Exercise #10.6: Nonlinear Pricing in Monopoly, based on Maskin and Riley (1984)^C

10.6 Consider a monopolist seeking to sell a product to a consumer with quasilinear utility function

$$u_i(x, y) = \theta_i \sqrt{x} + y,$$

where θ_i denotes his preference for quality (where $\theta_H > \theta_L$), x represents the quality of a particular good the monopolist sells, and y is money (representing the consumption of all other goods). The probability of a low-valuation consumer is α while that of a high valuation is $1 - \alpha$, where $\alpha \in [0, 1]$.

The monopolist offers the item at a price p and its cost of producing one unit of quality is c, where c satisfies $\theta_L > c > 0$. Assume that the reservation utility of both types of consumers is zero when they do not purchase the good. Since the monopolist cannot observe each type of consumer (i.e., the realization of parameter θ_i), it needs to screen customers by offering a menu that induces each type to self-select the offer meant for him. In the following sections of the exercise we first show that pricing strategies such as linear pricing or single two-part tariff yield a lower profit than a menu of two-part tariffs.

(a) *Uniform pricing.* Suppose that the monopolist can only offer one price p to every type of customer. What would be the profit maximizing price and profit?

- We can use backward induction in this setting, by first finding consumer *i*'s demand for a given price p (second stage), and then identifying the price that the seller sets in the first stage.
- *Second stage.* After observing a price and quality pair (p, x), consumer type *i* solves the utility maximization problem

$$\max_{x \geq 0} \ \theta_i \sqrt{x} + y - px.$$

Differentiating with respect to x, we obtain

$$\frac{\theta_i}{2\sqrt{x}} - p = 0.$$

Solving for x, we find that consumer *i*'s demand for quality is

$$x_i\left(p\right) = \frac{\theta_i^2}{4p^2}.$$

Such demand increases in consumer *i*'s preference for quality, θ_i, but decreases in the price p set by the monopolist. Hence, aggregate demand becomes

$$\left(1-\alpha\right) x_H\left(p\right) + \alpha x_L\left(p\right) = \frac{\left(1-\alpha\right)\theta_H^2 + \alpha\theta_L^2}{4p^2}.$$

- *First stage.* Anticipating such an aggregate demand, the monopolist sets a uniform price (the same price to all types of customers) that maximizes its profits as follows:

$$\max_{p\geq 0} \underbrace{(p-c)}_{\text{Margin}} \underbrace{\frac{\left((1-\alpha)\,\theta_H^2 + \alpha\theta_L^2\right)}{4p^2}}_{\text{Aggregate demand}}.$$

Differentiating with respect to p, we obtain

$$-\frac{\left[(1-\alpha)\,\theta_H^2 + \alpha\theta_L^2\right](p-2c)}{4p^3} = 0.$$

Solving for p, we find a uniform price of

$$p^U = 2c$$

which increases in cost c.

- In this context, equilibrium profits become

$$\begin{aligned}\pi^U &= \left(p^U - c\right)\frac{\left[(1-\alpha)\,\theta_H^2 + \alpha\theta_L^2\right]}{4\left(p^U\right)^2} \\ &= (2c-c)\frac{\left[(1-\alpha)\,\theta_H^2 + \alpha\theta_L^2\right]}{4\,(2c)^2} \\ &= \frac{(1-\alpha)\,\theta_H^2 + \alpha\theta_L^2}{16c}\end{aligned}$$

which is decreasing in cost c and the proportion of low-type consumers α.

(b) *Single two-part tariff.* Suppose the monopolist can offer a single two-part tariff consisting of an initial fee T and a unit price p (which does not depend on the quantity sold). What is the profit maximizing (T, p)-pair for the monopolist?

