EconS 501 - Micro Theory I
Assignment #3 - Answer key

1. **Exercises from Nicholson and Snyder (Chapter 6):**

   (a) Exercise 6.10 (Separable utility).

   - See answer key at the end of this handout.

2. **Compensating and equivalent variation - An application.** An individual consumes only good 1 and 2, and his preferences over these two goods can be represented by the utility function

$$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta} \quad \text{where } \alpha, \beta > 0 \;\text{ and } \alpha + \beta \gtrless 1$$

This individual currently works for a firm in a city where initial prices are $p^0 = (p_1, p_2)$, and his wealth is w.

   (a) Find the Walrasian demand for goods 1 and 2 of this individual, $x_1(p, w)$ and $x_2(p, w)$.

   - The Lagrangian of this UMP is then

$$\mathcal{L}(x_1, x_2; \lambda) = x_1^{\alpha} x_2^{\beta} - \lambda \left[p_1 x_1 + p_2 x_2 - w\right]$$

The first order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= \alpha x_1^{\alpha-1} x_2^{\beta} - \lambda p_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= \beta x_1^{\alpha} x_2^{\beta-1} - \lambda p_2 = 0 \end{aligned}$$

Solving for λ on both first order conditions, we obtain

$$\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{p_1} = \frac{\beta x_1^{\alpha} x_2^{\beta-1}}{p_2} \Longleftrightarrow x_2 = \frac{\beta p_1 x_1}{\alpha p_2}$$

Using the budget constraint (which is binding), we have

$$p_1 x_1 + p_2 x_2 = w \Longleftrightarrow x_1 = \frac{w}{p_1} - \frac{p_2 x_2}{p_1}$$

and plugging this expression of x_2 we found above, yields the Walrasian demand for good 1

$$x_1 = \frac{w}{p_1} - \frac{p_2 \left(\frac{\beta p_1 x_1}{\alpha p_2}\right)}{p_1} \Longleftrightarrow x_1 = \frac{\alpha w}{(\alpha + \beta)\, p_1}$$

and, hence, the Walrasian demand for good 2 is

$$x_2 = \frac{\beta p_1 \left(\frac{\alpha w}{(\alpha+\beta) p_1} \right)}{\alpha p_2} = \frac{\beta w}{(\alpha+\beta)\, p_2}$$

Hence, the Walrasian demand function is

$$x_1(p, w) = \frac{\alpha w}{(\alpha+\beta)\, p_1} \quad \text{and} \quad x_2(p, w) = \frac{\beta w}{(\alpha+\beta)\, p_2}$$

(b) Find his indirect utility function at price vector p, and denote it as $v(p, w)$.

- Plugging the above Walrasian demand functions in the consumer's utility function, we obtain

$$v(p, w) = \left[\frac{\alpha w}{(\alpha+\beta)\, p_1} \right]^{\alpha} \left[\frac{\beta w}{(\alpha+\beta)\, p_2} \right]^{\beta} = \left(\frac{w}{\alpha+\beta} \right)^{\alpha+\beta} \left(\frac{\alpha}{p_1} \right)^{\alpha} \left(\frac{\beta}{p_2} \right)^{\beta}$$

(c) The firm that this individual works for is considering moving its office to a different city, where good 1 has the same price, but good 2 (e.g., housing) is twice as expensive, i.e., the new price vector is $p' = (p_1, 2p_2)$. Find the value of the indirect utility function in the new location. Let us denote this indirect utility function $v(p', w)$.

- The indirect utility function $v(p', w)$ is

$$v(p', w) = \left(\frac{w}{\alpha+\beta} \right)^{\alpha+\beta} \left(\frac{\alpha}{p_1} \right)^{\alpha} \left(\frac{\beta}{2p_2} \right)^{\beta}$$

where, relative to $v(p, w)$, only the price of good 2 has changed (namely, it has doubled), while all other elements remain unaffected.

(d) This individual's expenditure function is[1]

$$e(p, u) = (\alpha+\beta) \left(\frac{p_1}{\alpha} \right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{p_2}{\beta} \right)^{\frac{\beta}{\alpha+\beta}} u^{\frac{1}{\alpha+\beta}}$$

Evaluate this expenditure function in the following cases:

1. Under initial prices, p, and maximal utility level $u \equiv v(p, w)$, and denote it by $e(p, u)$.

$$e(p, u) = (\alpha+\beta) \left(\frac{p_1}{\alpha} \right)^{\frac{\alpha}{\alpha+\beta}} \left(\frac{p_2}{\beta} \right)^{\frac{\beta}{\alpha+\beta}} \underbrace{\left[\left(\frac{w}{\alpha+\beta} \right)^{\alpha+\beta} \left(\frac{\alpha}{p_1} \right)^{\alpha} \left(\frac{\beta}{p_2} \right)^{\beta} \right]}_{u}^{\frac{1}{\alpha+\beta}} = w$$

[1] As a practice, you can set up the consumer's expenditure minimization problem (EMP), find the Hicksian demands that emerge from solving this EMP, $h_1(p, u)$ and $h_2(p, u)$, and afterwards plug them into $p_1 x_1 + p_2 x_2$ to obtain the expenditure function $e(p, u) \equiv p_1 h_1(p, u) + p_2 h_2(p, u)$. After some algebra, you should find an expression of $e(p, u)$ that coincides with that provided in the exercise.

