EconS 501 - Microeconomic Theory I
Midterm exam #2 - Answer key

1. **Gross Substitutes.** Consider an economy with two individuals, Amelia and Bernardo, with utility functions

$$\begin{aligned} u^A(x^A, y^A) &= \min\{x^A, 2y^A\} \text{ for Amelia, and} \\ u^B(x^B, y^B) &= \min\{2x^B, y^B\} \text{ for Bernardo,} \end{aligned}$$

and initial endowments are given by $\mathbf{e}^A = (1, 0)$ for Amelia and $\mathbf{e}^B = (0, 1)$ for Bernardo.

(a) Find the Walrasian demands of each individual.

- *Amelia.* The UMP of Amelia is

$$\begin{aligned} \max_{x^A, y^A \geq 0} \quad & \min\{x^A, 2y^A\} \\ \text{subject to} \quad & p_x x^A + p_y y^A \leq p_x \end{aligned}$$

since she only owns one unit of good x, $\mathbf{e}^A = (1, 0)$ the market value of her resources (as captured in the right-hand side of the budget constraint) is p_x. As she would consume (x^A, y^A) pairs at the kink of her L-shaped indifference curves, optimal consumption bundles satisfy $x^A = 2y^A$. Plugging $x^A = 2y^A$ into her budget line, $p_x x^A + p_y y^A = p_x$, yields

$$p_x \left(2y^A\right) + p_y y^A = p_x$$

and solving for y^A, we obtain Amelia's Walrasian demand of good y

$$y^A = \frac{p_x}{2p_x + p_y}$$

while her demand for good x is

$$x^A = 2y^A = \frac{2p_x}{2p_x + p_y}$$

- *Bernardo.* Similarly, Bernardo's utility maximizing bundles (x^B, y^B) satisfy $2x^B = y^B$ (bundles at the kink of his indifference curve) and $p_x x^B + p_y y^B = p_y$ (budget line since he only owns one unit of good y). Simultaneously solving for x^B and y^B yields

$$x^B = \frac{p_y}{p_x + 2p_y} \quad \text{and} \quad y^B = \frac{2p_y}{p_x + 2p_y}$$

(b) Find the excess demand functions, $z_x(p_x, p_y)$ and $z_y(p_x, p_y)$.

- The excess demand for good x is

$$z_x(p_x,p_y) = \frac{2p_x}{2p_x+p_y} + \frac{p_y}{p_x+2p_y} - 1 - 0 = \frac{p_x p_y - (p_y)^2}{(2p_x+p_y)(p_x+2p_y)}$$

while that of good y is

$$z_y(p_x,p_y) = \frac{p_x}{2p_x+p_y} + \frac{2p_y}{p_x+2p_y} - 0 - 1 = \frac{p_x p_y - (p_x)^2}{(2p_x+p_y)(p_x+2p_y)}$$

(c) Check that Walras' law holds.

- In order to check that $\mathbf{p}\cdot\mathbf{z}(\mathbf{p}) = 0$, we need

$$\begin{aligned} p_x z_x(p_x,p_y) + p_y z_x(p_x,p_y) &= p_x\left(\frac{p_x p_y - (p_y)^2}{(2p_x+p_y)(p_x+2p_y)}\right) + p_y\left(\frac{p_x p_y - (p_x)^2}{(2p_x+p_y)(p_x+2p_y)}\right) \\ &= \frac{(p_x)^2 p_y - p_x(p_y)^2 + p_x(p_y)^2 - (p_x)^2 p_y}{(2p_x+p_y)(p_x+2p_y)} = 0 \end{aligned}$$

thus confirming Walras' law.

(d) Check if goods are gross substitutes, i.e., for any two goods $k \neq j$ where $k, j = \{x, y\}$ their excess demand functions satisfy $\frac{\partial z_k(p_x,p_y)}{\partial p_j} > 0$.

