EconS 501 - Micro Theory I
Assignment #4 - Answer key

1. **Two factories.** A producer can use two factories, 1 and 2, to produce units of the same good. The production function of factory 1 is $q = \sqrt{z_1}$ and that of factory 2 is $q = \sqrt{z_2}$, where z_i denotes the amount of input used in factory $i = \{1, 2\}$. The price of each unit of input is 1, for both z_1 and z_2, and the cost of activating a factory is $k > 0$. Find this producer's cost function.

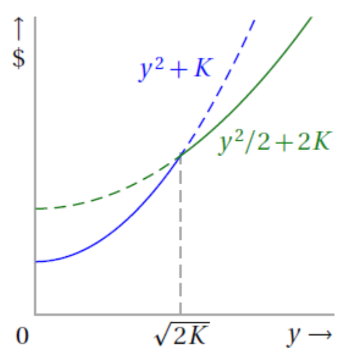
   - *One factory.* If the producer activates only one factory, his cost of producing y units is $y^2 + k$.
   - *Two factories.* If he activates both factories, he produces an aggregate output of y and, since factories have the same technology, he produces $\frac{y}{2}$ units in each factory, implying that the total cost of production is $2\frac{y^2}{4} + 2k = \frac{y^2}{2} + 2k$.
   - *Cost comparison.* Comparing our above results, we find that activating only one factory is less costly if

     $$y^2 + k \leq \frac{y^2}{2} + 2k,$$

     or after rearranging $y \leq \sqrt{2k}$. Otherwise, activating both factories is less costly. In summary, the total cost function is given by

     $$c(y) = \begin{cases} y^2 + k & \text{if } y \leq \sqrt{2k} \\ \frac{y^2}{2} + 2k & \text{otherwise.} \end{cases}$$

     as depicted in the figure below (lower envelope of curves $y^2 + k$ and $\frac{y^2}{2} + 2k$).



2. **A producer with a cost of firing workers.** A producer uses one input, workers, to produce output according to a production function f. She has already hired z_0 workers. She can fire some or all of them, or hire more workers. The wage of a worker is $w > 0$ and the price of output is $p > 0$.

   Compare the producer's behavior if she maximizes profit to her behavior if she also takes into account that firing workers causes her to fell as if she bears the cost $L > 0$ per fired worker.

- The producer who maximizes profits solves

$$\max_{z \geq 0} \; pf(z) - wz$$

with first-order condition $pf'(z) \geq w$. However, the producer who faces a cost when firing workers maximizes

$$pf(z) - wz - L(z_0 - z) \;\text{ if } z < z_0\text{, but } pf(z) - wz \;\text{ otherwise.}$$

with first-order condition $pf'(z) \geq w - L$ if $z < z_0$, but $pf'(z) \geq w$ otherwise.

- Therefore, we find three cases, according to the value of $pf'(z_0)$:
  - If $pf'(z_0) \geq w$, then both producers choose the value of z for which $pf'(z) = w$.
  - If $w > pf'(z_0) > w - L$, then the profit-maximizing producer chooses z such that $pf'(z) = w$, while the producer who faces a cost from firing workers chooses z_0.
  - If $pf'(z_0) \leq w - L$, then the profit-maximizing producer chooses z such that $pf'(z) = w$, while the producer who faces a cost from firing workers chooses z such that $pf'(z_0) = w - L$.

3. **Exercises from FMG (Chapter 6):**

6.4 **Exercise 6.4. Distribution of tax burden.** Consider a competitive market in which the government will be impossing an ad valorem tax of τ. Aggregate demand curve is $x(p) = Ap^{\varepsilon}$, where $A > 0$ and $\varepsilon < 0$, and aggregate supply curve $q(p) = \alpha p^{\gamma}$, where $\alpha > 0$ and $\gamma > 0$. Denote $\kappa = (1 + \tau)$. Assume that a partial equilibrium analysis is valid.

(a) Evaluate how the equilibrium price is affected by a marginal increase in the tax, i.e., a marginal increase in κ.