2. Under initial prices, p, and maximal utility level $u' \equiv v(p', w)$, and denote it by $e(p, u')$.

$$e(p,u') = (\alpha+\beta)\left(\frac{p_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{p_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\left[\left(\frac{w}{\alpha+\beta}\right)^{\alpha+\beta}\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{2p_2}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}} = \frac{1}{2^{\frac{\beta}{\alpha+\beta}}}w$$

3. Under new prices, p', and maximal utility level $u \equiv v(p, w)$, and denote it by $e(p', u)$.

$$e(p',u) = (\alpha+\beta)\left(\frac{p_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{2p_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\left[\left(\frac{w}{\alpha+\beta}\right)^{\alpha+\beta}\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}} = 2^{\frac{\beta}{\alpha+\beta}}w$$

4. Under new prices, p', and maximal utility level $u' \equiv v(p', w)$, and denote it by $e(p', u')$.

$$e(p',u') = (\alpha+\beta)\left(\frac{p_1}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\left(\frac{2p_2}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}\left[\left(\frac{w}{\alpha+\beta}\right)^{\alpha+\beta}\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{2p_2}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}} = w$$

(e) Find this individual's equivalent variation due to the price change. Explain how your result can be related with this proposal of the worker to his boss: "I would really prefer to stay in this city. In fact, I would accept a salary reduction if I could keep working for the firm in this city."

- The equivalent variation (EV) of a price change is given by

$$EV = e(p', u') - e(p, u')$$

using the results from the previous part, we have that $e(p', u') = w$, while $e(p, u') = \frac{1}{2^{\frac{\beta}{\alpha+\beta}}}w$, thus implying that the equivalent variation is

$$EV = w - \frac{1}{2^{\frac{\beta}{\alpha+\beta}}}w$$

That is, this individual would be willing to accept a reduction in his wealth of $w - \frac{1}{2^{\frac{\beta}{\alpha+\beta}}}w$ in order to avoid moving to a different city. [Alternatively, the individual is willing to accept a reduction of $\left(1 - \frac{1}{2^{\frac{\beta}{\alpha+\beta}}}\right)$ % of his weatlh.] Figure 1 depicts the EV for the case in which $\alpha = \beta = \frac{1}{2}$, yielding $EV = w\left(1 - \frac{1}{\sqrt{2}}\right)$. We also plot the wealth level of this individual, w, in the 45^0-line. Therefore, the region below the 45^0-line and above EV represents the remaining income that this individual would retain after giving up the amount found in the EV.

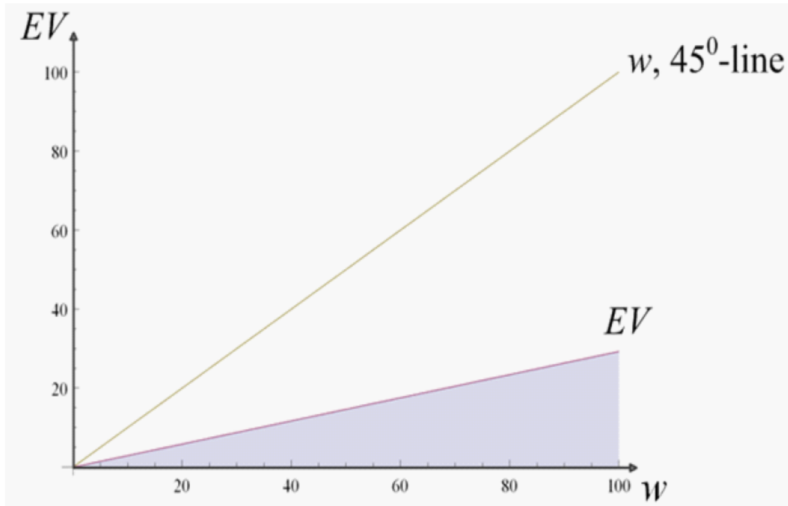

Figure 1. Equivalent variation (shaded area).

(f) Find this individual's compensating variation due to the price change. Explain how your result can be related with this statement from the individual to the media: "I really prefer to stay in this city. The only way I would accept to move to the new location is if the firm raises my salary."

