EconS 501 - Micro Theory I
Assignment #6 - Answer key

1. **Monopolist serving two interdependent markets.** Consider a monopolist producing two goods, 1 and 2, at a marginal cost $c > 0$. The demand function for product i is

$$q_i(p_i, p_j) = a - bp_i + gp_j \quad \text{where } i = \{1, 2\}$$

and parameters satisfy $a > c(b - g)$, $b, c > 0$, and $|b| > |g|$, entailing that own-price effects dominate cross-price effects. For generality, we allow for $g > 0$ and $g < 0$.

(a) Assume that the monopolist sets the price of good i separate to that of good j. Find the equilibrium price pair (p_i, p_j).

- The monopolist sets the price of good i by solving the following profit maximization problem:

$$\max_{p_i} \pi_i(p_i) = (a - bp_i + gp_j)(p_i - c)$$

Differentiating with respect to p_i, we obtain

$$a - bp_i + gp_j - b(p_i - c) = 0,$$

and solving for p_i, we find

$$p_i(p_j) = \frac{a + bc + gp_j}{2b},$$

which is analogous to a best response function in a standard duopoly Bertrand game of price competition where firms sell heterogeneous products. Graphically, this best response function originates at $\frac{a+bc}{2b}$ and increases at a slope of $\frac{g}{2b}$ if goods are substitutes ($g > 0$) but decreases at that slope if goods are complements ($g < 0$, respectively).

Following similar steps, the monopolist sets the price of good j by solving its own maximization problem and obtains the semetric best response function,

$$p_j(p_i) = \frac{a + bc + gp_i}{2b},$$

which has the same interpretation for the price of good i.

In a symmetric equilibrium, the monopolist sets the same price for both goods, i.e., $p_1 = p_2 = p_c$, implying that

$$p_c = \frac{a + bc + gp_c}{2b}$$

and solving for p_c yields an equilibrium price of

$$p_c = \frac{a + bc}{2b - g}.$$

(b) Suppose now that the monopolist sets the prices of goods i and j simultaneously. Find the equilibrium price pair (p_i, p_j).

- In this setting, the monopolist sets the prices of goods i and j by maximizing the joint profits across both goods,

$$\max_{p_i,p_j} \quad \pi_i(p_i) + \pi_j(p_j) = (a - bp_i + gp_j)(p_i - c) + (a - bp_j + gp_i)(p_j - c)$$

Differentiating with respect to p_i and p_j, we obtain

$$\begin{aligned} a - bp_i + gp_j - b(p_i - c) + g(p_j - c) &= 0 \\ a - bp_j + gp_i - b(p_j - c) + g(p_i - c) &= 0, \end{aligned}$$

and solving for p_i and p_j, we obtain

$$p_i = p_j = \frac{a + c(b-g)}{2(b-g)}.$$

(c) Under what condition will the distinct monopolist charge a higher (lower) price than a multi-product monopolist?

- As shown in the previous section, the multi-product monopolist will charge a price of $p_m = \frac{a+c(b-g)}{2(b-g)}$. Comparing this price against $p_c = \frac{a+bc}{2b-g}$, we find that

$$\frac{a + c(b-g)}{2(b-g)} > \frac{a+bc}{2b-g}$$

and rearranging, yields

$$\begin{aligned} [a + c(b-g)](2b-g) &> 2(a+bc)(b-g) \\ 2ab - ag + (b-g)(2bc - cg) &> (2a + 2bc)(b-g) \\ 2ab - ag &> (2a + cg)(b-g) \\ 2ab - ag &> 2ab - 2ag + cg(b-g) \\ ag &> cg(b-g) \\ g(a - c(b-g)) &> 0 \end{aligned}$$

Note that we do not cancel the parameter g out since it could be either positive or negative, which would influence the directionality of the inequality. Since $a > c(b-g)$ holds by assumption, the above inequality, $p_m > p_c$, holds (does not hold) if and only if $g > 0$ ($g < 0$).
Therefore, the multi-product monopolist charges a higher (lower) price than distinct monopolists if goods are substitutes (complements). Intuitively, when goods are substitutes, setting a higher price for good i decreases the demand for good j. This negative effect is ignored by the monopolist when setting prices as independent units, but internalized by the multi-product monopolist, such that independent units will set a lower price for good i. The opposite argument applies when good i and j are complements, and a monopolist with independent units setting each price would set a lower price for each good than the multi-product monopolist.

