EconS 501 - Micro Theory I
Assignment #5 - Answer key

1. **Additional lottery.** An individual faces the monetary lottery L, where $(p_1, p_2, ..., p_K)$ with $p_k \geq 0$ denoting the probability of outcome k, and $(z_1, z_2, ..., z_K)$ representing the profile of monetary outcomes. Consider that we make the offer of replacing every monetary outcome, z_k, in the support of L with the lottery that yields $z_k - 1$ and $z_k + 1$ each of them with equal probability.

   (a) Describe the lottery that he faces if he accepts the offer.

   - For every monetary payoff z_k, yields prize $z_k + 1$ and $z_k - 1$ with probability $\frac{1}{2}p_k$. Therefore, the expected value of this lottery is

   $$\sum_{k=1}^{K} \frac{p_k}{2} v(z_k + 1) + \frac{p_k}{2} v(z_k - 1) = \frac{1}{2} \sum_{k=1}^{K} p_k \left[v(z_k + 1) + v(z_k - 1) \right].$$

   (b) Show that if he is strictly risk-averse, he will reject the offer.

   - By risk aversion, the Bernouilli utility function $v(.)$ is concave, entailing that

   $$v(z_k) > \frac{1}{2} v(z_k + 1) + \frac{1}{2} v(z_k - 1)$$

   for every payoff z_k. Intuitively, the utility of the certain payoff, $v(z_k)$, is higher than the utility of the average payoff (note that the average of $z_k + 1$ and $z_k - 1$ is z_k). Multiplying by p_k on both sides of the above inequality, yields

   $$p_k v(z_k) > p_k \frac{1}{2} \left[v(z_k + 1) + v(z_k - 1) \right]$$

   and summing across all k's, we obtain that

   $$\sum_{k=1}^{K} p_k v(z_k) > \sum_{k=1}^{K} p_k \frac{1}{2} \left[v(z_k + 1) + v(z_k - 1) \right]$$

   which means that a strictly risk-averse individual would reject the offer (i.e., his expected utility of the lottery, on the left, is larger than that of the offer, on the right side of the inequality).

2. **Casino.** An individual has wealth $w > 0$ and has to choose an amount $x > 0$, after which a lottery is conducted in which with probability α he earns $2x$ and with probability $1 - \alpha$ he loses x (his payoff is zero). Show that the amount he chooses, x, increases in probability α.

   - The individual chooses x to solve

   $$\max_{x \geq 0} \;\; \alpha v(w + 2x) + (1 - \alpha) v(w - x)$$

since with probability α he earns $2x$, and with probability $1-\alpha$ he loses x. Since $v(.)$ is concave, the above objective function is also concave (it is just a linear combination of v). Differentiating with respect to x, yields

$$2\alpha v'(w+2x) = (1-\alpha)v'(w-x).$$

Intuitively, the individual increases x until the expected marginal utility of winning (left-hand side) coincides with her expected marginal utility of losing (right-hand side).

- After rearranging, we can express the above expression as

$$\frac{2v'(w+2x)}{v'(w-x)} = \frac{1-\alpha}{\alpha}$$

The left-hand side is decreasing in x because of the concavity of v, i.e., $v'' < 0$ implies that v' is decreasing. In addition, the right-hand side is decreasing in α, implying that, as we increase α, the individual increases x.

- *Graphical representation.* To see this result graphically, depict a figure with x on the horizontal axis and a downward sloping curve representing the left-hand side. Then, plot the right-hand side as a flat horizontal line. The crossing point of the left-hand and right-hand sides represents the individual's choice of x. If α increases, $\frac{1-\alpha}{\alpha}$ decreases as well, implying that the flat horizontal line shifts downwards, crossing the curve depicting the left-hand side at a higher value of x.

3. **IA implies monotonicity.** Consider an individual with preference relation $\succsim$ over monetary lotteries, which satisfies the IA. Show that, for every two monetary outcomes, x and y, where $x > y$, we must have that

$$\alpha x + (1-\alpha)y \succsim \beta x + (1-\beta)y$$

also holds if and only if probabilities α and β satisfy $\alpha > \beta$. Interpret your results.