- The compensating variation of a price change is given by

$$CV = e(p^1, u^0) - e(p^0, u^0) = 2^{\frac{\beta}{\alpha+\beta}} w - w$$

That is, we would need to raise this individuals' salary by $2^{\frac{\beta}{\alpha+\beta}} w - w$ in order to guarantee that his welfare level at the new city (with new prices) coincides with his welfare level at the initial city (at the initial price level). [Alternatively, the individual must receive an increase of $\left(2^{\frac{\beta}{\alpha+\beta}} - 1\right)$ of his wealth]

(g) How is this individual's consumer surplus affected by the price change? (The change in consumer surplus is often referred to as the "area variation (AV)"

- The area variation is given by the area below the Walrasian demand of good 2 (since only the price of this good changes), between the initial and final price level. That is,

$$AV = \int_{p_2}^{2p_2} x_2(p, w) dp = \int_{p_2}^{2p_2} \frac{\beta}{(\alpha + \beta)\, p} w \ dp$$

and rearranging

$$= \frac{\beta}{(\alpha + \beta)} w \int_{p_2}^{2p_2} \frac{1}{p} \ dp = \frac{\beta}{(\alpha + \beta)} w \ln 2$$

Hence, moving to the new city would imply a reduction in this individual's welfare of $\frac{\beta}{(\alpha+\beta)} w \ln 2$, or $\left(\frac{\beta}{(\alpha+\beta)} \ln 2\right)$ % of his wealth. Figure 2.6 depicts the

AV for the case in which $\alpha = \beta = \frac{1}{2}$, i.e, $AV = \frac{\ln 2}{2}w$, and compares it with the EV found in part (e) of the exercise.

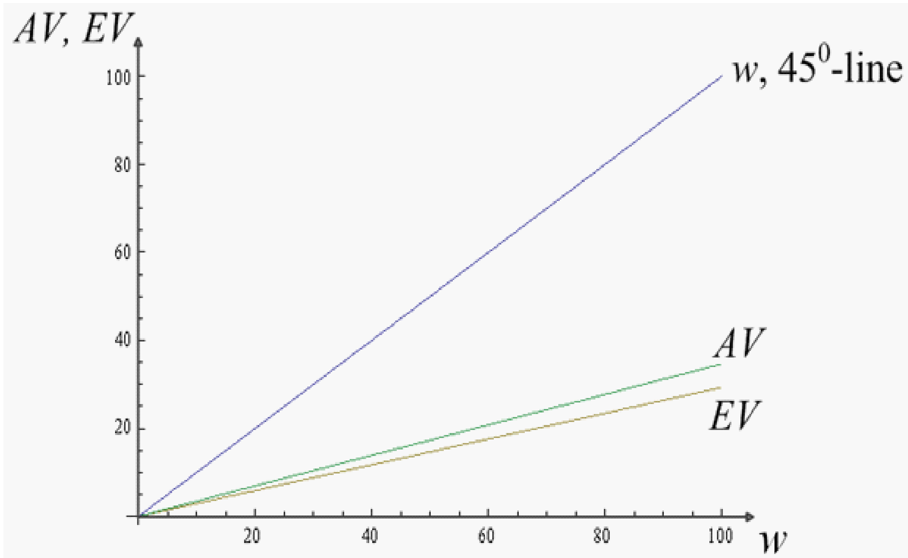

Figure 2.6. Area variation and equivalent variation.

(h) Which of the previous welfare measures in questions (e) and (f) coincide? Which of them do *not* coincide? Explain.

- None of them coincide, since this individual's preferences produces a positive income effect.

(i) Consider how the welfare measures from questions (e) and (f) would be modified if this individual's preferences were represented, instead, by the utility function $v(x_1, x_2) = \alpha \ln x_1 + \beta \ln x_2$.

- Since we have just applied a monotonic transformation to the initial utility function, $u(x_1, x_2)$, the new utility function $v(x_1, x_2)$ represents the same preference relation as utility function $v(x_1, x_2)$. Hence, the welfare results that we would obtain from function $v(x_1, x_2)$ would be the same as those with utility function $u(x_1, x_2)$. This is, in fact, one of the advantages of using monetary measures of welfare change (such as the equivalent, compensating, or area variation) rather than the simple difference in utility levels before and after the price change, i.e., $u' - u$. In particular, while the monetary measures are insensitive to monotonic transformations of the utility function, the utility difference when the consumer has utility function $u(x)$, i.e., $u' - u$, may differ from that when his utility experiences a monotonic transformation, $v' - v$.

3. [**Comprehensive exam, August 2011**] Consider a representative consumer in an economy with J goods, $j = 1, 2, ..., J$. Since we are mainly interested in this individual's consumption of goods 1 and 2, we group all the remaining goods $j = 3, 4, ..., J$ as good zero, q_0. The price of good zero is normalized to $p_0 = 1$ (i.e., good zero thus becomes the numeraire). The prices of goods 1 and 2 are p_1 and p_2, and income is $m > 0$. This consumer's preferences are represented by utility function

$$u(q_1, q_2, q_0) = q_1^{\frac{1}{4}} q_2^{\frac{1}{4}} + q_0$$

(a) Find the Walrasian demands and the associated indirect utility function.