- We first identify the profit maximizing two-part tariff when the monopolist serves both types of customers, or only the high-value customers, and subsequently compare the profits from each option.
- *Serving both types of customers.* When the monopolist sells to all types of consumers, we need that the low-type consumer participates, i.e., $T \leq S_L(p)$ where $S_L(p)$ represents the surplus for the low-valuation customers, as we next define for any type i

$$\begin{aligned}S_i\,(p) &\equiv u_i(x_i\,(p)\,, y) - px_i\,(p) \\ &= \theta_i\sqrt{\frac{\theta_i^2}{4p^2}} - p\frac{\theta_i^2}{4p^2} \\ &= \frac{\theta_i^2}{4p}.\end{aligned}$$

We also require a similar condition to hold for the high-type consumer, $T \leq S_H(p) = \frac{\theta_H^2}{4p}$. However, $S_L(p) \leq S_H(p)$ for every $p \geq 0$ given that $\theta_L < \theta_H$ by definition, implying that we only need to impose the condition on the low-type consumer. In other words, the participation constraint of the high-type consumer becomes slack.

- In this context, the monopolist's profit maximization problem becomes

$$\max_{T,p\geq 0} T + \underbrace{(p-c)}_{\text{Margin}}\underbrace{[\alpha x_L(p) + (1-\alpha)\, x_H(p)]}_{\text{Expected sales}}$$

$$\text{subject to } T \leq S_L(p) \qquad (PC_L)$$

In addition, note that PC$_L$ must bind. Otherwise, the monopolist would have incentives to increase the fee T and still achieve participation of the low type. Hence, $T = S_L(p) = \frac{\theta_L^2}{4p}$, implying that the above problem can be expressed as the following unconstrained maximization problem:

$$\max_{p\geq 0} \underbrace{\frac{\theta_L^2}{4p}}_{T} + (p-c)\left[\frac{\alpha\theta_L^2 + (1-\alpha)\,\theta_H^2}{4p^2}\right].$$

Differentiating with respect to p, we obtain

$$-\frac{\theta_L^2}{4p^2} - \frac{(p-2c)[\alpha\theta_L^2 + (1-\alpha)\theta_H^2]}{4p^3} = 0.$$

Solving for p, we obtain

$$p^{ST} = \frac{4c\left(\alpha\theta_L^2 + (1-\alpha)\,\theta_H^2\right)}{(1+2\alpha)\,\theta_L^2 + 2\,(1-\alpha)\,\theta_H^2},$$

where the superscript ST denotes "single two-part tariff." Plugging p^{ST} into the fee $T^{ST} = \frac{\theta_L^2}{4p^{ST}}$ yields

$$T^{ST} = \frac{\theta_L^2}{4p^{ST}} = \frac{\theta_L^2\left[(1+2\alpha)\,\theta_L^2 + 2\,(1-\alpha)\,\theta_H^2\right]}{16c\left[\alpha\theta_L^2 + (1-\alpha)\,\theta_H^2\right]}$$

entailing profits of

$$\begin{aligned}\pi^{ST} &= \frac{\theta_L^2}{4p^{ST}} + \left(p^{ST}-c\right)\left[\frac{\alpha\theta_L^2 + (1-\alpha)\,\theta_H^2}{4\left(p^{ST}\right)^2}\right] \\ &= \frac{\theta_L^2 p^{ST} + \left(p^{ST}-c\right)\left(\alpha\theta_L^2 + (1-\alpha)\,\theta_H^2\right)}{4\left(p^{ST}\right)^2} \\ &= \frac{\left[(3+2\alpha)\,\theta_L^2 + 2\,(1-\alpha)\,\theta_H^2\right]\left[(1+2\alpha)\,\theta_L^2 + 2\,(1-\alpha)\,\theta_H^2\right]}{64c\left[\alpha\theta_L^2 + (1-\alpha)\,\theta_H^2\right]}.\end{aligned}$$

- *Serving only the high-type customer.* We now check if the monopolist's profits from serving both types of customers are larger than its profits from selling to the high-type customer alone. We next find the tariff and price if the monopolist only sells to the high-type customer. In this context, its profit maximization problem becomes

$$\max_{T,p\geq 0} \; (1-\alpha)\left[T+(p-c)\,x_H\,(p)\right]$$

$$\text{subject to } \; T \leq S_H\,(p) \qquad (PC_H)$$

Note that now the only *PC* constraint we consider is that of the high type, where $S_H\,(p) = \frac{\theta_H^2}{4p}$. In addition, PC$_H$ must bind (otherwise the monopolist could increase the fee T and still achieve participation of this type of consumer), so we must have that $T = S_H\,(p) = \frac{\theta_H^2}{4p}$. The monopolist's problem then becomes the following unconstrained program:

$$\max_{p\geq 0} \; (1-\alpha)\left[\frac{\theta_H^2}{4p}+(p-c)\,\frac{\theta_H^2}{4p^2}\right].$$