2. **Multiproduct monopoly with economies of scope.** Consider Ferdinand's food company, a monopolist producing two goods, ice cream (good 1) and cheese (good 2), which are regarded as substitutes for consumers. The inverse demand function of good i is

$$p_i(q_i, q_j) = a - bq_i - gq_j$$

where $a, b, g > 0$ and $|b| > |g|$, entailing that own-price effects dominate cross-price effects.

In addition, Ferdinand's cost function is

$$C(q_1, q_2) = \frac{c}{2}\left(q_1^2 + q_2^2\right) - \beta q_1 q_2$$

where $c > 0$, and $\beta > 0$ indicates that the marginal cost of producing one good decreases in the output of another good, i.e., there are cost complementarities in production often referred as "economies of scope." When $\beta = 0$, the cost of one output is independent of the other.

(a) Find the profit-maximizing output and associated profits of Ferdinand's food company.

- Ferdinand's food company chooses q_1 and q_2 to solve

$$\max_{q_1, q_2} \pi(q_1, q_2) = (a - bq_1 - gq_2) q_1 + (a - bq_2 - gq_1) q_2 - \frac{c}{2}\left(q_1^2 + q_2^2\right) + \beta q_1 q_2$$

Differentiating with respect to q_1 and q_2, and assuming interior solutions, we find

$$a - 2bq_1 - 2gq_2 - cq_1 + \beta q_2 = 0, \text{ and}$$
$$a - 2bq_2 - 2gq_1 - cq_2 + \beta q_1 = 0.$$

Invoking symmetry, where $q^* = q_1 = q_2$, we obtain

$$a - 2bq^* - 2gq^* - cq^* + \beta q^* = 0.$$

Rearranging, equilibrium output becomes

$$q^* = \frac{a}{2b + 2g + c - \beta}.$$

- Substituting equilibrium output into Ferdinand's profit function, we have

$$\begin{aligned}
\pi^* &= 2\left(a - (b+g) q^*\right) q^* + (\beta - c)(q^*)^2 \\
&= \frac{2a^2}{2b + 2g + c - \beta} - (2b + 2g + c - \beta)\left(\frac{a}{2b + 2g + c - \beta}\right)^2 \\
&= \frac{a^2}{2b + 2g + c - \beta}.
\end{aligned}$$

(b) How does equilibrium output change in parameters β and g? Interpret your results.

- Differentiating the equilibrium output, q^*, with respect to β, we obtain

$$\frac{\partial q^*}{\partial \beta} = \frac{a}{(2b + 2g + c - \beta)^2} > 0,$$

so that as cost complementarity, β, increases, the monopolist increases the output of both goods. Intuitively, producing more units of one good makes the other good less costly to produce, increasing the incentive to produce further units of both goods.

- Differentiating the equilibrium output, q^*, with respect to g, we find

$$\frac{\partial q^*}{\partial g} = -\frac{2a}{(2b + 2g + c - \beta)^2} < 0.$$

Therefore, as the cross-price effect is strengthened, goods are more easily substitutable (more homogeneous). In this context, the monopolist reduces output of both goods.

(c) *Numerical example.* Evaluate your results in parts (a) and (b) assuming parameter values $a = 1$, $\beta = 1/2$, $c = 1/3$, and $g = 1/4$. How do they change with b? Interpret.

- Substituting $a = 1$, $\beta = 1/2$, $c = 1/3$, and $g = 1/4$ into equilibrium output, we find

$$q^* = \frac{1}{2b + 2 \times \frac{1}{4} + \frac{1}{3} - \frac{1}{2}} = \frac{3}{1 + 6b}.$$

- Similarly, substituting the above parameter values into equilibrium profit, we have

$$\pi^* = \frac{1^2}{2b + 2 \times \frac{1}{4} + \frac{1}{3} - \frac{1}{2}} = \frac{3}{1 + 6b}.$$

Therefore, both equilibrium output and profit decrease in own price effect b. Intuitively, the more sensitive is own price to the quantity demanded of one good, the fewer units will this good be produced in equilibrium.