- To compute the change in the price received by producers, we can use the results from Example 6.2

$$\begin{aligned} p^{*\prime}(0) &= -\frac{x'(p_*)}{x'(p_*) - q'(p_*)} = -\frac{A\varepsilon p_*^{\varepsilon-1}}{A\varepsilon p_*^{\varepsilon-1} - \alpha\gamma p_*^{\gamma-1}} = -\frac{A\varepsilon p_*^{\varepsilon}}{A\varepsilon p_*^{\varepsilon} - \alpha\gamma p_*^{\gamma}} = \\ &= -\frac{\varepsilon x(p^*)}{\varepsilon x(p^*) - \gamma q(p^*)} = -\frac{\varepsilon}{\varepsilon - \gamma}. \end{aligned}$$

(We have multiplied both the numerator and the denominator by p^* and used the fact that p^* is an equilibrium price, which entails $x(p^*) = q(p^*)$.) The price paid by consumers is $(p^*) + t$, and its derivative with respect to t at $t = 0$ is

$$p'(0) + 1 = -\frac{\varepsilon}{\varepsilon - \gamma} + 1 = -\frac{\gamma}{\varepsilon - \gamma}.$$

(b) Describe the incidence of the tax when $\gamma = 0$.

- From the above expression,

$$p'(0) + 1 = -\frac{\varepsilon}{\varepsilon - \gamma} + 1 = -\frac{\gamma}{\varepsilon - \gamma}.$$

we can see that when $\gamma = 0$ (supply is perfectly inelastic) or $\varepsilon \to -\infty$ (demand is perfectly elastic), the price paid by consumers is unchanged, and the price received by producers decreases by the amount of the tax. That is, producers bear the full effect of the tax while consumers are essentially unaffected.

(c) What is the tax incidence when, instead, $\epsilon = 0$?

- On the other hand, when $\varepsilon = 0$ (demand is perfectly inelastic) or $\gamma \to \infty$ (supply is perfectly elastic), the price received by producers is unchanged and the price paid by consumers increases by the amount of the tax. That is, consumers bear now the full burden of the tax.

(d) What happens when each of these elastiticities approaches ∞ in absolute value?

- As suggested above, when $\varepsilon \to -\infty$ (demand is perfectly elastic), the price paid by consumers is unchanged, and the price received by producers decreases by the amount of the tax. In contrast, when $\gamma \to \infty$ (supply is perfectly elastic), the price received by producers is unchanged and the price paid by consumers increases by the amount of the tax.

6.8 **Exercise 6.8, Barter economies.** Consider the following indirect utility functions for consumers A and B

$$\begin{aligned} v^A(\mathbf{p}, m) &= \ln m - \frac{1}{2}\ln p_1 - \frac{1}{2}\ln p_2 \\ v^B(\mathbf{p}, m) &= \left(\frac{1}{p_1} + \frac{1}{p_2}\right) m \end{aligned}$$

Initial endowments coincide across consumers, $\mathbf{e}^A = \mathbf{e}^B = (5.8, 2.1)$. Assuming good 1 is the numeraire, $p_1 = 1$, find the equilibrium price vector $\mathbf{p}^*$.

- By Walras' law we know that if the market for good 1 clears, $z_1(\mathbf{p}) = 0$ then so does the market of good 2, $z_2(\mathbf{p}) = 0$. Let us then take the market of good 1, where $z_1(\mathbf{p}) = 0$ implies

$$e_1^A + e_1^B = x_1^A(\mathbf{p}, m) + x_1^B(\mathbf{p}, m)$$

where $e_1^A + e_1^B = 5.8 + 5.8$. The Walrasian demand functions can be recovered from the indirect utility function using Roy's identity, as follows

$$x_1^A(\mathbf{p}, m^A) = -\frac{\frac{\partial v^A(\mathbf{p}, m^A)}{\partial p_1}}{\frac{\partial v^A(\mathbf{p}, m^A)}{\partial m^A}} = -\frac{-\frac{1}{2p_1}}{\frac{1}{m^A}} = \frac{m^A}{2p_1}$$

for consumer A, and similarily for consumer B,

$$x_1^B(\mathbf{p}, m^B) = -\frac{\frac{\partial v^B(\mathbf{p}, m^B)}{\partial p_1}}{\frac{\partial v^B(\mathbf{p}, m^B)}{\partial m^B}} = -\frac{-\frac{m^B}{2p_1^2}}{\frac{1}{p_1} + \frac{1}{p_2}} = \frac{\frac{m^B}{2p_1^2}}{\frac{1}{p_1} + \frac{1}{p_2}}$$