- **UMP:** In order to solve this problem, we use a standard argument for additively separate utility functions: define $e^R(\mathbf{p}, m) \equiv p_1 q_1^W + p_2 q_2^W$ to be the amount of money spent on purchasing the Walrasian demand of goods 1 and 2 alone. Then, the pair $\left(q_1^W, q_2^W\right)$ must solve the auxiliary problem

$$\max_{q_1, q_2} \; q_1^{\frac{1}{4}} q_2^{\frac{1}{4}} \tag{1}$$

$$\text{subject to} \quad p_1 q_1 + p_2 q_2 = e^R(\mathbf{p}, m)$$

Solving for q_2 in the constraint, $q_2 = \frac{e^R}{p_2} - \frac{p_1}{p_2} q_1$, and plugging it into the objective function, the maximization problem reduces to one with a single choice variable, q_1, as follows

$$\max_{q_1} \; q_1^{\frac{1}{4}} \left(\frac{e^R}{p_2} - \frac{p_1}{p_2} q_1 \right)^{\frac{1}{4}}$$

Taking first order conditions with respect to q_1,

$$\frac{e^R - 2p_1 q_1}{4p_2 q_1^{\frac{3}{4}} \left(\frac{e^R - p_1 q_1}{p_2} \right)^{\frac{3}{4}}} = 0$$

and solving for q_1 yields

$$q_1^W(\mathbf{p}, e^R) = \frac{e^R}{2p_1}$$

Plugging $q_1^W(\mathbf{p}, e^R) = \frac{e^R}{2p_1}$ into the constraint, $q_2 = \frac{e^R}{p_2} - \frac{p_1}{p_2} q_1$, we obtain

$$q_2^W(\mathbf{p}, e^R) = \frac{e^R}{2p_2}$$

In addition, note that $q_1^W(\mathbf{p}, e^R)$ and $q_2^W(\mathbf{p}, e^R)$ do not depend on the overall income of the individual, m, but on the amount of income he spends on good 1 and 2 alone, e^R. Expressions $q_1^W(\mathbf{p}, e^R)$ and $q_2^W(\mathbf{p}, e^R)$ yield an associated utility level of

$$v^R(\mathbf{p}, e^R) = \left(\frac{1}{p_1} \right)^{1/4} \left(\frac{1}{p_2} \right)^{1/4} \left(\frac{e^R}{2} \right)^{1/2}$$

which can be interpreted as the indirect utility function of the auxiliary maximization problem (1).

- Given these results for goods 1 and 2, we can analyze good 0. In particular, the Walrasian demand for good 0, q_0^W, and the amount of income spent on goods 1 and 2, $e^R(\mathbf{p}, m)$, must solve

$$\max_{q_0, e^R} \; v^R(\mathbf{p}, e^R) + q_0$$

$$\text{subject to} \quad e^R(\mathbf{p}, m) + q_0 = m$$

Furthermore, since $q_0 = m - e^R(\mathbf{p}, m)$, the above program can be simplified to the following maximization problem (with only one choice variable):

$$\max_{e^R} \; g(e^R, \mathbf{p}) = v^R(\mathbf{p}, e^R) + \left[m - e^R(p, m)\right]$$

Taking first order conditions with respect to e^R, we obtain

$$\frac{\partial g(e^R, \mathbf{p})}{\partial e^R} = \frac{\left(\frac{1}{p_1}\right)^{1/4}\left(\frac{1}{p_2}\right)^{1/4}}{2\sqrt{2}\sqrt{e^R}} - 1 \tag{2}$$

and second order conditions

$$\frac{\partial^2 g(e^R, \mathbf{p})}{\partial e^{R\ 2}} = -\frac{\left(\frac{1}{p_1}\right)^{1/4}\left(\frac{1}{p_2}\right)^{1/4}}{4\sqrt{2}\,(e^R)^{3/2}} < 0 \tag{3}$$

showing that the objective function $g(e^R, \mathbf{p})$ is strictly concave.

- Therefore, from the first order conditions in (2), the value of $e^R(\mathbf{p}, m)$ that maximizes $g(e^R, \mathbf{p})$ is $e^*(\mathbf{p}) = \frac{1}{8\sqrt{p_1 p_2}}$. This implies that:
  - When $m > \frac{1}{8\sqrt{p_1 p_2}}$, Walrasian demands are

$$q_1^W = \frac{\frac{1}{8\sqrt{p_1 p_2}}}{2p_1} = \frac{1}{16\sqrt{p_1^3 p_2}} \quad \text{and} \quad q_2^W = \frac{\frac{1}{8\sqrt{p_1 p_2}}}{2p_2} = \frac{1}{16\sqrt{p_1 p_2^3}}$$

    for goods 1 and 2, and the rest of income, $q_0^W = m - \frac{1}{8\sqrt{p_1 p_2}}$, is spent on good 0. (Interior solutions).
  - By contrast, when $m \leq \frac{1}{8\sqrt{p_1 p_2}}$, no income is spent on good 0, $q_0^W = 0$, but only on goods 1 and 2, that is

