EconS 501 - Microeconomic Theory I
Midterm Exam #1 - Answer key

1. **Checking properties of preference relations.** Consider the following preference relation defined in $X = \mathbb{R}^2_+$. A bundle (x_1, x_2) is weakly preferred to another bundle (y_1, y_2), i.e., $(x_1, x_2) \succsim (y_1, y_2)$, if and only if

$$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} > \min\{3y_1 + 2y_2, 2y_1 + 3y_2\}$$

   (a) For any given bundle (y_1, y_2), draw the upper contour set, the lower contour set, and the indifference set of this preference relation.

   - *Upper contour set.* Take a bundle $(2, 1)$. Then,

$$\min\{3 * 2 + 2 * 1, 2 * 2 + 3 * 1\} = \min\{8, 7\} = 7.$$

   The upper contour set of this bundle is given by

$$\begin{aligned} UCS(2,1) &= \{(x_1, x_2) \succsim (2,1)\} \\ &= \{\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} > 7 \equiv \min\{8,7\}\} \end{aligned}$$

   which is graphically represented by all those bundles in $\mathbb{R}^2_+$ which are strictly above *both* lines $3x_1 + 2x_2 = 7$ and $2x_1 + 3x_2 = 7$. That is, for all (x_1, x_2) strictly above both lines

$$x_2 = \frac{7}{2} - \frac{3}{2}x_1 \text{ and } x_2 = \frac{7}{3} - \frac{2}{3}x_1.$$

   (See figure 1, which depicts these two lines and shades the set of bundles lying above both lines.)

   - *Lower contour set.* On the other hand, the lower contour set is defined as

$$\begin{aligned} LCS(2,1) &= \{(2,1) \succsim (x_1, x_2)\} \\ &= \{7 > \min\{3x_1 + 2x_2, 2x_1 + 3x_2\}\}, \end{aligned}$$

   which is graphically represented by all bundles (x_1, x_2) strictly below the maximum of the lines described above. For instance, bundle $(y_1, y_2) = (2.5, 0)$, which lies on the horizontal axis and between both lines' horizontal intercept, implies

$$\min\{3 \cdot 2.5 + 2 \cdot 0, 2 \cdot 2.5 + 3 \cdot 0\} = \min\{7.5, 5\} = 5$$

   Thus implying that this consumer prefers bundle $(x_1, x_2) = (2, 1)$ than $(y_1, y_2) = (2.5, 0)$. A similar argument applies to all other bundles lying above $x_2 = \frac{7}{2} - \frac{3}{2}x_1$ and below $x_2 = \frac{7}{3} - \frac{2}{3}x_1$, where bundle $(2.5, 0)$ also belongs; see the triangle that both lines form at the right-hand side of the figure. Similarly, bundles such as $(0, 2.5)$ yield

$$\min\{3 \cdot 0 + 2 \cdot 2.5, 2 \cdot 0 + 3 \cdot 2.5\} = \min\{5, 7.5\} = 5$$

which implies that the consumer also prefers bundle $(2,1)$ to $(0,2.5)$. An analogous argument applies to all bundles above line $x_2 = \frac{7}{2} - \frac{3}{2}x_1$ but below $x_2 = \frac{7}{3} - \frac{2}{3}x_1$ in the triangle at the left-hand side of figure 1.

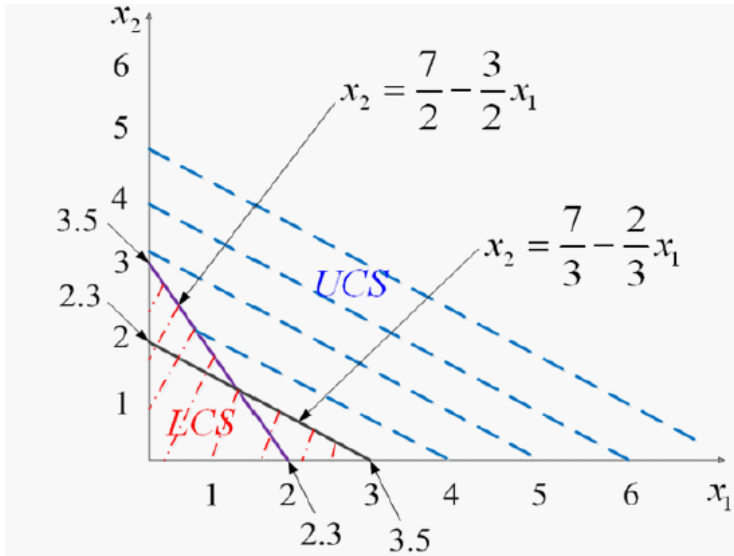

Figure 1. UCS and LCS of bundle $(2,1)$.