In addition, since their initial endowments coincide $m^A = m^B = m$. In particular, the market value of their endowments, m, is

$$m = p_1 e_1^A + p_2 e_2^A = 5.8 + 2.1p_2$$

since good 1 is the numeraire, i.e., $p_1 = 1$. Plugging $m = 5.8 + 2.1p_2$ into the Walrasian demands found above, and using $p_1 = 1$, yields

$$x_1^A(\mathbf{p}, m^A) = \frac{5.8 + 2.1p_2}{2} \text{ and } x_1^B(\mathbf{p}, m^B) = \frac{5.8 + 2.1p_2}{1 + \frac{1}{p_2}}$$

Therefore, the initial market clearing condition for good 1, $e_1^A + e_1^B = x_1^A(\mathbf{p}, m) + x_1^B(\mathbf{p}, m)$ becomes

$$5.8 + 5.8 = \frac{5.8 + 2.1p_2}{2} + \frac{5.8 + 2.1p_2}{1 + \frac{1}{p_2}}$$

where, solving for p_2, yields an equilibrium price of $p_2^* = 2$. Since good 1 acted as the numeraire, this result implies that the equilibrium price of good 2 needs to be double that of good 1, i.e., the equilibrium price ratio is $\frac{p_2^*}{p_1^*} = 1.98$.

6.16 **Exercise 6.16. Concave/convex contract curve.** Consider an economy with two consumers, A and B, with utility functions

$$\begin{aligned} u^A(x^A, y^A) &= \left(x^A\right)^{\alpha} \left(y^A\right)^{1-\alpha} \text{ and} \\ u^B(x^B, y^B) &= \left(x^B\right)^{\beta} \left(y^B\right)^{1-\beta} \text{ where } \alpha, \beta > 0 \end{aligned}$$

where consumer A's endowment is (e_1^A, e_2^A), and that of individual B is (e_1^B, e_2^B).

(a) Find their contract curve, expressing it as a function of x^A, that is, $y^A = f(x^A)$.

- Starting with consumer A's UMP

$$\begin{aligned} &\max \quad \left(x^A\right)^{\alpha} \left(y^A\right)^{1-\alpha} \\ \text{subject to} \quad &p_1 x^A + p_2 y^A = p_1 e_1^A + p_2 e_2^A \end{aligned}$$

with FOCs

$$\begin{aligned} \alpha \left(x^A\right)^{\alpha-1} \left(y^A\right)^{1-\alpha} - p_1 \lambda^A &= 0 \\ (1-\alpha) \left(x^A\right)^{\alpha} \left(y^A\right)^{-\alpha} - p_2 \lambda^A &= 0 \\ p_1 e_1^A + p_2 e_2^A - p_1 x^A - p_2 y^A &= 0 \end{aligned}$$

Combining the first two FOCs yields

$$\frac{p_1}{p_2} = \frac{\alpha}{1-\alpha} \cdot \frac{y^A}{x^A}$$

- Next, we use consumer B's UMP

$$\begin{array}{rl} \max & \left(x^B\right)^{\beta}\left(y^B\right)^{1-\beta} \\ \text{subject to} & p_1 x^B + p_2 y^B = p_1 e_1^B + p_2 e_2^B \end{array}$$

with FOCs

$$\begin{aligned} \beta\left(x^B\right)^{\beta-1}\left(y^B\right)^{1-\beta} - p_1\lambda^B &= 0 \\ (1-\beta)\left(x^B\right)^{\beta}\left(y^B\right)^{-\beta} - p_2\lambda^B &= 0 \\ p_1 e_1^B + p_2 e_2^B - p_1 x^B - p_2 y^B &= 0 \end{aligned}$$