$$q_1^W = \frac{m}{2p_1} \quad \text{and} \quad q_2^W = \frac{m}{2p_2}$$

    at a corner solution.
- Hence, the Walrasian demand correspondence can be summarized as

$$\left(q_1^W, q_2^W, q_0^W\right) = \begin{cases} \left(\frac{1}{16\sqrt{p_1^3 p_2}}, \frac{1}{16\sqrt{p_1 p_2^3}}, m - \frac{1}{8\sqrt{p_1 p_2}}\right) & \text{if } m > \frac{1}{8\sqrt{p_1 p_2}}, \text{ and} \\ \left(\frac{m}{2p_1}, \frac{m}{2p_2}, 0\right) & \text{if } m \leq \frac{1}{8\sqrt{p_1 p_2}}. \end{cases}$$

  Note that, at the interior solution, the Walrasian demands of goods 1 and 2 do not depend on income, implying that these goods do not exhibit income effects, since all additional income effect is entirely spent on the numeraire good.
- From the above Walrasian demands, it is easy to obtain the associated indirect utility function

$$v(\mathbf{p}, m) = \begin{cases} m + \frac{1}{8\sqrt{p_1 p_2}} & \text{if } m > \frac{1}{8\sqrt{p_1 p_2}}, \text{ and} \\ \left(\frac{m^2}{4p_1 p_2}\right)^{1/4} & \text{if } m \leq \frac{1}{8\sqrt{p_1 p_2}}. \end{cases}$$

(b) Invert the indirect utility function $v(\mathbf{p}, m)$ to obtain the expenditure function $e(\mathbf{p}, u)$.

- Note that in order to obtain the expenditure function $e(\mathbf{p}, u)$, we just need to invert the indirect utility function $v(\mathbf{p}, m)$, i.e., solving for m, which yields

$$e(\mathbf{p}, u) = \begin{cases} u - \frac{1}{8\sqrt{p_1 p_2}} & \text{if } u > \frac{1}{4\sqrt{p_1 p_2}}, \text{ and} \\ 2u^2\sqrt{p_1 p_2} & \text{if } u \leq \frac{1}{4\sqrt{p_1 p_2}}. \end{cases}$$

(c) Consider that the price vector increases from $\mathbf{p^0} = (p_1^0, p_2^0) = (1, 1)$ to $\mathbf{p}^1 = (p_1^1, p_2^1) = (2, 1)$, i.e., only the price of good 1 doubles. Let us next use the equivalent variation (EV) to evaluate the welfare loss that the consumer suffers from the increase in the price of good 1. In order to keep track of the possible corner solutions that arise at different income levels, we separately evaluate the EV at different values of m.

1. What is the EV when income satisfies $m > \frac{1}{8}$, i.e., the consumer is relatively rich?

   - In this case, the consumer is at the interior solution both *before* and *after* the price change. In particular,

   $$u^0 = v(\mathbf{p}^0, m) = m + \frac{1}{8} \text{ and } u^1 = v(\mathbf{p}^1, m) = m + \frac{1}{8\sqrt{2}}$$

   and the corresponding expenditure functions are

   $$e(\mathbf{p}^0, u^0) = u^0 - \frac{1}{8} = m \text{ and } e(\mathbf{p}^1, u^1) = u^1 - \frac{1}{8\sqrt{2}} = m$$

   and

   $$e(\mathbf{p}^0, u^1) = u^1 - \frac{1}{8} = m + \frac{1}{8\sqrt{2}} - \frac{1}{8} \text{ and } e(\mathbf{p}^1, u^0) = u^0 - \frac{1}{8\sqrt{2}} = m + \frac{1}{8} - \frac{1}{8\sqrt{2}}$$

   - Therefore, the equivalent variation (EV) is

   $$EV = e(\mathbf{p}^0, u^0) - e(\mathbf{p}^0, u^1) = m - \left(m + \frac{1}{8\sqrt{2}} - \frac{1}{8}\right) \simeq 0.036$$

   where note that we define the EV as the negative of the standard definition, since in this case we measure a loss in consumer welfare. Intuitively, the EV measures the additional income that we need to give to this consumer after the price increase, for him to maintain the same utility level he reached before the price increase.