3. **Persuasive advertising in monopoly.** Consider a monopolist facing the demand function

$$Q(p, A) = a - pA^{-\frac{1}{2}}$$

where a is the market size, and the firm spends advertising dollars A in promoting its products. For simplicity, assume that production cost is zero, and we consider $a, A > 0$.

(a) Find the price elasticity of demand $\varepsilon_{Q,P}$. Does advertising make demand more inelastic?

- The price elasticity of demand is

$$\varepsilon_{Q,P} = -\frac{\partial Q(p, A)}{\partial p} \frac{p}{Q(p, A)} = \frac{p}{A^{\frac{1}{2}} \left(a - pA^{-\frac{1}{2}}\right)} = \frac{p}{a\sqrt{A} - p}$$

Differentiating $\varepsilon_{Q,P}$ with respect to A, yields

$$\frac{\partial \varepsilon_{Q,P}}{\partial A} = -\frac{ap}{2\sqrt{A}\left(a\sqrt{A}-p\right)^2}$$

that is unambiguously negative. Therefore, advertising expenditure A makes demand more inelastic and consumers are less likely to switch from the good sold by this firm to other goods regarded as close substitutes. The advertising that makes demand more inelastic is known as "persuasive."

(b) Consider that the firm simultaneously chooses its price p and advertising expenditure, A. Setup the firm's profit maximization problem and solve for the firm's equilibrium price p^*, advertising A^*, and the resulting equilibrium profits π^*. Then, evaluate your results assuming $a = 2$.

- The firm chooses p and A to solve the following profit maximization problem (recall that, for simplicity, the model assumes no production costs),

$$\max_{p,A>0} \pi(p,A) = p\underbrace{\left(a - pA^{-\frac{1}{2}}\right)}_{Q(p,A)} - A$$

Differentiating with respect to p, we obtain

$$a - \frac{2p}{\sqrt{A}} = 0$$

which we rearrange to yield

$$p(A) = \frac{a\sqrt{A}}{2}.$$

Similarly, if we differentiate with respect to A, to find

$$\frac{p^2}{2A^{\frac{3}{2}}} - 1 = 0$$

which we rearrange to yield

$$A(p) = \left(\frac{p^2}{2}\right)^{\frac{2}{3}}.$$

Substituting $A(p)$ into $p(A)$, we find

$$p = \frac{a}{2}\sqrt[3]{\frac{p^2}{2}}$$

Rearranging, we obtain the equilibrium price, as follows

$$p^* = \frac{a^3}{16}$$

Substituting $p^* = \frac{a^3}{16}$ into $A(p)$ yields equilibrium advertising,

$$A^* = \frac{a^4}{64}$$

where both equilibrium price and advertising expenditure are increasing in market demand, a.

- Plugging $p^* = \frac{a^3}{16}$ and $A^* = \frac{a^4}{64}$ into the profit function, equilibrium profits become

$$\begin{aligned}\pi^* &= p^* \left[a - p^* (A^*)^{-\frac{1}{2}}\right] - A^* \\ &= \frac{a^3}{16}\left[a - \frac{\frac{a^3}{16}}{\sqrt{\frac{a^4}{64}}}\right] - \frac{a^4}{64} \\ &= \frac{a^4}{64}\end{aligned}$$

which is increasing in market demand, a.

- Finally, substituting $a = 2$ into the above results, yields

$$\begin{aligned}p^* &= \frac{2^3}{16} = \frac{1}{2} \\ A^* &= \frac{2^4}{64} = \frac{1}{4} \\ \pi^* &= \frac{2^4}{64} = \frac{1}{4}\end{aligned}$$

4. **Regulating a natural monopoly.** A water supply company provides water to Pullman. The demand for water in Pullman is $p(q) = 10 - q$, and this company's costs are $c(q) = 1 + 2q$.