Combining the first two FOCs yields

$$\frac{p_1}{p_2} = \frac{\beta}{1-\beta}\cdot\frac{y^B}{x^B}$$

This leaves us with two expressions for $\frac{p_1}{p_2}$, one for each consumer

$$\frac{p_1}{p_2} = \frac{\alpha}{1-\alpha}\cdot\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\cdot\frac{y^B}{x^B}$$

Using our feasibility constraints

$$\begin{aligned} x^A + x^B &= e_1^A + e_1^B \\ y^A + y^B &= e_2^A + e_2^B \end{aligned}$$

we can substitute for x^B and y^B in our price ratio equations yielding

$$\frac{\alpha}{1-\alpha}\cdot\frac{y^A}{x^A} = \frac{\beta}{1-\beta}\cdot\frac{e_2^A + e_2^B - y^A}{e_1^A + e_1^B - x^A}$$

and solving this expression for y^A yields the contract curve,

$$y^A = x^A\left[\frac{\beta(1-\alpha)(e_2^A + e_2^B)}{\alpha(1-\beta)(e_1^A + e_1^B) + x^A(\beta-\alpha)}\right]$$

(b) Show that such contract curve is convex if $\alpha > \beta$ but concave otherwise.

- Starting with our contract curve, the numerator is trivially positive. Looking at the denominator, we can substitute the feasibility condition to have

$$\alpha(1-\beta)(x^A + x^B) + x^A(\beta-\alpha) = \beta(1-\alpha)x^A + \alpha(1-\beta)x^B > 0$$

and therefore, the denominator is unambiguously positive. Hence, $y^A = f(x^A) > 0$. Taking the first derivative,

$$f'(x^A) = \frac{\alpha\beta(1-\alpha)(1-\beta)(e_1^A + e_1^B)(e_2^A + e_2^B)}{\left[\alpha(1-\beta)(e_1^A + e_1^B) + x^A(\beta-\alpha)\right]^2}$$

Again, the numerator is trivially positive, and we have already shown that the denominator is also positive. Hence, $f'(x^A) > 0$. Lastly, we look at the second derivative of the contract curve

$$f''(x^A) = (\alpha - \beta) \cdot \frac{2\alpha\beta(1-\alpha)(1-\beta)(e_1^A + e_1^B)(e_2^A + e_2^B)}{\left[\alpha(1-\beta)(e_1^A + e_1^B) + x^A(\beta - \alpha)\right]^3}$$

Looking at the fraction, both the numerator and denominator are unambiguously positive. All that remains is the $\alpha - \beta$ term before the ratio, which is positive when $\alpha > \beta$. For the contract curve to be convex, we require $f''(x^A)$ to be positive, and thus $\alpha > \beta$. In contrast, when $\alpha < \beta$, $f''(x^A) < 0$ and the contract curve is concave.

6.24 **Exercise 6.24, Equilibrium with production.** Consider an economy with two goods, 1 and 2, both of them being produced by using capital and labor. Firms are price takers, and output prices are determined in the international market. The output factors of goods 1 and 2 are

$$\begin{aligned} q_1 &= (K_1)^{\frac{1}{4}} (L_1)^{\frac{3}{4}} \\ q_2 &= (K_2)^{\frac{3}{4}} (L_2)^{\frac{1}{4}} \end{aligned}$$

(a) Find the marginal cost for each firm.

- *Firm 1.* Starting with firm 1's cost minimization problem,

$$\begin{aligned} &\min_{K_1, L_1 \geq 0} \quad w_K K_1 + w_L L_1 \\ &\text{subject to} \quad q_1 = (K_1)^{\frac{1}{4}} (L_1)^{\frac{3}{4}} \end{aligned}$$

We can express this minimization problem as a maximization problem of the negative of the objective function, or

$$\begin{aligned} &\max_{K_1, L_1 \geq 0} \quad - w_K K_1 - w_L L_1 \\ &\text{subject to} \quad q_1 = (K_1)^{\frac{1}{4}} (L_1)^{\frac{3}{4}} \end{aligned}$$