- Let $p_x = \alpha x + (1-\alpha)y$. Because the preference relation satisfies IA, $p_x \succ \alpha y + (1-\alpha)y = y$. Using the IA again, we obtain

$$p_x = \frac{\beta}{\alpha}p_x + \left(1-\frac{\beta}{\alpha}\right)p_x \succ \frac{\beta}{\alpha}p_x + \left(1-\frac{\beta}{\alpha}\right)\beta = \beta x + (1-\beta)y$$

implying that $p_x \succ \beta x + (1-\beta)y$.

- Intuitively, we say that a preference relation over lotteries is monotonic if, for two monetary outcomes x and y such that $x > y$, the preference relation ranks such lotteries by the probability that outcome x occurs. Alternatively, if $x > y$, an individual with monotonic preferences deems the compound lottery $\alpha x + (1-\alpha)y$ more attractive when the probability of outcome x, α, increases.

4. **Non-constant coefficient of absolute risk aversion**. Suppose that the utility function is given by

$$u(w) = aw - bw^2,$$

where $a, b > 0$, and $w > 0$ denotes income.

(a) Find the coefficient of absolute risk-aversion, $r_A(w,u)$. Does it increase or decrease in wealth? Interpret.

- First, note that $u' = a - 2bw$ and $u'' = -2b$. Hence, the Arrow-Pratt coefficient of absolute risk-aversion is

$$r_A(w,u) = -\frac{u''(w)}{u'(w)} = \frac{2b}{a - 2bw}$$

  Note that, as w rises, the denominator decreases, and as a consequence $r_A(w,u)$ rises, i.e., the decision maker becomes more risk averse as his wealth increases.

- Importantly, this exercise illustrates that, while the decision maker can have a concave utility function (indeed, $u'' = -2b < 0$, as illustrated in Figure 5.12, which depicts utility function $u(w) = aw - bw^2$ evaluated at parameters $a = 80$ and $b = 1$), the Arrow-Pratt coefficient of absolute risk aversion, $r_A(w,u)$, can increase as he becomes richer.

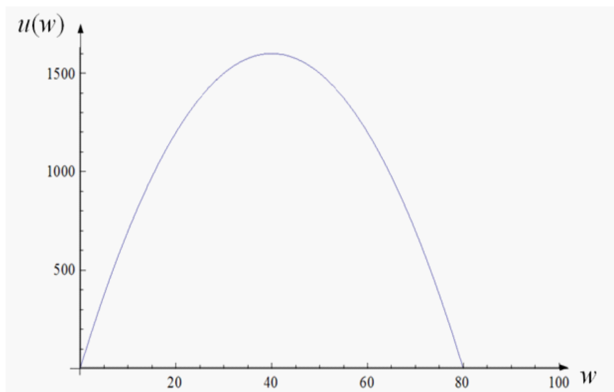

Figure 5.12. Utility function $u(w) = aw - bw^2$

(b) Let us now consider that this decision maker is deciding how much to invest in a risky asset. This risky asset is a random variable R, with mean $\overline{R} > 0$ and variance σ_R^2. Assuming that his initial wealth is w, state the decision maker's expected utility maximization problem, and find first order conditions.

- First, note that the decision maker's wealth (W in his utility function) is now a random variable $w + xR$, where x is the amount of risky asset that he acquires. Inserting this expression in the decision maker's utility function, and taking expectations we obtain that the decision maker selects his optimal investment in risky asset, x, in order to solve

$$\max_x \; E\left[a(w + xR) - b(w + xR)^2\right]$$

  And the associated first order condition with respect to x is

$$E\left[aR - 2bR(w + x^* R)\right] = 0$$

- We can use the definition of the variance of random variable R, $\sigma_R^2 = E[R^2] - \overline{R}^2$, to obtain $E[R^2] = \overline{R}^2 + \sigma_R^2$. Hence, the above first order condition can be simplified to

$$\begin{aligned} E\left[aR - 2bR\left(w + x^* R\right)\right] &= a\overline{R} - 2b\overline{R}w - E\left[2bR^2 x^*\right] = \\ &= a\overline{R} - 2b\overline{R}w - 2bx^*\left(\overline{R}^2 + \sigma_R^2\right) = 0 \end{aligned}$$

(c) What is the optimal investment in risky assets?