Differentiating with respect to p, we obtain

$$(1-\alpha)\left[\frac{(c-p)\theta_H^2}{2p^3}\right]=0.$$

Solving for p yields

$$p^H = c,$$

where superscript H denotes that the monopolist only serves the high-type consumer. Then, the fee in this context becomes

$$T^H = \frac{\theta_H^2}{4p^H} = \frac{\theta_H^2}{4c},$$

entailing profits of

$$\pi^H = (1-\alpha)\left[\overbrace{\frac{\theta_H^2}{4c}}^{T^H} + \;(c-c)\,\frac{\theta_H^2}{4c^2}\right]$$
$$= \frac{(1-\alpha)\,\theta_H^2}{4c}.$$

- *Comparison.* Finally, we can now compare the profits of using a single two-part tariff that serves both customers, π^{ST}, against those from selling to the high-type customer alone, π^H, finding that $\pi^{ST} < \pi^H$ if

$$\frac{\left[(3+2\alpha)\,\theta_L^2+2\,(1-\alpha)\,\theta_H^2\right]\left[(1+2\alpha)\,\theta_L^2+2\,(1-\alpha)\,\theta_H^2\right]}{64c\left[\alpha\theta_L^2+(1-\alpha)\,\theta_H^2\right]} < \frac{(1-\alpha)\,\theta_H^2}{4c}.$$

This inequality simplifies to

$$12(1-\alpha)^2\theta_H^4 - 8(1-\alpha)^2\theta_H^2\theta_L^2 - (1+2\alpha)(3+2\alpha)\theta_L^4 > 0,$$

which we can rearrange as

$$12(1-\alpha)^2\left(\frac{\theta_H}{\theta_L}\right)^4 - 8(1-\alpha)^2\left(\frac{\theta_H}{\theta_L}\right)^2 - (1+2\alpha)(3+2\alpha) > 0.$$

Solving for ratio $\frac{\theta_H}{\theta_L}$, we obtain

$$\frac{\theta_H}{\theta_L} > \sqrt{\frac{2(1-\alpha)+\sqrt{13+16\alpha+16\alpha^2}}{6(1-\alpha)}}.$$

For example, when both types of consumers are equally likely, $\alpha = \frac{1}{2}$, the above inequality simplifies to $\frac{\theta_H}{\theta_L} > \sqrt{2} \approx 1.414$, meaning that when the high-type consumers place a sufficiently higher valuation than the low-type consumers, it is more profitable for the monopolist to only serve the high type than to serve both types.

(c) *Menu of two-part tariffs*. Let us now find the menu of offers (contracts), (x_L, T_L) and (x_H, T_H), meant for low-type and high-type customer, respectively.

- The problem for the monopolist is now the following (as usual in screening problems, we need to include the participation constraint for both types of customers, along with the incentive compatibility conditions for both of them):

$$\max_{x_H, T_H, x_L, T_L \geq 0} \; (1-\alpha)(T_H - cx_H) + \alpha(T_L - cx_L)$$

subject to

$$\theta_L\sqrt{x_L} - T_L \geq 0 \qquad (PC_L)$$

$$\theta_H\sqrt{x_H} - T_H \geq 0 \qquad (PC_H)$$

$$\theta_L\sqrt{x_L} - T_L \geq \theta_L\sqrt{x_H} - T_H \qquad (IC_L)$$

$$\theta_H\sqrt{x_H} - T_H \geq \theta_H\sqrt{x_L} - T_L \qquad (IC_H)$$

Since

$$\theta_H\sqrt{x_H} - T_H \underbrace{\geq}_{\text{From } IC_H} \theta_H\sqrt{x_L} - T_L \underbrace{>}_{\text{From } \theta_H > \theta_L} \theta_L\sqrt{x_L} - T_L \geq 0,$$

combining the first and last inequality yields $\theta_H\sqrt{x_H} - T_H > 0$ so that the PC_H constraint must be slack. In other words, the PC_L constraint binds, as the monopolist can charge a higher tariff to this type of consumer and still achieve participation.