- *Indifference set.* Finally, there are no bundles for which the consumer is just indifferent between bundle (2,1) and any other bundle (note that there are no bundles for which the upper and lower contour set coincide or overlap). Hence, the indifference set is empty,

$$IND(2,1) = \varnothing$$

Here is an alternative approach to show that the indifference set is empty. First, note that both of the elements in the $\min\{\cdot\}$ operator are real numbers, i.e., $(3x_1 + 2x_2) \in \mathbb{R}_+$ and $(2x_1 + 3x_2) \in \mathbb{R}_+$, thus implying that the minimum

$$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} = a$$

exists and it is also a real number, $a \in \mathbb{R}_+$. Similarly, the minimum

$$\min\{3y_1 + 2y_2, 2y_1 + 3y_2\} = b$$

exists and $b \in \mathbb{R}_+$. Therefore, we can easily compare a and b, obtaining that either $a > b$, which implies $(x_1, x_2) \succsim (y_1, y_2)$; or $a < b$, which implies $(y_1, y_2) \succsim (x_1, x_2)$. Finally, note that, for this preference relation, we cannot find that both $a > b$ and $b > a$. Therefore, we cannot have that the individual is indifferent between bundles x and y, confirming that the indifference set is nil.

(b) Check if this preference relation satisfies: (i) completeness, (ii) transitivity, and (iii) weak convexity.

- *Completeness*. From our analysis of the UCS, LCS, and IND in figure 1, we can claim that two bundles on the lower bound of the UCS, such as $(2, 1)$ and $(3.5, 0)$, cannot be ranked according to this preference relation. This occurs because the lower bound of the UCS does not belong to the UCS and, similarly, the upper bound of the LCS in the figure does not belong to the LCS.
- *Transitivity*. We need to show that, for any three bundles (x_1, x_2), (y_1, y_2) and (z_1, z_2) such that

  $$(x_1, x_2) \succsim (y_1, y_2) \text{ and } (y_1, y_2) \succsim (z_1, z_2), \text{ then } (x_1, x_2) \succsim (z_1, z_2)$$

  First, note that $(x_1, x_2) \succsim (y_1, y_2)$ implies

  $$a \equiv \min\{3x_1 + 2x_2, 2x_1 + 3x_2\} > \min\{3y_1 + 2y_2, 2y_1 + 3y_2\} \equiv b$$

  and $(y_1, y_2) \succsim (z_1, z_2)$ implies that

  $$b \equiv \min\{3y_1 + 2y_2, 2y_1 + 3y_2\} > \min\{3z_1 + 2z_2, 2z_1 + 3z_2\} \equiv c$$

  Combining both conditions we have that $a > b > c$, which implies that $a > c$. Hence, we have that

  $$\min\{3x_1 + 2x_2, 2x_1 + 3x_2\} > \min\{3z_1 + 2z_2, 2z_1 + 3z_2\}$$

  and thus $(x_1, x_2) \succsim (z_1, z_2)$, implying that this preference relation is transitive.
- *Weak Convexity*. This property implies that the upper contour set must be convex. That is, if bundle (x_1, x_2) is weakly preferred to (y_1, y_2), $(x_1, x_2) \succsim (y_1, y_2)$, then the linear combination of these two bundles is also weakly preferred to (y_1, y_2),

  $$\lambda(x_1, x_2) + (1 - \lambda)(y_1, y_2) \succsim (y_1, y_2) \quad \text{for any } \lambda \in [0, 1]$$

  For compactness, let $a \equiv 3x_1 + 2x_2$, $b \equiv 2x_1 + 3x_2$, $c \equiv 3y_1 + 2y_2$ and $d \equiv 2y_1 + 3y_2$. Hence, the property that $(x_1, x_2) \succsim (y_1, y_2)$ implies $\min\{a, b\} > \min\{c, d\}$. We therefore need to show that

  $$\min\{\lambda a + (1 - \lambda)c, \lambda b + (1 - \lambda)d\} > \min\{c, d\}$$

