 2\cdot\frac{3}{2}\sqrt{rw}q^{\frac{1}{2}} = 0$$

which, after rearranging, becomes

$$q^{\frac{1}{2}} = \frac{p}{3\sqrt{rw}}$$

Solving for q, we find the firm's supply function

$$q(p,r,w) = \frac{p^2}{9rw}$$

which is increasing in output price p (so the Law of Supply holds) but decreases in input prices r and w.

(c) Assume that the price of capital r increases marginally. Show that cross-price effects on labor satisfy

$$TE = SE + OE,$$

that is, the total effect (TE) is the sum of substitution effect (SE) and output effect (OE).

- *Finding unconditional input demands.* Let us first find the unconditional demand function for labor and capital. We can find these functions evaluating the conditional factor demands found in part (a) at the supply function $q(p,r,w) = \frac{p^2}{9rw}$ found in part (b), as follows

$$\begin{aligned} l(p,r,w) &= l^c(r,w,q(p,r,w)) = \sqrt{\frac{r}{w}}q(p,r,w)^{\frac{3}{2}} = \sqrt{\frac{r}{w}}\overbrace{\left(\frac{p^2}{9rw}\right)}^{q}{}^{\frac{3}{2}} = \frac{p^3}{27rw^2} \\ k(p,r,w) &= k^c(r,w,q(p,r,w)) = \sqrt{\frac{w}{r}}q(p,r,w)^{\frac{3}{2}} = \sqrt{\frac{w}{r}}\underbrace{\left(\frac{p^2}{9rw}\right)}_{q}{}^{\frac{3}{2}} = \frac{p^3}{27r^2w} \end{aligned}$$

- Therefore, the cross-price *total* effect of a change in the price of capital is measured using the unconditional labor demand, as follows,

$$TE = \frac{\partial l\left(p, r, w\right)}{\partial r} = \frac{\partial}{\partial r}\left[\frac{p^3}{27rw^2}\right] = -\frac{p^3}{27r^2w^2}$$

- Next, we find the cross-price *substitution* effect of a marginal change in r using the conditional labor demand, that is,

$$\begin{aligned} SE = \frac{\partial l^c\left(r, w, q\right)}{\partial r} &= \frac{\partial}{\partial r}\left[\sqrt{\frac{r}{w}}q^{\frac{3}{2}}\right] \\ &= \frac{1}{2\sqrt{rw}} \cdot \left(\frac{p^2}{9rw}\right)^{\frac{3}{2}} = \frac{p^3}{54r^2w^2} \end{aligned}$$

where, in the third line, we inserted supply function $q\left(p, r, w\right) = \frac{p^2}{9rw}$.

- Finally, we find the cross-price *output* effect of a marginal increase in r using again the conditional labor demand, as follows,

$$\begin{aligned} OE &= \frac{\partial l^c\left(r, w, q\right)}{\partial q} \cdot \frac{\partial q}{\partial r} \\ &= \frac{\partial}{\partial q}\left[\sqrt{\frac{r}{w}}q^{\frac{3}{2}}\right] \cdot \frac{\partial}{\partial r}\left[\frac{p^2}{9rw}\right] \\ &= -\frac{3}{2}\sqrt{\frac{r}{w}}q^{\frac{1}{2}} \cdot \frac{p^2}{9r^2w} \\ &= -\frac{p^2}{6\left(rw\right)^{\frac{3}{2}}} \cdot \left(\frac{p}{3\sqrt{rw}}\right) \\ &= -\frac{p^3}{18r^2w^2} \end{aligned}$$

where, in the third line, we also inserted supply function $q\left(p, r, w\right) = \frac{p^2}{9rw}$.

- *Slutsky equation.* We can now check if condition $TE = SE + OE$ holds, that is,

$$\underbrace{\frac{\partial l\left(p, r, w\right)}{\partial r}}_{\text{cross-price TE}} = \underbrace{\frac{\partial l^c\left(r, w, q\right)}{\partial r}}_{\text{cross-price SE}} + \underbrace{\frac{\partial l^c\left(r, w, q\right)}{\partial q} \cdot \frac{\partial q}{\partial r}}_{\text{cross-price OE}}$$

The right-side of the above equality is

$$\begin{aligned} SE + OE &= \frac{p^3}{54r^2w^2} + \left(-\frac{p^3}{18r^2w^2}\right) \\ &= -\frac{p^3}{r^2w^2}\left(\frac{1}{18} - \frac{1}{54}\right) \\ &= -\frac{p^3}{27r^2w^2} = TE \end{aligned}$$

which coincides with the left-side, so that condition $TE = SE + OE$ holds; as expected.