2. What is the EV when income satisfies $\frac{1}{8} > m > \frac{1}{8\sqrt{2}}$, i.e., the consumer is moderately rich?

   - *Utility levels.* In this case, the initial equilibrium *before* the price change is at a corner solution, while the equilibrium *after* the price change is interior. In particular,

   $$u^0 = v(\mathbf{p}^0, m) = \left(\frac{m^2}{4}\right)^{1/4} \text{ and } u^1 = v(\mathbf{p}^1, m) = m + \frac{1}{8\sqrt{2}}$$

- *Expenditure function* $e(\mathbf{p}^0, u^0)$. The expenditure functions that we need to use in each case depend on whether the utility level we are using (u^0 or u^1) exceed the cutoff $\frac{1}{4\sqrt{p_1 p_2}}$, as we described in the previous part of the exercise when we found the piecewise expenditure function $e(p,u)$. In particular, for utility level $u^0 = \left(\frac{m^2}{4}\right)^{1/4}$, we have that $u^0 \leq \frac{1}{4\sqrt{p_1 p_2}}$ since $\left(\frac{m^2}{4}\right)^{1/4} < \frac{1}{4}$ holds given that $m < \frac{1}{8}$. Hence, for utility level u^0 we need to use expenditure function $2u^2\sqrt{p_1 p_2}$, as follows

$$e(\mathbf{p}^0, u^0) = 2\left[\left(\frac{m^2}{4}\right)^{1/4}\right]^2 = 2\sqrt{\frac{m^2}{4}} = m$$

- *Expenditure function* $e(\mathbf{p}^1, u^1)$. For the case of utility level u^1 we have that $u^1 > \frac{1}{4\sqrt{p_1 p_2}}$ holds given that $m + \frac{1}{8\sqrt{2}} > \frac{1}{4\sqrt{2}}$ is satisfied for all $m > \frac{1}{8\sqrt{2}}$. Since this part of the exercise assumes that m satisfies $\frac{1}{8} > m > \frac{1}{8\sqrt{2}}$, we have that we need to use $u - \frac{1}{8\sqrt{p_1 p_2}}$ as the expenditure function. In particular,

$$e(\mathbf{p}^1, u^1) = \underbrace{\left(m + \frac{1}{8\sqrt{2}}\right)}_{u^1} - \frac{1}{8\sqrt{2}} = m$$

- *Expenditure function* $e(\mathbf{p}^0, u^1)$. Similarly, in order to find expenditure function $e(\mathbf{p}^0, u^1)$, notice that for utility level u^1 we have that $u^1 > \frac{1}{4\sqrt{p_1 p_2}}$ holds (as discussed above). Hence, we need to use $u - \frac{1}{8\sqrt{p_1 p_2}}$ as the expenditure function. Importantly, note that our testing of whether u^1 exceeds cutoff $\frac{1}{4\sqrt{p_1 p_2}}$ must always be evaluated at the prices at which u^1 is evaluated ($\mathbf{p}^1$ price vector), regardless of the prices at which we afterwards seek to evaluate the expenditure function. In particular, for expenditure function $e(\mathbf{p}^0, u^1)$, which is evaluated at the original price vector $\mathbf{p}^0$,.we have

$$e(\mathbf{p}^0, u^1) = u_1 - \frac{1}{8\sqrt{1}} = \underbrace{\left(m + \frac{1}{8\sqrt{2}}\right)}_{u^1} - \frac{1}{8}$$

- Therefore, the equivalent variation (EV) is

$$EV = e(\mathbf{p}^0, u^0) - e(\mathbf{p}^0, u^1) = m - \left(m + \frac{1}{8\sqrt{2}} - \frac{1}{8}\right) = \frac{1}{8} - \frac{1}{8\sqrt{2}} \simeq 0.036$$

3. What is the EV when income satisfies $\frac{1}{8\sqrt{2}} > m$, i.e., the consumer is poor?
   - In this case, the equilibrium is at a corner solution, both *before* and *after* the price change. In particular,

$$u^0 = v(\mathbf{p}^0, m) = \left(\frac{m^2}{4}\right)^{1/4} \quad \text{and } u^1 = v(\mathbf{p}^1, m) = \left(\frac{m^2}{8}\right)^{1/4}$$

and the corresponding expenditure functions are

$$e(\mathbf{p}^0, u^0) = 2\sqrt{\frac{m^2}{4}} = m \quad \text{and} \quad e(\mathbf{p}^1, u^1) = 2\sqrt{2\frac{m^2}{8}} = m$$

and

$$e(\mathbf{p}^0, u^1) = 2\sqrt{\frac{m^2}{8}} = \frac{m}{\sqrt{2}}$$

- Therefore, the equivalent variation (EV) is

$$EV = e(\mathbf{p}^0, u^0) - e(\mathbf{p}^0, u^1) = m - \frac{m}{\sqrt{2}} = \left(1 - \frac{1}{\sqrt{2}}\right) m \simeq 0.29m$$

4. **Exercises from Jehle and Reny (3rd edition):**

    (a) Chapter 3: Exercises 3.6, 3.7, and 3.21.

    - See answer key at the end of this handout.

Exercises from JR

Exercise 3.6 Let $f(x_1, x_2)$ be non-decreasing and homogenous of degree one.

a) Show that the isoquants of *f* are radially parallel, with equal slope at all paths along any given ray from the origin.

b) Use this to demonstrate that the marginal rate of technical substitution depends only on input proportions.

c) Further, show that MP_1 is non-decreasing and MP_2 is non-increasing in input proportions, $R \equiv x_2 / x_1$.

d) Show that the same is true when the production function is homothetic.