   (a) Depict the following in a figure: the demand curve $p(q)$, the associated marginal revenue $MR(q)$, the marginal cost of production $MC(q)$ and the average cost of production $AC(q)$. Discuss why this situation illustrates a "natural monopoly."

   - Figure 7.8 depicts the information provided in the exercise. The average cost curve is decreasing in output, implying that multiple producers are more costly than a single monopolist, i.e., the sum of their average costs will be larger than the monopolist's average costs. Then, a monopolist naturally becomes more cost-efficient than having several producers with a similar cost structure as the monopolist.

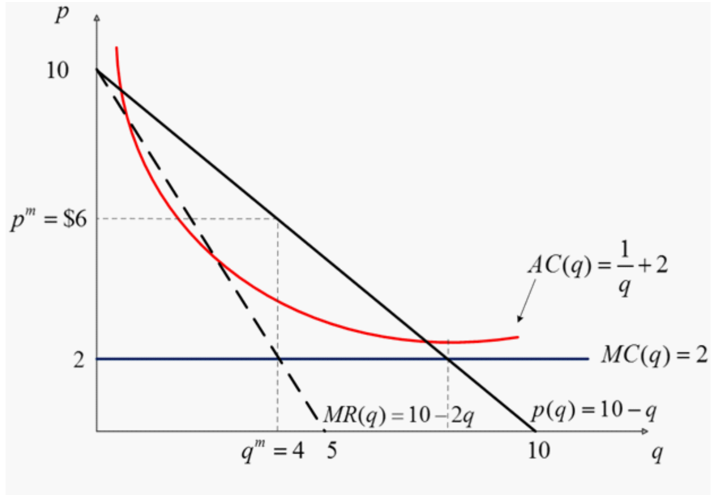

Figure 7.8. Natural monopoly.

- *Unregulated monopolist.* Find the amount of water that this firm will produce if left unregulated as a monopolist. Determine the corresponding prices and profits for the firm.
- The monopolist maximizes

$$\max_q \ (10 - q)q - (1 + 2q)$$

Taking first order conditions with respect to q, we find $10 - 2q^m - 2 = 0$. Solving for q we obtain a monopoly output of $q^m = 4$ units, which are sold at a price of $p^m = 10 - 4 = \$6$, with associated monopoly profits of

$$\pi^m = (10 - 4)4 - (1 + (2 \times 4)) = \$15.$$

(b) *Marginal cost pricing.* Determine the amount of water that this firm will produce if a regulatory agency in Pullman forces the firm to price according to marginal cost (i.e., to produce an amount of output q^* that solves $p(q^*) = MC(q^*)$). Find the corresponding prices and profits for the firm.

- In that case, the monopolist sets $10 - q^* = 2$, i.e., $q^* = 8$ units, at a price $p^* = 10 - 8 = \$2$, with corresponding losses of

$$\pi = (10 - 8)8 - (1 + (2 \times 8)) = -1.$$

This result arises because the presence of decreasing average costs, i.e., in figure 7.9 the production of $q^* = 8$ units yields per unit losses of

$$AC(8) - MC(8) = \left(\frac{1}{8} + 2\right) - 2 = \$0.125.$$

(c) *Price discrimination.* Consider now that the regulatory agency allows the monopoly to charge two different prices: a high price p_1 for the first q_1 units, and a low price $p(q^*)$ for the remaining $(q^* - q_1)$ (i.e., the units from q_1 up to the output level you found in part (c), q^*). In addition, the regulatory agency imposes the condition that the firm cannot make any profits, $\pi = 0$, when charging these two prices.

1. Find the value of q_1 and the associated price $p(q_1)$.
   - First, note that the value of q_1 must satisfy the "no profits" condition, that is

$$\pi = [p(q_1) - AC(q_1)]\, q_1 + [p(q^*) - AC(q^*)]\, (q^* - q_1) = 0$$

   and since we know from part (c) that $q^* = 8$ units, and that $p(q) = 10 - q$ and $AC(q) = \frac{1+2q}{q} = \frac{1}{q} + 2$, we can rewrite the above condition as

$$\pi = \left[(10 - q_1) - \left(\frac{1}{q_1} + 2\right)\right] q_1 + \left[(10 - 8) - \left(\frac{1}{8} + 2\right)\right] (8 - q_1) = 0$$