  1. *First case*: $\min\{a, b\} = a$, $\min\{c, d\} = c$ and without loss of generality, $a > c$. Therefore,

     $$\min\{\lambda a + (1 - \lambda)c, \lambda b + (1 - \lambda)d\} = \lambda a + (1 - \lambda)c$$

     and $\lambda a + (1 - \lambda)c > \min\{c, d\} = c$. For this case, convexity is satisfied.
  2. *Second case*: $\min\{a, b\} = a$, $\min\{c, d\} = d$ and without loss of generality, $a > d$. Hence,

     $$\min\{\lambda a + (1 - \lambda)c, \lambda b + (1 - \lambda)d\} = \lambda a + (1 - \lambda)c$$

     and $\lambda a + (1 - \lambda)c > \min\{c, d\} = d$ given that $a > d$ and $c > d$. For this case, convexity is satisfied as well. An analogous argument applies in the other two cases, in which $\min\{a, b\} = b$ and $\min\{c, d\} = c$, and in which $\min\{a, b\} = b$ but $\min\{c, d\} = d$.

2. **Finding the compensating and equivalent variation with little information.** Consider a consumer who, facing a initial price vector $p^0 \in \mathbb{R}^n_{++}$ for n commodities, purchases a bundle $x \in \mathbb{R}^n_+$ with an income of w dollars. Assume that the price of all goods experience a common increase measured by factor $\theta > 1$.

   (a) Compute the compensating variation (CV) of this price increase.

   - Using the expenditure function, the CV is

   $$CV = e(p^1, u^0) - e(p^0, u^0)$$

   where p^1 and p^0 denote the final and initial price vector, respectively, and u^0 represents the utility level that the consumer achieves at the initial price-wealth pair (p^0, w). In this exercise, we are informed that final prices p^1 satisfy $p^1 = \theta p^0$, thus implying that the above expression for CV can be rewritten as

   $$CV = e(\theta p^0, u^0) - e(p^0, u^0)$$

   Recall that the expenditure function is homogeneous of degree one in prices, i.e., $e(\theta p^0, u^0) = \theta e(p^0, u^0)$. In words, increasing the prices of all goods by a common factor θ increases the consumer's minimal expenditure (the expenditure he needs to reach utility level u^0) by exactly θ. In addition, the consumer spends w dollars, i.e., $e(p^0, u^0) = w$. These properties reduce the expression of the CV to

   $$\begin{aligned} CV &= e(\theta p^0, u^0) - e(p^0, u^0) = \\ &= \theta \underbrace{e(p^0, u^0)}_{w} - \underbrace{e(p^0, u^0)}_{w} = \\ &= \theta w - w = w(\theta - 1) \end{aligned}$$

   For instance, increasing all prices by 50%, i.e., $\theta = 1.5$, yields a compensating variation of $CV = 0.5w$, which implies that the consumer needs to receive half of his initial wealth in order to be able to reach the same utility level as before the price change.

   (b) Compute the equivalent variation (EV) of this price increase.

   - Using the expenditure function, the EV is

   $$EV = e(p^1, u^1) - e(p^0, u^1)$$

   where u^1 represents the utility level that the consumer achieves at the final price-wealth pair (p^1, w). In this exercise, we are informed that $p^1 = \theta p^0$, or $p^0 = \frac{1}{\theta} p^1$, implying that the above expression for EV can be rewritten as

   $$EV = e(p^1, u^1) - e\left(\frac{1}{\theta} p^1, u^0\right)$$

   Since the expenditure function is homogeneous of degree one in prices, i.e., $e\left(\frac{1}{\theta} p^1, u^0\right) = \frac{1}{\theta} e\left(p^1, u^0\right)$, and the consumer spends w dollars, $e(p^1, u^1) = w$.

These properties reduce the EV to

$$
\begin{aligned}
EV &= e(p^1, u^1) - e\left(\frac{1}{\theta}p^1, u^0\right) = \\
&= \underbrace{e(p^0, u^0)}_{w} - \frac{1}{\theta}\underbrace{e\left(p^1, u^0\right)}_{w} = \\
&= w - \frac{1}{\theta}w = w\left(1 - \frac{1}{\theta}\right)
\end{aligned}
$$

Following the same numerical example as in section (a), if all prices experience a 50% increase, i.e., $\theta = 1.5$, the equivalent variation would be $EV = 0.3w$, thus suggesting that, before the price increase, the consumer would need to give up a third of his wealth in order to be as worse off as he will be after the price increase.