- Solving for x^* in the above expression, we obtain

$$x^* = \frac{(a - 2bw)\,\overline{R}}{2b\left(\overline{R}^2 + \sigma_R^2\right)}$$

(d) Show that the optimal amount of investment in risky assets is a decreasing function in wealth. Interpret.

- Differentiating x^* with respect to wealth, yields

$$\frac{\partial x^*}{\partial w} = -\frac{\overline{R}}{\left(\overline{R}^2 + \sigma_R^2\right)}$$

which is negative, since $\overline{R}$, $\sigma_R^2 > 0$. Intuitively, the larger the decision maker's wealth, the lower is the amount of risky assets he wants to hold. This explanation is consistent with his coefficient of absolute risk aversion found at the beginning of the exercise, where we showed that the individual becomes more risk averse as his wealth increases.

5. **Exercises 4 from Chapter 7 in Rubinstein.** A decision maker is to choose an action from a set A. The set of consequences is Z. For every action $a \in A$, the consequence z^* is realized with probability α and any $z \in Z \setminus z^*$ is realized with probability $r(a,z) = (1-\alpha)q(a,z)$. Assume that after making his choice he is told that z^* will not occur and is given a chance to change his decision.

(a) Show that if the decision maker obeys the Bayesian updating rule and follows vNM axioms, he will not change his decision.

- By the vNM Theorem, preferences exhibit expected utility representation. Before learning the information, the decision maker solves

$$\max_{a\in A} \sum_{z\in Z\setminus z^*} r\left(a,z\right)v(z) + \alpha v(z^*).$$

After learning that z^* will not occur, the decision maker updates his beliefs so that

$$r'\left(a,z\right) = \frac{r(a,z)}{1-\alpha} = q(a,z)$$

for all $z \in Z \setminus z^*$ and the decision maker solves

$$\max_{a \in A} \sum_{z \in Z \setminus z^*} r^{'}(a, z)\, v(z),$$

which yields the same solution. Intuitively, the decision maker cannot affect the probability of outcome z^* occuring, since it happen with probability p for all effort levels e. Hence, his optimal choice of a (e.g., effort) is unaffected by the information he received. (Of course, this result would not apply if outcome z^* was affected by the level of a chosen by the decision maker, i.e., if $r(a, z)$ was non-constant in a.

(b) Give an example where a decision maker who follows nonexpected utility preference relation or obeys a non-Bayesian updating rule is not time consistent.

- **Example 1.** Assume the decision maker has a "worst case" preference relation, where z_1 is the best prize, z_2 is the second best, and z^* is the worst. Let action a_1 yield z_1 for sure, and action a_2 yield z_1 and z_2 with equal probability, conditional on z^* not occurring. Then the decision maker will initially be indifferent between a_1 and a_2, but will strictly prefer a_1 after the information is revealed.
- **Example 2.** Assume that $Z = \{1, 2, 3\}$, that $z^* = 0$, and that the Bernouilli utility function is linear, $v(z) = z$. Assume that initially his beliefs are: $q(a_1, 2) = 1$, $q(a_2, 3) = 0.4$ and $q(a_2, 1) = 0.6$. In words, when the decision maker chooses action a_1, he believes that outcome 2 occurs with certainty; whereas when he chooses action a_2, he believes that outcome 3 occurs with probability 0.4 and outcome 1 happens with the remaining probability (0.6). Contingentally the decision maker chooses a_1. If he updates his beliefs and after he was lucky to avoid z^* he believes that he will be fortunate again, that is $q^{'}(a_2, 3) = 1$, then he will change his mind and choose a_2.