- Substituting the binding PC_L into IC_L, we obtain $0 \geq \theta_L\sqrt{x_H} - T_H$, meaning that it is unprofitable for the low type to take on a high-type contract, so that IC_L is slack. From the binding PC_L and IC_H, we obtain

$$\theta_L\sqrt{x_L} - T_L = 0$$
$$\theta_H\sqrt{x_H} - T_H = \theta_H\sqrt{x_L} - T_L.$$

Rearranging, we obtain

$$T_L = \theta_L \sqrt{x_L}$$

and using this expression in $\theta_H \sqrt{x_H} - T_H = \theta_H \sqrt{x_L} - T_L$ yields

$$\begin{aligned} T_H &= \theta_H \sqrt{x_H} - \theta_H \sqrt{x_L} + \overbrace{\theta_L \sqrt{x_L}}^{=T_L} \\ &= \theta_H \left(\sqrt{x_H} - \sqrt{x_L}\right) + \theta_L \sqrt{x_L}. \end{aligned}$$

Inserting T_L and T_H, we simplify the monopolist's expected profit maximization problem to the following unconstrained program which, in addition, has only two choice variables (x_H and x_L) rather than the original four choice variables:

$$\max_{x_H, x_L \geq 0} \; (1-\alpha) \left(\overbrace{\theta_H \left(\sqrt{x_H} - \sqrt{x_L}\right) + \theta_L \sqrt{x_L}}^{=T_H} - c x_H \right) + \alpha \left(\overbrace{\theta_L \sqrt{x_L}}^{=T_L} - c x_L \right).$$

Differentiating with respect to x_H and x_L yields

$$\begin{aligned} \frac{\partial E[\pi]}{\partial x_H} &= \frac{\theta_H}{2\sqrt{x_H}} - c = 0 \\ \frac{\partial E[\pi]}{\partial x_L} &= \frac{\theta_L - (1-\alpha)\theta_H}{2\sqrt{x_L}} - \alpha c = 0. \end{aligned}$$

Simplifying, we obtain

$$\begin{aligned} x_H &= \frac{\theta_H^2}{4c^2} \\ x_L &= \frac{\left(\theta_L - (1-\alpha)\theta_H\right)^2}{4\alpha^2 c^2}. \end{aligned}$$

- Substituting these results, we find that optimal tariffs are

$$T_L = \theta_L \sqrt{x_L} = \theta_L \sqrt{\underbrace{\frac{\left(\theta_L - (1-\alpha)\theta_H\right)^2}{4\alpha^2 c^2}}_{x_L}} = \frac{\theta_L \left(\theta_L - (1-\alpha)\theta_H\right)}{2\alpha c}$$

and

$$\begin{aligned} T_H &= \theta_H \left(\sqrt{x_H} - \sqrt{x_L}\right) + \theta_L \sqrt{x_L} \\ &= \theta_H \left(\sqrt{\underbrace{\frac{\theta_H^2}{4c^2}}_{x_H}} - \sqrt{\underbrace{\frac{\left(\theta_L - (1-\alpha)\theta_H\right)^2}{4\alpha^2 c^2}}_{x_L}} \right) + \theta_L \sqrt{\underbrace{\frac{\left(\theta_L - (1-\alpha)\theta_H\right)^2}{4\alpha^2 c^2}}_{x_L}} \end{aligned}$$

$$= \theta_H \left(\frac{\theta_H}{2c} - \frac{\theta_L - (1-\alpha)\,\theta_H}{2\alpha c} \right) + \theta_L \frac{\theta_L - (1-\alpha)\,\theta_H}{2\alpha c}$$
$$= \frac{\theta_H^2 - (2-\alpha)\,\theta_H \theta_L + \theta_L^2}{2\alpha c}.$$