3. **Marginal cost being independent of an input price.** Consider the production function $f(h(z_1) + z_2)$, where $f(\cdot)$ is increasing, $h(\cdot)$ is an increasing concave function which satisfies $h'(0) = \infty$ and $h'(\infty) = 0$.

   (a) Given the input price vector w, show that for large enough output levels, the input demand correspondence of input 2, $z_2(w, q)$, must be strictly positive.

   - Define $v = h(z_1) + z_2$. Consider the profit maximization problem,

$$\max_{z \geq 0} \quad p \cdot f(h(z_1) + z_2)$$

$$\text{subject to} \quad f(z) \leq q$$

Since we seek to show that $z_2(w, q) > 0$, let's operate by contradiction, by supposing that the optimal level of input 2 is zero. Then $z_1 > 0$ and FOC are as follows:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial z_1} &= pf'(v)h'(z_1) - \lambda w_1 = 0. \\
\frac{\partial \mathcal{L}}{\partial z_2} &= pf'(v) - \lambda w_2 \leq 0.
\end{aligned}
$$

Solving for the shadow price λ, it follows that $\frac{pf'(v)h'(z_1)}{w_1} = \lambda \geq \frac{pf'(v)}{w_2}$, or $h'(z_1) \geq \frac{w_1}{w_2}$. In addition $h(\cdot)$ is concave, i.e, $h''(z_1) < 0$, and $h'(\infty) = 0$; as depict on the figure below. It follows that for all sufficiently large output and hence z_1, $h'(z_1) < \frac{w_1}{w_2}$ (see figure 2). But this contradicts our earlier conclusion. Thus for sufficiently large output the FOC cannot hold with

$z_2 = 0$.

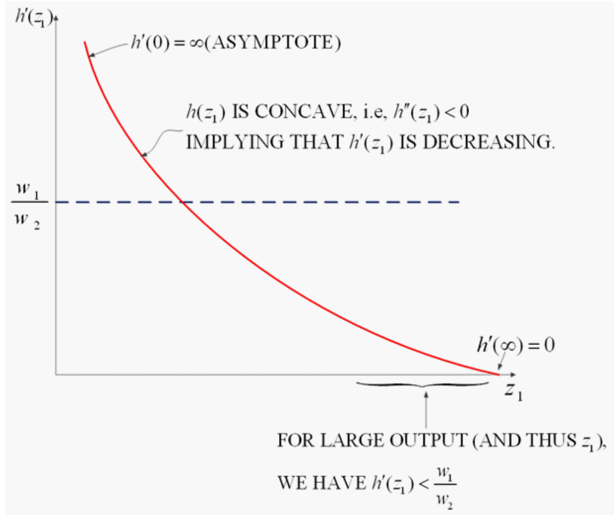

Figure 2. Function $h'(z_1)$

(b) Assuming that $z_2(w, q) > 0$ (as shown in the previous part), write down the firm's cost function as a function of $f^{-1}(q)$ and z_1 alone. Hence, show that the input demand correspondence of input 1, $z_1(w, q)$, is independent of q.

- From part (a), q is sufficiently large so both inputs are strictly positive. From the production function, we obtain $f^{-1}(q) = h(z_1) + z_2$. Solving for z_2, yields $z_2 = f^{-1}(q) - h(z_1)$ and so total cost is

$$\begin{aligned} c &= w_1 z_1 + w_2 z_2 \\ &= w_1 z_1 + w_2(f^{-1}(q) - h(z_1)). \end{aligned}$$

The FOC for minimizing total cost is $w_1 - w_2 h^{'}(z_1) = 0$ or, after rearranging, $h^{'}(z_1) = \frac{w_1}{w_2}$ Thus the cost minimizing level of input 1 is a function only of the input price ratio and not of output.

(c) Show that the marginal cost is independent of w_1.

- Appealing to the Envelope Theorem, we can write the marginal cost as

$$\frac{\partial c}{\partial q} = w_2 \frac{d}{dq} f^{-1}(q).$$

which is independent of w_1.