Answer:

a) Compare MRTS at *x* and *kx*

$$MRTS_{12}(x) = \frac{\partial f(x_1, x_2)/\partial x_1}{\partial f(x_1, x_2)/\partial x_2} = \frac{f_1(x)}{f_2(x)} \quad MRTS_{12}(kx) = \frac{\partial f(kx_1, kx_2)/\partial x_1}{\partial f(kx_1, kx_2)/\partial x_2} = \frac{f_1(kx_1)}{f_2(kx_2)} = \frac{f_1(x_1)}{f_2(x_2)}$$

MRTS(*x*)=MRTS(*kx*) implies that the slope at *x* and *kx* are the same. So isoquants are radially parallel.

b) $MRTS_{12}(x) = \dfrac{f_1(kx_1, kx_2)}{f_2(kx_1, kx_2)} = \dfrac{f_1(x_1, x_2)}{f_2(x_1, x_2)}$

Let's choose $k = 1/x_2$ then $MRTS_{12}(x) = \dfrac{f_1(\frac{1}{x_2}\cdot x_1, \frac{1}{x_2}\cdot x_2)}{f_2(\frac{1}{x_2}\cdot x_1, \frac{1}{x_2}\cdot x_2)} = \dfrac{f_1(\frac{x_1}{x_2}, 1)}{f_2(\frac{x_1}{x_2}, 1)}$

Now our MRTS just depend on $\dfrac{x_1}{x_2}$.

c) $MP_1 = f_1(x_1, x_2)$ and $MP_2 = f_2(x_1, x_2)$

given $R = \frac{x_2}{x_1}$, then $MP_1 = f_1(x_1, Rx_1)$ and $MP_2 = f_2(\frac{x_2}{R}, x_2)$

$$\Leftrightarrow x_2 = Rx_1 \Leftrightarrow x_1 = x_2 / R$$

So $f(x_1, x_2) = f(x_1, Rx_1)$, since $x_2 = Rx_1$ then $\frac{dx_2}{dx_1} = R$ and $\frac{dx_1}{dx_2} = \frac{1}{R}$

$$MP_1 = \frac{df}{dx_1} = \frac{df}{dx_1} + \frac{df}{dx_2} \cdot \frac{dx_2}{dx_1} = f_1 + f_2 \cdot R$$
$$MP_2 = \frac{df}{dx_2} = \frac{df}{dx_2} \cdot \frac{dx_1}{dx_2} + \frac{df}{dx_2} = f_2 \cdot \frac{1}{R} + f_1$$

Then MP_1 is a linear function in R. If R increases given that f_1 and f_2 positive, MP_1 is nondecreasing.

MP_2 is also a linear function in ($1/R$). If R increases given that f_1 and f_2 positive, MP_2 is nonincreasing.

d) For homotheticity, we need that $f(x_1, x_2) = f(g(x_1, x_2))$.

Slope $-\frac{f_1'(g(x_1, x_2))}{f_2'(g(x_1, x_2))} = -\frac{f'(g(x_1, x_2)) \cdot g_1'(x_1, x_2)}{f'(g(x_1, x_2)) \cdot g_2'(x_1, x_2)} = -\frac{g_1'(x_1, x_2)}{g_2'(x_1, x_2)}$

$\frac{f_1(kx_1, kx_2)}{f_2(kx_1, kx_2)} = \frac{g_1(kx_1, kx_2)}{g_2(kx_1, kx_2)} = \frac{g_1(1, \frac{x_2}{x_1})}{g_2(1, \frac{x_2}{x_1})}$ with $k = 1/x_2$

Hence, MRTS depends on input proportions.

Exercise 3.7 Goldman & Uzawa (1964) have shown that the production function is weakly separable with respect to the partition $\{N_1, ..., N_S\}$ if and only if it can be written in the form

$$f(x) = g(f^1(x^{(1)}), ..., f^S(x^{(S)}))$$

where g is some function of S variables, and, for each i, $f^i(x^{(i)})$ is a function of the subvector $x^{(i)}$ of inputs from group i alone. They have also shown that the production function will be strongly separable if and only if it is of the form

$$f(x) = G(f^1(x^{(1)}) + ... + f^S(x^{(S)}))$$

where G is a strictly increasing function of one variable, and the same conditions on the subfunctions subvectors apply. Verify their results by showing that each is separable as they claim.