Differentiating yields

$$\begin{aligned} -w_K + \frac{1}{4}\lambda_1 (K_1)^{-\frac{3}{4}} (L_1)^{\frac{3}{4}} &= 0 \\ -w_L + \frac{3}{4}\lambda_1 (K_1)^{\frac{1}{4}} (L_1)^{-\frac{1}{4}} &= 0 \\ (K_1)^{\frac{1}{4}} (L_1)^{\frac{3}{4}} - q_1 &= 0 \end{aligned}$$

Combining the first two first-order conditions and rearranging yields

$$\frac{w_K}{w_L} = \frac{1}{3} \cdot \frac{L_1}{K_1} \implies L_1 = \frac{3w_K}{w_L} K_1$$

and substituting this into the third first-order condition gives

$$(K_1)^{\frac{1}{4}}\left(\frac{3w_K}{w_L}K_1\right)^{\frac{3}{4}} - q_1 = 0$$

Solving this expression, gives firm 1's factor demand for capital,

$$K_1(w_K, w_L, q_1) = q_1\left(\frac{w_L}{3w_K}\right)^{\frac{3}{4}}$$

and plugging this value in the expression above will give firm 1's factor demand for labor,

$$L_1(w_K, w_L, q_1) = \frac{3w_K}{w_L}\left[q_1\left(\frac{w_L}{3w_K}\right)^{\frac{3}{4}}\right] = q_1\left(\frac{3w_K}{w_L}\right)^{\frac{1}{4}}$$

We can determine firm 1's total cost function by substituting our factor demands into the objective function (i.e., the minimal cost of producing output level q_1 when input prices are w_K and w_L)

$$C_1(w_K, w_L, q_1) = w_K K_1(w_K, w_L, q_1) + w_L L_1(w_K, w_L, q_1) = \frac{4}{3^{\frac{3}{4}}}q_1(w_K)^{\frac{1}{4}}(w_L)^{\frac{3}{4}}$$

with marginal cost

$$MC_1 = \frac{4}{3^{\frac{3}{4}}}(w_K)^{\frac{1}{4}}(w_L)^{\frac{3}{4}}$$

- *Firm 2.* Next, we set up firm 2's cost minimization problem,

$$\begin{aligned} &\max_{K_2, L_2 \geq 0} \quad - w_K K_2 - w_L L_2 \\ \text{subject to} \quad & q_2 = (K_2)^{\frac{3}{4}}(L_2)^{\frac{1}{4}} \end{aligned}$$

Differentiating yields

$$\begin{aligned} -w_K + \frac{3}{4}\lambda_2 (K_2)^{-\frac{1}{4}}(L_2)^{\frac{1}{4}} &= 0 \\ -w_L + \frac{1}{4}\lambda_2 (K_2)^{\frac{3}{4}}(L_1)^{-\frac{3}{4}} &= 0 \\ (K_1)^{\frac{3}{4}}(L_1)^{\frac{1}{4}} - q_2 &= 0 \end{aligned}$$

Combining the first two first-order conditions and rearranging yields

$$\frac{w_K}{w_L} = 3 \cdot \frac{L_2}{K_2} \implies L_2 = \frac{w_K}{3w_L}K_2$$

and substituting this into the third first-order condition gives

$$(K_2)^{\frac{3}{4}}\left(\frac{w_K}{3w_L}K_2\right)^{\frac{1}{4}} - q_2 = 0$$

Solving this expression, gives firm 2's factor demand for capital,

$$K_2(w_K, w_L, q_2) = q_2 \left(\frac{3w_L}{w_K}\right)^{\frac{1}{4}}$$

and plugging this value in the expression above will give firm 1's factor demand for labor,

$$L_2(w_K, w_L, q_2) = \frac{w_K}{3w_L}\left[q_2 \left(\frac{3w_L}{w_K}\right)^{\frac{1}{4}}\right] = q_2 \left(\frac{w_K}{3w_L}\right)^{\frac{3}{4}}$$

We can determine firm 1's total cost function by substituting our factor demands into the objective function (i.e., minimal cost of producing output level q_2)

$$C_2(w_K, w_L, q_2) = w_K K_2(w_K, w_L, q_2) + w_L L_2(w_K, w_L, q_2) = \frac{4}{3^{\frac{3}{4}}} q_2 (w_K)^{\frac{3}{4}} (w_L)^{\frac{1}{4}}$$

with marginal cost

$$MC_2 = \frac{4}{3^{\frac{3}{4}}} (w_K)^{\frac{3}{4}} (w_L)^{\frac{1}{4}}$$

(b) Use the results from part (a) to connect your result with the Stopler-Samuelson theorem.