(d) How do the conditional and unconditional labor demands vary differently to a marginal increase in the price of capital, r? Explain.

- *Unconditional labor demand.* From part (c), we found that the unconditional labor demand decreases in the price of capital, r, because the cross-price total effect is negative, $TE < 0$. Intuitively, the firm demands fewer units of labor when capital becomes more expensive.
- *Conditional labor demand.* However, conditional labor demand increases in r because the cross-price substitution effect is negative, $SE > 0$, since the firm substitutes labor for capital along the same isoquant as before the price change. Intuitively, to produce the same output as before the increase in the price of capital, the firm changes its relative use of inputs, towards a more intense use of the relatively cheaper input (labor) and a less intense use of the input that become relatively more expensive (capital).
- The output effect is negative, $OE = -\frac{p^3}{18r^2w^2}$, indicating that the firm responds decreasing its profit-maximizing output. Importantly, this output effect is larger in absolute value than the positive substitution effect, ultimately producing a negative total effect. In other words, despite labor becoming relatively cheaper, the firm scales down its output so much, that its demand for labor decreases. Graphically, conditional labor demand curve shifts rightward when the price of capital, r , increases, whereas the unconditional labor demand curve shifts leftward.

2. **Quasilinear utility function in a pure exchange economy.** Consider a pure exchange economy with two individuals, A and B, whose utility functions are

$$u^A\left(x_1^A, x_2^A\right) = \log x_1^A + x_2^A$$
$$u^B\left(x_1^B, x_2^B\right) = x_1^B x_2^B$$

with endowments of $\omega^A = \left(\omega_1^A, \omega_2^A\right) = (3,5)$ and $\omega^B = \left(\omega_1^B, \omega_2^B\right) = (6,4)$, respectively.

(a) Find the Walrasian demand functions of individuals A and B.

- *UMP for A.* Individual A chooses x_1^A and x_2^A to solve the utility maximization problem (UMP),

$$\max_{x_1^A, x_2^A \geq 0} \; u^A\left(x_1^A, x_2^A\right) = \log x_1^A + x_2^A$$
$$\text{subject to } \; p_1 x_1^A + p_2 x_2^A = 3p_1 + 5p_2$$

Rearranging the budget constraint, and substituting into individual A's utility function, we obtain the following unconstrained UMP,

$$\max_{x_1^A \geq 0} \; \log x_1^A + \frac{3p_1 + 5p_2 - p_1 x_1^A}{p_2}$$

which is a function of x_1^A alone. Differentiating with respect to x_1^A, and assuming interior solutions, we obtain

$$\frac{1}{x_1^A} - \frac{p_1}{p_2} = 0$$

Rearranging, individual A's Walrasian demand for good 1 is

$$x_1^A = \frac{p_2}{p_1}$$

Substituting $x_1^A = \frac{p_2}{p_1}$ into the budget constraint of individual A, we obtain

$$p_1 \cdot \frac{p_2}{p_1} + p_2 x_2^A = 3p_1 + 5p_2$$

Rearranging, we have

$$p_2 \left(1 + x_2^A\right) = 3p_1 + 5p_2$$

Simplifying, individual A's Walrasian demand for good 2 is

$$x_2^A = 3\frac{p_1}{p_2} + 4$$

- *UMP for B*. Individual B chooses x_1^B and x_2^B to solve the UMP,

$$\max_{x_1^B, x_2^B \geq 0} u^B \left(x_1^B, x_2^B\right) = x_1^B x_2^B$$

$$\text{subject to} \quad p_1 x_1^B + p_2 x_2^B = 6p_1 + 4p_2$$

Rearranging the budget constraint, and substituting into individual B's utility function, we obtain the following unconstrained UMP as follows,

$$\max_{x_1^B \geq 0} x_1^B \cdot \frac{6p_1 + 4p_2 - p_1 x_1^B}{p_2}$$

which is a function of x_1^B alone. Differentiating with respect to x_1^B, and assuming interior solutions, we obtain

$$\frac{6p_1 + 4p_2 - 2p_1 x_1^B}{p_2} = 0$$

Rearranging, individual B's Walrasian demand for good 1 is

$$x_1^B = 2\frac{p_2}{p_1} + 3$$

Substituting $x_1^B = 2\frac{p_2}{p_1} + 3$ into the budget constraint of individual B, we obtain

$$p_1 \left(2\frac{p_2}{p_1} + 3\right) + p_2 x_2^B = 6p_1 + 4p_2$$

Rearranging, we have

$$p_2 x_2^B = 3p_1 + 2p_2$$

Simplifying, individual B's Walrasian demand for good 2 is

$$x_2^B = 3\frac{p_1}{p_2} + 2$$

(b) Characterize the set of Pareto efficient allocations (PEAs).