Therefore, the monopolist's profits from the menu of two-part tariffs are

$$\begin{aligned}
\pi^{MT} &= (1-\alpha)\,(T_H - cx_H) + \alpha\,(T_L - cx_L) \\
&= (1-\alpha) \left(\frac{\theta_H^2 - (2-\alpha)\,\theta_H \theta_L + \theta_L^2}{2\alpha c} - \frac{c\theta_H^2}{4c^2} \right) \\
&+ \alpha \left(\frac{\theta_L\,(\theta_L - (1-\alpha)\,\theta_H)}{2\alpha c} - \frac{c\,(\theta_L - (1-\alpha)\,\theta_H)^2}{4\alpha^2 c^2} \right) \\
&= \frac{(1-\alpha)\,\theta_H^2 - 2\,(1-\alpha)\,\theta_L \theta_H + \theta_L^2}{4\alpha c}.
\end{aligned}$$

(d) Rank the profits of the monopolist in parts (a)-(c). Interpret your results. For simplicity, assume that both types of consumers are equally likely, $\alpha = \frac{1}{2}$.

- We have relaxed the constraints in steps. As we move from part (a) to part (c), the monopolist derives more and more rent (surplus) as he develops more sophisticated pricing strategies. We can then have two cases, depending on whether $\frac{\theta_H}{\theta_L} > \sqrt{2}$ or $\frac{\theta_H}{\theta_L} \leq \sqrt{2}$.
- *Case 1, High-type consumers assign a relatively high value to quality.* When condition $\frac{\theta_H}{\theta_L} > \sqrt{2}$ holds, in part (b) we found that the monopolist earns a higher profit when serving high-type customers alone, so that $\pi^{ST} < \pi^H$. Therefore, we can rank profits in parts (a)-(c) as follows:

$$\pi^U < \pi^H < \pi^{MT}$$

which holds if

$$\frac{\theta_L^2 + \theta_H^2}{32c} < \frac{\theta_H^2}{8c} < \frac{\theta_H^2 - 2\theta_L\theta_H + 2\theta_L^2}{4c}.$$

For the first inequality to hold, we need $4\theta_H^2 > \theta_L^2 + \theta_H^2$, or $\frac{\theta_H}{\theta_L} > \frac{1}{\sqrt{3}}$, which is satisfied since $\frac{\theta_H}{\theta_L} > 1$ given that $\theta_L < \theta_H$. For the second inequality to hold, we need $\theta_H^2 - 4\theta_H\theta_L + 4\theta_L^2 > 0$, which holds because this is factorized into $(\theta_H - 2\theta_L)^2 > 0$.
- *Case 2, High-type consumers assign a relatively low value to quality.* When condition $\frac{\theta_H}{\theta_L} \leq \sqrt{2}$ holds, in part (b) we found that the monopolist earns a higher profit when serving both types of consumers, so that $\pi^{ST} > \pi^H$. Therefore, we can rank profits in parts (a)–(c) as follows:

$$\pi^U < \pi^{ST} < \pi^{MT}$$

which holds if

$$\frac{\theta_L^2+\theta_H^2}{32c} < \frac{\left(4\theta_L^2+\theta_H^2\right)\left(2\theta_L^2+\theta_H^2\right)}{32c\left(\theta_L^2+\theta_H^2\right)} < \frac{\theta_H^2-2\theta_L\theta_H+2\theta_L^2}{4c}.$$

For the first inequality to hold, we need $\theta_L^2\left(7\theta_L^2+4\theta_H^2\right)>0$, which is satisfied. For the second inequality to hold, we need

$$7\left(\frac{\theta_H}{\theta_L}\right)^4-16\left(\frac{\theta_H}{\theta_L}\right)^3+18\left(\frac{\theta_H}{\theta_L}\right)^2-16\left(\frac{\theta_H}{\theta_L}\right)+8>0,$$

which holds for all values of $\frac{\theta_H}{\theta_L}\leq\sqrt{2}$.

- Combining the above cases, the monopolist obtains the highest (lowest) profit in practicing menu (uniform) pricing for all values of θ_L and θ_H.

(e) *Welfare comparison.* Assume again that both types of consumers are equally likely, $\alpha=\frac{1}{2}$, and that $c=\frac{1}{4}$, and $\frac{\theta_H}{\theta_L}>\sqrt{2}$. Evaluate the expected social welfare that emerges when the monopolist practices uniform pricing (as in part a), limited pricing by serving only high-type customers (as in part b), and offers a menu of two-part tariffs (as in part c). Which pricing strategy yields the highest expected social welfare? Compare your welfare ranking with the profit ranking obtained in part (d). Interpret.