- From part (a), we found two ratios for $\frac{w_L}{w_K}$,

$$\frac{w_K}{w_L} = \frac{1}{3} \cdot \frac{L_1}{K_1} \text{ for firm 1, and } \frac{w_K}{w_L} = 3 \cdot \frac{L_2}{K_2} \text{ for firm 2}$$

Therefore,

$$\frac{L_1}{K_1} = 9\frac{L_2}{K_2}$$

Naturally, this implies that

$$\frac{L_1(w_K, w_L, q_1)}{K_1(w_K, w_L, q_1)} > \frac{L_2(w_K, w_L, q_2)}{K_2(w_K, w_L, q_2)}$$

That is, firm 1 is more labor intensive than firm 2, or alternatively,

$$L_1(w_K, w_L, q_1)K_2(w_K, w_L, q_2) - L_2(w_K, w_L, q_2)K_1(w_K, w_L, q_1) > 0$$

Applying the Stopler-Samuelson theorem, we can see what happens to input prices when one of the output prices rise, i.e.,

$$\frac{dw_L}{dp_1} = \frac{K_2(w_K, w_L, q_2)}{L_1(w_K, w_L, q_1)K_2(w_K, w_L, q_2) - L_2(w_K, w_L, q_2)K_1(w_K, w_L, q_1)} > 0$$
$$\frac{dw_K}{dp_1} = -\frac{L_2(w_K, w_L, q_2)}{L_1(w_K, w_L, q_1)K_2(w_K, w_L, q_2) - L_2(w_K, w_L, q_2)K_1(w_K, w_L, q_1)} < 0$$
$$\frac{dw_L}{dp_2} = \frac{K_1(w_K, w_L, q_1)}{L_2(w_K, w_L, q_2)K_1(w_K, w_L, q_1) - L_1(w_K, w_L, q_1)K_2(w_K, w_L, q_2)} < 0$$
$$\frac{dw_K}{dp_2} = -\frac{L_1(w_K, w_L, q_1)}{L_2(w_K, w_L, q_2)K_1(w_K, w_L, q_1) - L_1(w_K, w_L, q_1)K_2(w_K, w_L, q_2)} > 0$$

In words, when the good that firm 1 produces becomes more expensive the input more (less) used by firm 1 becomes more expensive (cheaper, respectively), i.e., $\frac{\partial w_L}{\partial p_1} > 0$ and $\frac{\partial w_K}{\partial p_1} < 0$. A similar argument applies to firm 2, where $\frac{\partial w_L}{\partial p_2} < 0$ and $\frac{\partial w_L}{\partial p_2} > 0$.

(c) Show that if $p_1 = 2p_2$, then in equilibrium $w_L = 4w_K$.

- Since the firms are price takers, $p_1 = MC_1$ and $p_2 = MC_2$. Taking the ratio of our marginal costs,

$$\frac{MC_1}{MC_2} = \frac{\frac{4}{3^{\frac{3}{4}}}(w_K)^{\frac{1}{4}}(w_L)^{\frac{3}{4}}}{\frac{4}{3^{\frac{3}{4}}}(w_K)^{\frac{3}{4}}(w_L)^{\frac{1}{4}}} = \left(\frac{w_L}{w_K}\right)^{\frac{1}{2}}$$

and if $p_1 = 2p_2$, then $\frac{p_1}{p_2} = 2$. Substituting in $\frac{MC_1}{MC_2} = \frac{p_1}{p_2}$ yields,

$$\left(\frac{w_L}{w_K}\right)^{\frac{1}{2}} = 2 \implies w_L = 4w_K$$