   which simplifies to

$$-q_1^2 + \frac{65}{8} q_1 - 2 = 0$$

   Solving for q_1 we obtain two solutions:
   - *First solution:* $q_1 = 0.82$ with a corresponding price of $p_1 = \$9.18$. This means that the first 0.82 units are sold at \$9.18 each, while the remaining 7.18 (up to 8 units) are sold at a price of \$2.
   - *Second solution:* $q_1 = 0.30$ with a corresponding price of $p_1 = \$9.7$. This means that the first 0.30 units are sold at \$9.7 each, while the remaining 9.7 (up to 8 units) are sold at a price of \$2.
2. Depict these two prices and quantities in a figure and shade the areas of benefits and losses for the firm.
   - Figure 7.9 depicts out above results: for the first q_1 units, the monopolist makes a profit of $p(q_1) - AC(q_1)$ per unit. For instance, if $q_1 = 0.82$, then the monopolist makes a profit of $(10 - 0.82) - \left(\frac{1}{0.82} + 2\right) = \$5.97$ per unit. In contrast, for the remaining $8 - q_1$ units, the monopolist incurs a loss measured by the distance between the average and marginal cost curves, i.e.,

$$AC(8) - MC(8) = \left(\frac{1}{8} + 2\right) - 2 = \$0.125$$

   per unit.

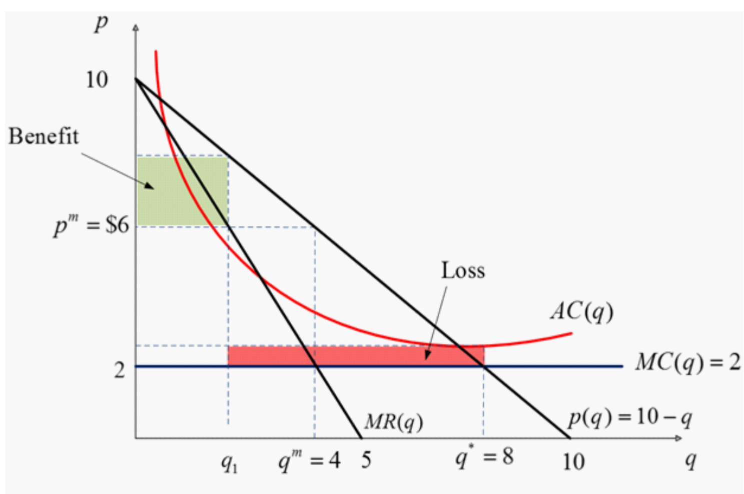

Figure 7.9. Discriminating natural monopoly.

5. **Monopsony, a general approach.** Consider a firm with production technology $f(x)$, where $x \in [0,1]$ stands for the input (e.g, gas supply), and $f(x)$ represents the output that is sold in a competitive market at a price p. The firm is subject to an input cost function $g(x)$ that is increasing and convex in input x. This happens, for example, when there is only one company supplying natural gas.

   (a) Define $\varepsilon_g \equiv \frac{\partial x}{\partial g(x)} \frac{g(x)}{x}$ to be the price elasticity of gas supply, measuring the percentage change of gas supply given one percent change in gas price. Setup the firm's profit-maximization problem to maximize $\pi(x) = pf(x) - g(x)x$, and show that

   $$p'f(x) = g(x)\left[1 + \frac{1}{\varepsilon_g}\right].$$

   - The firm chooses input x to solve the following profit maximization problem,

     $$\max_{x \geq 0} \pi(x) = pf(x) - g(x)x$$

     Differentiating with respect to x, and assuming interior solutions,

     $$pf'(x) - g'(x)x - g(x) = 0$$

     Rearranging, we obtain

     $$pf'(x) = g(x)\left[1 + \frac{g'(x)x}{g(x)}\right]$$

     Since $g'(x) = \frac{\partial g(x)}{\partial x}$, the above equality can be written as

     $$pf'(x) = g(x)\left[1 + \frac{\partial g(x)}{\partial x}\frac{x}{g(x)}\right]$$

     Simplifying, we have

     $$pf'(x) = g(x)\left[1 + \frac{1}{\varepsilon_g}\right]$$

   (b) Let $f(x) = x$ and $g(x) = x^{\beta}$, where $\beta > 1$. Use the expression found in part (a) to identify the optimal gas supply x^*. For simplicity, you may assume that $p = 1$ in the remainder of this exercise.