- *Uniform pricing.* Expected social welfare is

$$\begin{aligned}
W^U &= \overbrace{(1-\alpha)\,x_H\left(p^U\right)(\theta_H-c)}^{\text{High-type cons.}}+\overbrace{\alpha x_L\left(p^U\right)(\theta_L-c)}^{\text{Low-type cons.}}\\
&=(1-\alpha)\overbrace{\frac{\theta_H^2}{4p^2}}^{=x_H\left(p^U\right)}(\theta_H-c)+\alpha\overbrace{\frac{\theta_L^2}{4p^2}}^{=x_L\left(p^U\right)}(\theta_L-c)\\
&=\frac{\theta_H^2(4\theta_H-1)+\theta_L^2(4\theta_L-1)}{32p^2}\\
&=\frac{\theta_H^2\left(4\theta_H-1\right)+\theta_L^2\left(4\theta_L-1\right)}{8},
\end{aligned}$$

where the last step considers that $p^U=2c=\$\frac{1}{2}$.

- *Limited pricing.* Expected social welfare in this case, as the monopolist only serves the high-type consumers, is given by

$$\begin{aligned}
W^H &= (1-\alpha)\,x_H\left(p^H\right)(\theta_H-c)\\
&=(1-\alpha)\overbrace{\frac{\theta_H^2}{4p^2}}^{=x_H\left(p^H\right)}(\theta_H-c)\\
&=\frac{\theta_H^2(4\theta_H-1)}{32p^2}\\
&=\frac{\theta_H^2\left(4\theta_H-1\right)}{2},
\end{aligned}$$

where the last step inserts $p_H = c = \$\frac{1}{4}$.

- *Menu pricing.* Expected social welfare in this case, where the monopolist sells to both types of consumers, is

$$W^{MT} = \overbrace{(1-\alpha)\,x_H^{MT}\,(\theta_H - c)}^{\text{High-type cons.}} + \overbrace{\alpha x_L^{MT}\,(\theta_L - c)}^{\text{Low-type cons.}}$$
$$= (1-\alpha)\overbrace{\left(\frac{\theta_H^2}{4c^2}\right)}^{=x_H^{MT}}(\theta_H - c) + \alpha\overbrace{\left(\frac{(\theta_L - (1-\alpha)\,\theta_H)^2}{4\alpha^2 c^2}\right)}^{=x_L^{MT}}(\theta_L - c)$$
$$= \frac{\theta_H^2\,(4\theta_H - 1) + (2\theta_L - \theta_H)^2\,(4\theta_L - 1)}{2}.$$

- *Welfare comparison.* Comparing across different pricing strategies, we find that:
  - $W^H > W^U$, that is, social welfare under limited pricing is unambiguously higher than uniform pricing because

$$\frac{\theta_H^2\,(4\theta_H - 1)}{2} > \frac{\theta_H^2\,(4\theta_H - 1) + \theta_L^2\,(4\theta_L - 1)}{8}$$

    holds if $3\theta_H^2\,(4\theta_H - 1) > \theta_L^2\,(4\theta_L - 1)$, which is true for $\theta_H > \theta_L$ by definition.
  - $W^{MT} > W^H$, that is, social welfare under limited pricing falls below menu pricing because

$$\frac{\theta_H^2\,(4\theta_H - 1) + (2\theta_L - \theta_H)^2\,(4\theta_L - 1)}{2} > \frac{\theta_H^2\,(4\theta_H - 1)}{2}$$

    which simplifies to

$$(2\theta_L - \theta_H)^2\,(4\theta_L - 1) > 0$$

    that holds as long as $\theta_L > c = \frac{1}{4}$ that holds by definition.
  - In summary, we obtain that social welfare satisfies

$$W^{MT} > W^H > W^U,$$

    which coincides with the profit ranking when $\frac{\theta_H}{\theta_L} > \sqrt{2}$, where

$$\pi^{MT} > \pi^H > \pi^U.$$

    because the monopolist can extract the most consumer surplus from all consumers when they are served under menu pricing, which is more preferred from the social and the firm's perspective of only serving high-type consumers in limit pricing. This is, in turn, better off than uniform pricing in which the loss in the monopolist's profits does not offset the gains in consumer surplus.