- Using $z_x(p_x,p_y)$, we find

$$\frac{\partial z_x(p_x,p_y)}{\partial p_y} = \frac{2(p_x)^3 - 4(p_x)^2 p_y - 7p_x(p_y)^2}{(2(p_x)^2 + 2(p_y)^2 + 5p_x p_y)^2}$$

which is positive if the numerator is positive, that is,

$$p_y < \frac{2p_x}{2+3\sqrt{2}} \simeq 0.32p_x$$

Similarly, using $z_y(p_x,p_y)$, we find that

$$\frac{\partial z_y(p_x,p_y)}{\partial p_x} = \frac{2(p_y)^3 - 4p_x(p_y)^2 - 7(p_x)^2 p_y}{(2(p_x)^2 + 2(p_y)^2 + 5p_x p_y)^2}$$

which is positive if the numerator is positive, that is,

$$p_y > \frac{p_x}{2}\left(3\sqrt{2}+2\right) \simeq 3.12p_x$$

Figure 1 depicts in the (p_x,p_y)-quadrant the two cutoffs we identified:

- price pairs in area C entail that good x is a gross substitute of good y; whereas
- price pairs in area A imply that good y is a gross substitute of good x.

In other words, the conditions for goods to be gross substitutes are asymmetric, as there is no region where both good x is a gross substitute of y and vice versa.

Last, note that price pairs in area B entail that good x is a gross complement of good y and, simultaneously, good y is a gross complement of good x.

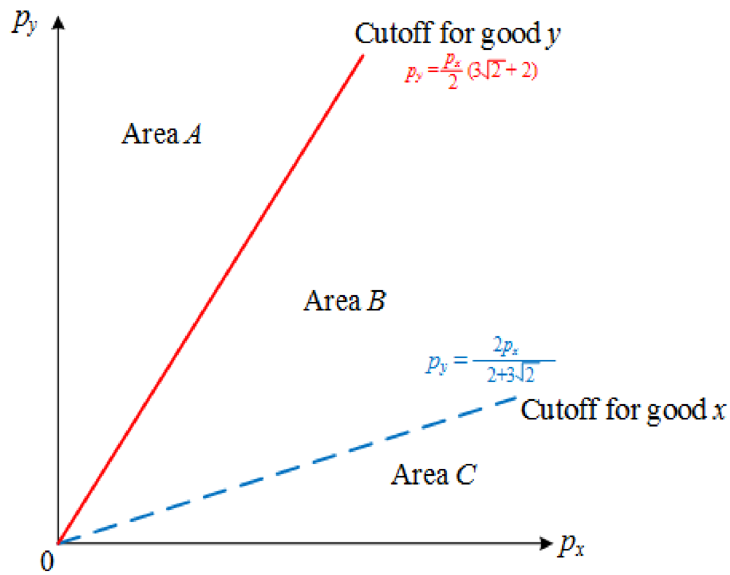

Figure 1. Areas for which goods x and y can be gross substitutes.

2. **Concave transformation of a utility function.** Consider an individual with the utility function,

$$u(x) = x^{\alpha}$$

where $0 \leq \alpha \leq 1$, and $x > 0$ represents the outcome that the individual receives.

   (a) Consider the concave transformation $g(y) = \ln y$, where $y > 0$. Find the Arrow-Pratt coefficient of absolute risk aversion for the composite function $g(u(x))$. Show that this function exhibits stronger risk aversion than the utility function $u(x)$.

   - The composite function is

$$\begin{aligned} h(x) &\equiv g(u(x)) \\ &= \ln(x^{\alpha}) \\ &= \alpha \ln x \end{aligned}$$

Differentiating the composite function, $h(x) \equiv g(u(x))$, with respect to x, we obtain

$$\begin{aligned} h'(x) &\equiv \frac{dh(x)}{dx} = \frac{\alpha}{x} \\ h''(x) &\equiv \frac{d^2h(x)}{dx^2} = -\frac{\alpha}{x^2} \end{aligned}$$

so the Arrow-Pratt coefficient of absolute risk aversion of the composite function is

$$r_A(x,h) = -\frac{h''(x)}{h'(x)} = -\frac{-\frac{\alpha}{x^2}}{\frac{\alpha}{x}} = \frac{1}{x}.$$