   - Since $f(x) = x$ and $g(x) = x^{\beta}$, we have $f'(x) = 1$ and $g'(x) = \beta x^{\beta-1}$, yielding a price elasticity of gas supply of

     $$\varepsilon_g = \frac{g(x)}{xg'(x)} = \frac{x^{\beta}}{x\beta x^{\beta-1}} = \frac{1}{\beta}$$

     Substituting $\varepsilon_g = \frac{1}{\beta}$ into the expression we found in part (a), yields

     $$1 = x^{\beta}(1+\beta)$$

     Rearranging, the optimal gas supply, x^*, solves

     $$x^* = (1+\beta)^{-\frac{1}{\beta}}$$

(c) *Comparative statics.* How does x^* change with β? Explain.

- Differentiating x^* with respect to β, and using the fact that

$$(1+\beta)^{-\frac{1}{\beta}} = \exp\left[-\frac{1}{\beta}\log(1+\beta)\right],$$

we can use Chain rule to derive

$$\begin{aligned}\frac{\partial x^*}{\partial \beta} &= \exp\left[-\frac{1}{\beta}\log(1+\beta)\right]\left[\frac{1}{\beta^2}\log(1+\beta) - \frac{1}{\beta(1+\beta)}\right] \\ &= (1+\beta)^{-\frac{1}{\beta}}\left[\frac{(1+\beta)\log(1+\beta)-\beta}{\beta^2(1+\beta)}\right]\end{aligned}$$

Since $(1+\beta)\log(1+\beta) - \beta > 0$ for all $\beta \geq 1$, we have that $\frac{\partial x^*}{\partial \beta} > 0$, indicating that, as the supply function becomes more convex, gas becomes less costly (remembering that $x \in [0,1]$), so that the firm can use more gas in producing output.

(d) *Numerical example.* Evaluate the firm's optimal gas supply x^* when $\beta = 1$, $\beta = 2$, $\beta = 4$, and $\beta \to +\infty$. Interpret.

- Substituting $\beta = 1$ into x^* yields

$$x^* = (1+1)^{-\frac{1}{1}} = 0.5.$$

- When $\beta = 2$, we obtain that

$$x^* = (1+2)^{-\frac{1}{2}} = 0.58.$$

- When $\beta = 4$, we have that

$$x^* = (1+4)^{-\frac{1}{4}} = 0.67.$$

- When $\beta \to +\infty$, we find that

$$\begin{aligned}\lim_{\beta\to\infty} x^* &= \lim_{\beta\to\infty} \exp\left[-\frac{\log(1+\beta)}{\beta}\right] \\ &= \exp\left[-\lim_{\beta\to\infty}\frac{\log(1+\beta)}{\beta}\right] \\ &= \exp\left[-\frac{\lim_{\beta\to\infty}\frac{d\log(1+\beta)}{d\beta}}{\lim_{\beta\to\infty}\frac{d\beta}{d\beta}}\right] \\ &= \exp\left[-\lim_{\beta\to\infty}\frac{1}{1+\beta}\right] \\ &= \exp[0] = 1.\end{aligned}$$

where we apply the Continuous Mapping Theorem in the second line since the exponential function is continuous in the limit, and the L'Hôpital's rule in the third line since both the numerator and denominator approach positive infinity with β.

- Intuitively, the supply curve, $g(x)$, bends away from the 45^o-line (where $\beta = 1$) when β increases (becoming more convex), ultimately allowing the firm to use more gas as this input becomes less expensive. A similar argument applies with the marginal cost curve, $g'(x)\,x + g(x)$, which becomes more convex as β increases. In the limit, when $\beta \rightarrow +\infty$, the marginal cost curve intersects with the marginal revenue product curve, $pf'(x)$, at $x = 1$ unit.