- The feasibility constraints in this pure exchange economy are

$$x_1^A + x_1^B = 3 + 6 = 9 \text{ for good 1}$$
$$x_2^A + x_2^B = 5 + 4 = 9 \text{ for good 2}$$

which are rearranged as follows

$$x_1^A = 9 - x_1^B$$
$$x_2^A = 9 - x_2^B$$

The contract curve, which defines the set of PEAs, is the locus of tangency of indifference curves between individuals A and B, satisfying

$$MRS_{12}^A = \frac{MU_1^A}{MU_2^A} = \frac{MU_1^B}{MU_2^B} = MRS_{12}^B$$

which we rearrange as

$$\frac{1}{x_1^A} = \frac{x_2^B}{x_1^B}$$

Substituting the feasibility constraints into the above tangency condition, we obtain

$$\frac{1}{9 - x_1^B} = \frac{x_2^B}{x_1^B}$$

which, after rearranging, yields the contract curve as follows,

$$x_2^B = \frac{x_1^B}{9 - x_1^B}$$

(c) Identify the Walrasian equilibrium allocation (WEA).

- Substituting the Walrasian demands for good 1 that we found in part (a) into the feasibility constraint of good 1, $x_1^A + x_1^B = 9$, we obtain

$$\underbrace{\frac{p_2}{p_1}}_{x_1^A} + \underbrace{\left(2\frac{p_2}{p_1} + 3\right)}_{x_1^B} = 9$$

Rearranging, yields

$$3\frac{p_2}{p_1} = 6$$

that gives the equilibrium price ratio of

$$\frac{p_1}{p_2} = \frac{1}{2}$$

Substituting $\frac{p_1}{p_2} = \frac{1}{2}$ into the Walrasian demand functions found in part (a), we obtain the equilibrium allocations

$$x_1^{A*} = \frac{p_2}{p_1} = 2$$
$$x_2^{A*} = 3\frac{p_1}{p_2} + 4 = \frac{11}{2}$$
$$x_1^{B*} = 2\frac{p_2}{p_1} + 3 = 7$$
$$x_2^{B*} = 3\frac{p_1}{p_2} + 2 = \frac{7}{2}$$

Therefore, the Walrasian equilibrium allocation (WEA) can be summarized as follows

$$\left(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2}\right) = (2, 5.5; 7, 3.5; 0.5)\,.$$

(d) Show that the WEA found in part (c) is a PEA, as found in part (b).

- Inserting the WEA of part (c) in the condition for an allocation to be a PEA (the contract curve of part b), $x_2^B = \frac{x_1^B}{9 - x_1^B}$, we obtain that

$$x_2^B = \frac{7}{9-7} = \frac{7}{2} = 3.5$$

which exactly coincies with x_2^{B*} in the WEA. Therefore, the WEA is a PEA, confirming the first welfare theorem.

3. **Socially excessive exploitation.** Consider a setting with N individuals, where every individual i simultaneously and independently chooses his exploitation level $e_i \geq 0$. The marginal cost of effort is symmetric across individuals, $c > 0$. For compactness, denote by $e_{-i} = (e_1, ..., e_{i-1}, e_{i+1}, ...e_N)$ the profile of effort levels by i's rivals, E the sum of all individuals' efforts, and $E_{-i} = E - e_i$ the aggregate effort of all i's rivals. The utility function for individual i is given by

$$u_i(e_i, e_{-i}) = A(e_i + E_{-i})e_i - ce_i$$

In addition, assume that $A(\cdot)$ represents an outcome function which is strictly decreasing in aggregate effort E. This outcome function can represent several economic contexts, such as: (1) a common-pool resource, where $A(E) = \frac{f(E)}{E}$ indicates the average appropriation accruing to every player i, with $f(E)$ capturing total appropriation (e.g., total catches by all fishermen), as in Dasgupta and Heal (1979); (2) Cournot competition, where $A(E)$ represents the inverse demand function, which decreases in aggregate output, e.g., $A(E) = a - bE$; and (3) rent-seeking contests where $A(E) = \frac{e_i}{E}$ indicates the probability that player i wins the prize (e.g., promotion in a company), which is also decreasing in total effort.