Answer: To show that the first equation is weakly separable with respect to the partitions, we need to show that $\frac{\partial[f_i(x)/f_j(x)]}{\partial x_k} = 0 \forall i, j \in N_S$ and $k \notin N_S$. Calculate the marginal products of the first equation for two arbitrary inputs *i* and *j*:

$$f_i(x) = \frac{\partial g}{\partial f^S}\frac{\partial f^S}{\partial x_i} \quad f_j(x) = \frac{\partial g}{\partial f^S}\frac{\partial f^S}{\partial x_j}$$

The marginal rate of technical substitution between these two inputs is

$$\frac{f_i(x)}{f_j(x)} = \frac{\frac{\partial f^S}{\partial x_i}}{\frac{\partial f^S}{\partial x_j}}$$

This expression is independent of any other input which is not in the same partition N^S and, therefore, the production function is weakly separable.

$$\frac{\partial(f_i / f_j)}{\partial x_k} = 0 \text{ for } k \notin N^S$$

To show that the second equation is strongly separable we have to perform the same exercise, however, assuming that the three inputs are elements of three different partitions $i \in N_S, j \in N_T$ and $k \notin N_S \cup N_T$. The marginal products of the two inputs *i* and *j* are:

$$f_i(x) = G' \frac{\partial f^S(x^S)}{\partial x_i} \quad f_i(x) = G' \frac{\partial f^T(x^{(T)})}{\partial x_j}$$

The MRTS is:

$$\frac{f_i(x)}{f_j(x)} = \frac{\partial f^S / \partial x_i}{\partial f^T / \partial x_j}.$$

It follows for $k \notin N_S \cup N_T$

$$\frac{\partial (f_i / f_j)}{\partial x_k} = 0.$$

Exercise 3.21

i) We need to show the superadditivity of the cost function, that is

$$c(w^1 + w^2, y) \geq c(w^1, y) + c(w^2, y)$$

By definition of the cost function, there must be some cost minimizing bundles for each price vector

$$c(w^1 + w^2, y) \equiv \min_{x \geq 0} \{(w^1 + w^2) \cdot x^*\} s.t. f(x) = y$$
$$c(w^1, y) \equiv \min_{x \geq 0} \{w^1, x^1\} s.t. f(x) = y$$
$$c(w^2, y) \equiv \min_{x \geq 0} (w^2, x^2) s.t. f(x) = y$$

From cost minimization

$$c(w^1, y) \leq w^1 \cdot x^*$$
$$c(w^2, y) \leq w^2 \cdot x^*$$

Adding the last two equations gives:

$$w^1 \cdot x^* + w^2 \cdot x^* \geq c(w^1, y) + c(w^2, y)$$
$$\Rightarrow (w^1 + w^2) \cdot x^* \geq c(w^1, y) + c(w^2, y)$$
$$\Rightarrow c(w^1 + w^2, y) \geq c(w^1, y) + c(w^2, y)$$

ii) We now need to show that the cost function is non-decreasing in input prices, *w*:

Suppose $\Delta w \geq 0$, that is $\Delta w_i \geq 0$ for all *i* and there exists *i* such that $\Delta w_i > 0$. For $\Delta w_i = 0$, we have $c(\Delta w_i, y) = 0$. Then by superadditivity of the cost function

$$c(w + \Delta w, y) \geq c(w, y) + c(\Delta w, y)$$

Thus, $c(w + \Delta w, y) \geq c(w, y)$.

EconS 501 – Homework #3

Answer key

Exercise 6.10 from NS. Separable Utility. This problem shows that many of the complications in a many good utility function can be greatly simplified if utility is assumed to be separable.

a. This functional form assumes $U''_{xy} = 0$ That is, since its marginal utility, $\frac{\partial U(x,y)}{\partial x} = U'_1(x)$, is only a function of good *x*, then the cross-partial derivative is

$$\frac{\partial^2 U(x,y)}{\partial x \partial y} = \frac{\partial U'_1(x)}{\partial y} = 0$$

Intuitively, the marginal utility of x does not depend on the amount of *y* consumed. Though unlikely in a strict sense, this independence might hold for large consumption aggregates such as "food" and "housing."

b. Because utility maximization requires $\frac{MU_x}{MU_y} = \frac{p_x}{p_y} \Rightarrow \frac{MU_x}{p_x} = \frac{MU_y}{p_y}$, any increase in income with no change in p_x or p_y must cause both x and y to increase to maintain this equality (assuming $U_i > 0$ and $U_{ii} < 0$). Neither good's consumption decreases, so both goods must normal.

c. Again, using $\frac{MU_x}{p_x} = \frac{MU_y}{p_y}$, a rise in p_x will cause the consumption of x to fall and, as a consequence, its marginal utility MU_x to rise, because MU_x is decreasing in *x*. So the direction of change in $\frac{MU_x}{p_x}$ is indeterminate. Hence, the change in y is also indeterminate.

d. If the utility function is a Cobb-Douglas, $U = x^{\alpha} y^{\beta}$, then the marginal utility is $MU_x = \alpha x^{\alpha-1} y^{\beta}$, implying that the Cobb-Douglas utility function is not separable, since its marginal utility depends on both *x* and *y*.

But, the monotonic transformation $\ln U = \alpha \ln x + \beta \ln y$, yields a marginal utility of $MU_x = \dfrac{\alpha}{x}$. Hence, its monotonic transformation is separable.