- We can now differentiate the original utility function, $u(x)$, with respect to x, finding

$$u'(x) \equiv \frac{du(x)}{dx} = \alpha x^{\alpha-1}$$
$$u''(x) \equiv \frac{d^2u(x)}{dx^2} = -\alpha(1-\alpha)x^{\alpha-2}$$

Therefore, the Arrow-Pratt coefficient of absolute risk aversion of $u(x)$ is

$$r_A(x,u) = -\frac{u''(x)}{u'(x)} = -\frac{-\alpha(1-\alpha)x^{\alpha-2}}{\alpha x^{\alpha-1}} = \frac{1-\alpha}{x}$$

Comparing the Arrow-Pratt coefficients of absolute risk aversion, we obtain that

$$r_A(x,h) = \frac{1}{x} \geq \frac{1-\alpha}{x} = r_A(x,u)$$

for all $\alpha \in [0,1]$. Therefore, the composite utility function, $g(u(x))$, exhibits stronger risk aversion than the simple utility function $u(x)$. Graphically, the concave transformation makes $h(x)$ more concave than $u(x)$, thus making the individual more averse to playing lotteries.

(b) Suppose the individual plays a lottery with equal probability of winning x and $3x$. Find the certainty equivalent of this lottery for the individual with (i) the utility function $u(x)$, and (ii) the composite utility function $g(u(x))$.

- The certainty equivalent is the amount that makes this individual indifferent to the expected utility of the lottery. Therefore, for the initial utility function $u(x)$, the certainty equivalent solves

$$u(CE(\alpha,x,u)) = E(u(x)) = \frac{1}{2}x^\alpha + \frac{1}{2}(3x)^\alpha$$
$$(CE(\alpha,x,u))^\alpha = \frac{1+3^\alpha}{2}x^\alpha$$

and, solving for CE, we find

$$CE(\alpha,x,u) = \left(\frac{1+3^\alpha}{2}\right)^{\frac{1}{\alpha}} x.$$

- For the composite utility function $g(u(x))$, the certainty equivalent solves a similar problem:

$$u(CE(\alpha,x,h)) = E(h(x)) = \frac{1}{2}\alpha \ln x + \frac{1}{2}\alpha \ln 3x$$
$$\alpha \ln(CE(\alpha,x,h)) = \frac{\alpha}{2}(2\ln x + \ln 3)$$

and, solving for CE, we find

$$CE(\alpha, x, h) = \exp\left(\frac{2\ln x + \ln 3}{2}\right)$$

(c) Let $\alpha = \frac{1}{2}$. Does the individual require a lower certainty equivalent with utility function $u(x)$ or with the composite utility function $g(u(x))$? Interpret.

- Substituting $\alpha = \frac{1}{2}$ into the certainty equivalents we found in part (b), we obtain that

$$CE\left(\frac{1}{2}, x, u\right) = \left(\frac{1+3^{1/2}}{2}\right)^{\frac{1}{1/2}} x = \left(\frac{1+\sqrt{3}}{2}\right)^2 x$$
$$CE\left(\frac{1}{2}, x, h\right) = \exp\left(\frac{2\ln x + \ln 3}{2}\right)$$

Let us check if $CE\left(\frac{1}{2}, x, h\right) < CE\left(\frac{1}{2}, x, u\right)$, which entails

$$\exp\left(\frac{2\ln x + \ln 3}{2}\right) < \left(\frac{1+\sqrt{3}}{2}\right)^2 x$$
$$\ln x + \frac{\ln 3}{2} < \ln x + 2\ln\frac{1+\sqrt{3}}{2}$$
$$\ln 3 < 4\ln\frac{1+\sqrt{3}}{2}$$

which simplifies to $1.1 < 1.25$ that holds. Intuitively, the individual is more risk adverse under the composite utility function $g(u(x))$ than under the initial utility function $u(x)$, so that this individual is willing to accept a lower certainty equivalent under the composite utility function than the initial utility function.

3. **Monopolist interested in fairness.** Consider a monopolist who faces a market with two segments, with demand functions $q_1(p_1) = a_1 - p_1$ and $q_2(p_2) = a_2 - p_2$, where $a_2 > a_1$, and production costs are normalized to zero, $c = 0$. Suppose that she maximizes profit subject to the constraint that the outcome is fair in the sense that consumer surpluses coincide, that is,

$$CS_1(q_1) = CS_2(q_2).$$

(a) Formulate the optimization problem of this monopolist and solve the problem.