(a) *Competitive equilibrium.* Find the implicit function that defines the equilibrium effort level that every individual i chooses in this setting.

- Every individual i solves

$$\max_{e_i} \quad A(e_i + E_{-i})e_i - ce_i$$

Taking first-order condition with respect to effort e_i, we obtain

$$A(E) + e_i A'(E) = c$$

In words, every individual i chooses an effort level e_i such that the marginal benefit from *individual* effort (left-hand side of the above equation) coincides with its own marginal cost from effort, c (right-hand side).

- Taking the second-order condition with respect to e_i, we obtain

$$2A'(E) + e_i A''(E)$$

Therefore, for the equilibrium effort to be a maximum, this second-order condition must be negative, i.e., $2A'(E) + e_i A''(E) < 0$. By definition, we know that the outcome of aggregate effort is decreasing, $A'(E) < 0$. If we assume that it is concave, $A''(E) < 0$, the second-order condition holds. Alternatively, function $A(E)$ can be convex as long as it is not "extremely convex", that is, $2A'(E) < e_i A''(E)$.

(b) *Social optimum.* Assume that a social planner considers the sum of every individual's utility as a measure of social welfare. (Alternatively, this setting can represent a cooperative solution where individuals seek to maximize their joint utility.) Find the profile of effort levels that maximize social welfare (again, an implicit equation).

- The social planner's problem can be stated as

$$\max_{e_1, e_2, \ldots e_N} \quad \sum_{i=1}^{N} u_i(e_i, e_{-i}) = \sum_{i=1}^{N} A(e_i + E_{-i})e_i - ce_i$$

Taking the first-order conditions with respect to every e_i, we obtain

$$A(E) + A'(E)(e_1 + e_2 + ... + e_N) - c = 0$$

where $E = \sum_{i=1}^{N} e_i$ denotes aggregate effort. Rearranging this expression, we obtain the profile of effort levels that maximize social welfare, which is the solution to the following implicit equation

$$A(E) + EA'(E) = c$$

- Intuitively, this equations says that the social planner chooses the profile of effort levels $e^* = (e_1, e_2, ... e_N)$ such that the marginal benefit of the *aggregate* effort (left-hand side of the equation) coincides with the marginal cost, c (right-hand side).

(c) *Comparison.* Compare your results from parts (a) and (b), showing that equilibrium effort is socially excessive.

- In part (a), considering the welfare of each player, the equilibrium effort is determined by the equation

$$A(E) + e_i A'(E) - c = 0 \tag{1}$$

In part (b), considering the sum of every individual's utility, the equilibrium effort is determined by the equation

$$A(E) + EA'(E) - c = 0 \tag{2}$$

Comparing equations (1) and (2), since $A(\cdot)$ is strictly decreasing in E, we can see that players' effort in the competitive equilibrium is socially excessive. As a result, the aggregate socially optimal effort, E^{SO}, is lower than the competitive equilibrium effort, E^*, implying that, $A(E^{SO}) > A(E^*)$. The high quantity of aggregate effort generated by the competitive equilibrium results in a welfare loss due to the presence of negative externalities.

- For illustration purposes, we next evaluate our results in the three economic contexts described at the beginning of this exercise:
  - *Common-pool resource interpretation.* When outcome function $A(E) = \frac{f(E)}{E}$ represents the average appropriation accruing to every player i (e.g., tons of fish captured by fisherman i), the payoff function for each individual i can be written as

$$u_i(e_i, e_{-i}) = \frac{e_i}{E} f(E) - ce_i \tag{3}$$

  - *Rent-seeking interpretation.* Alternatively, equation (3) can be interpreted as the payoff function in a rent-seeking contest with $\frac{e_i}{E}$ as the probability that player i wins the prize. Intuitively, individual i's probability of winning increases in his own effort e_i, but decreases in the effort other players exert, as captured by aggregate effort E. A higher aggregate effort implies a lower probability of winning the prize. Hence, given that $E^* > E^{SO}$, the probability of winning the contest is higher when maximizing the social welfare than when maximizing individual welfare.
  - *Cournot competition interpretation.* When outcome function $A(E)$ represents the inverse demand function, the inefficiency of the noncooperative equilibrium relative to the social optimum is equivalent to showing that collusion leads to lower aggregate production ($E^{SO} < E^*$) but higher prices ($A(E^{SO}) > A(E^*)$). Hence, collusive equilibrium results in higher profits.