- The monopolist solves

$$\max_{q_1, q_2 \geq 0} \; p_1 q_1 + p_2 q_2$$
$$\text{subject to } \; CS_1(q_1) = CS_2(q_2).$$

Since demand functions are linear,

$$\begin{aligned} CS_i(q_i) &= \int (a_i - q_i)\, dq_i - \underbrace{(a_i - q_i)}_{p_i} q_i \\ &= \left[a_i q_i - \frac{1}{2} q_i^2 \right] - (a_i - q_i)\, q_i \\ &= \frac{1}{2} q_i^2, \end{aligned}$$

Therefore, the constraint implies that $\frac{1}{2}q_1^2 = \frac{1}{2}q_2^2$, which holds only if $q_1 = q_2 = q$, meaning that the monopolist sells the same quantity in each segment. Using $q_1 = q_2 = q$ in the objective function, the monopolist problem above simplifies to

$$\max_{q \geq 0} \; p_1 q + p_2 q = (a_1 - q)q + (a_2 - q)\, q$$

Differentiating with respect to q, yields

$$a_1 - 2q + a_2 - 2q = 0$$

and, solving for q, we obtain

$$q^* = \frac{a_1 + a_2}{4}.$$

- Inserting q^* into the inverse demand functions of each segment, we find the equilibrium prices

$$\begin{aligned} p_1(q^*) &= a_1 - \frac{a_1 + a_2}{4} = \frac{3a_1 - a_2}{4}, \text{ and} \\ p_2(q^*) &= a_2 - \frac{a_1 + a_2}{4} = \frac{3a_2 - a_1}{4}. \end{aligned}$$

(b) Find equilibrium output and prices if, instead, the monopolist seeks to maximize profits without the fairness constraint.

- If the monopolist seeks to maximize profits without the fairness constraint, she solves

$$\max_{q_1, q_2 \geq 0} \; p_1 q_1 + p_2 q_2 = (a_1 - q_1)q_1 + (a_2 - q_2)\, q_2$$

Differentiating with respect to q_1, yields $a_1 - 2q_1 = 0$, which entails $q_1^m = \frac{a_1}{2}$. Similarly, differentiating with respect to a_2, we obtain $a_2 - 2q_2 = 0$, which entails $q_2^m = \frac{a_2}{2}$. Inserting these output levels in the inverse demand functions of each segment, we find that equilibrium prices are

$$\begin{aligned} p_1(q_1^m) &= a_1 - \frac{a_1}{2} = \frac{a_1}{2}, \text{ and} \\ p_2(q_2^m) &= a_2 - \frac{a_2}{2} = \frac{a_2}{2}. \end{aligned}$$

(c) Compare your results in parts (a) and (b). Rank equilibrium prices, output, and interpret.

- Comparing equilibrium prices, we find that the price high-demand customers pay satisfies

$$p_1(q^*) = \frac{3a_1 - a_2}{4} > \frac{a_1}{2} = p_1(q_1^m)$$

which simplifies to $a_1 > a_2$, which holds by assumption. In contrast, the price that low-demand customers pay satisfies

$$p_2(q^*) = \frac{3a_2 - a_1}{4} < \frac{a_2}{2} = p_1(q_1^m)$$

meaning that high-demand customers pay more when the monopolist is contrained by fairness considerations than otherwise. Low-demand customers, however, pay less when the monopolist is contrained by fairness considerations than otherwise.

- Comparing equilibrium output, we find that the sales to high-demand customers satisfies

$$q^* = \frac{a_1 + a_2}{4} < \frac{a_1}{2} = q_1^m$$

since $a_2 < a_1$. In contrast, the sales to low-demand customers satisfies

$$q^* = \frac{a_1 + a_2}{4} > \frac{a_2}{2} = q_2^m$$

meaning that the high-demand customers purchase fewer units when the monopolist is constrained by fairness than otherwise, while low-demand customers buy more units.