# EconS 501 - Microeconomic Theory I
# Homework #2 - Answer Key

1. **Exercise from FMG.** Chapter 2, exercise 26. Consider an individual facing price vector $p = (p_1, p_2) >> 0$ and income $w > 0$. If, after solving his UMP, his indirect utility function is $v(p, w) = \left(p_1^{\alpha} p_2^{1-\alpha}\right)^{-1} w$, show that his utility function $u(x)$ must have a Cobb-Douglas representation, where $x = (x_1, x_2)$.

   - *Proof (EMP approach).* First, recall from duality that
   $$u(x) \equiv \min_{p \in \mathbb{R}^2_+} v(p, p \cdot x) \quad (1)$$
   Since $v(p, w)$ is homogeneous of degree zero, we can divide both arguments by $p \cdot x$ to obtain that the indirect utility function $v(p, w)$ is unaffected, i.e., $v(p, w) = v\left(\frac{p}{p \cdot x}, 1\right)$. Let $\overline{p} \equiv \frac{p}{p \cdot x}$, and thus $v(p, w) = v(\overline{p}, 1)$. As a consequence, if price vector $p^*$ minimizes $v(p, p \cdot x)$ for an income level $p \cdot x = w$, then price vector $\overline{p}$ minimizes $v(p, 1)$ for an income level $p \cdot x = 1$. That is, we can rewrite program (1) as
   $$u(x) \equiv \min_{p \in \mathbb{R}^2_+} v(p, 1) \quad \text{subject to } p \cdot x = 1 \quad (2)$$

   - We can now find the price vector $p = (p_1, p_2)$ that solves program (2). Plugging them afterwards in the indirect utility function $v(p, 1)$ will yield the original utility function $u(x)$ that this consumer maximized in his UMP (as stated in (2)). Since program (2) is a constrained minimization problem, we set up the Lagrangian
   $$\mathcal{L} = \left(p_1^{\alpha} p_2^{1-\alpha}\right) - \lambda(p_1 x_1 + p_2 x_2 - 1)$$
   Taking first-order conditions with respect to $p_1$ and $p_2$ yields, respectively
   $$\frac{d\mathcal{L}}{dp_1} = \alpha p_1^{\alpha-1} p_2^{1-\alpha} - \lambda x_1 = 0$$
   $$\frac{d\mathcal{L}}{dp_2} = (1-\alpha) p_1^{\alpha} p_2^{-\alpha} - \lambda x_2 = 0$$
   and
   $$\frac{d\mathcal{L}}{d\lambda} = -p_1 x_1 - p_2 x_2 + 1 = 0$$
   and simultaneously solving for $p_1$ and $p_2$ we obtain
   $$p_1^* = \frac{\alpha}{x_1} \quad \text{and} \quad p_2^* = \frac{1-\alpha}{x_2}$$
   We can finally plug these two prices, which solve (2), into the indirect utility funciton $v(p, 1)$, yielding
   $$v(p_1^*, p_2^*, 1) = \left(\left(\frac{\alpha}{x_1}\right)^{\alpha} \left(\frac{1-\alpha}{x_2}\right)^{1-\alpha}\right)^{-1} (1) = \underbrace{\alpha^{-\alpha}(1-\alpha)^{\alpha-1}}_{\text{constant, } A} x_1^{\alpha} x_2^{1-\alpha}.$$
   which is clearly of the Cobb-Douglas type. For instance, labeling $A \equiv \alpha^{-\alpha}(1-\alpha)^{\alpha-1}$ yields $v(p_1^*, p_2^*, 1) = A x_1^{\alpha} x_2^{1-\alpha}$, thus taking a more familiar format.

- *Proof (Roy's identify).* Using Roy's identity, we find that the Walrasian demand for good 1 is

$$x_1(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_1}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{(-\alpha)\, p_1^{-\alpha-1} p_2^{\alpha-1} w}{p_1^{-\alpha} p_2^{\alpha-1}} = \frac{\alpha}{p_1} w$$

which, solving for $p_1$, gives us the indirect demand function $p_1 = \frac{\alpha}{x_1} w$. Similarly, the Walrasian demand of good 2 is

$$x_2(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_2}}{\frac{\partial v(p,w)}{\partial w}} = -\frac{(\alpha-1)\, p_1^{-\alpha} p_2^{\alpha-2} w}{p_1^{-\alpha} p_2^{\alpha-1}} = \frac{1-\alpha}{p_2} w$$

which, solving for $p_2$, gives us the indirect demand function $p_2 = \frac{1-\alpha}{x_2} w$. Inserting these indirects demands into the utility function, we obtain

$$v(p_1^*, p_2^*, w) = \left(\left(\frac{\alpha}{x_1} w\right)^{\alpha} \left(\frac{1-\alpha}{x_2} w\right)^{1-\alpha}\right)^{-1} (w) = \underbrace{\alpha^{-\alpha}(1-\alpha)^{\alpha-1}}_{\text{constant, } A} x_1^{\alpha} x_2^{1-\alpha}.$$

which is clearly of the Cobb-Douglas type. For instance, labeling $A \equiv \alpha^{-\alpha}(1-\alpha)^{\alpha-1}$ yields $v(p_1^*, p_2^*, w) = A x_1^{\alpha} x_2^{1-\alpha}$, thus taking a more familiar format.

2. **Short proofs.** Consider an individual with $X = \mathbb{R}_+^2$ and budget line $p_1 x_1 + p_2 x_2 = w$, where $p_1$, $p_2$, and $w$ are all strictly positive.

   (a) If the preference relation is continuous then the consumer's problem has a solution.

   - If the preference relation is continuous then it has a continuous utility representation. Given that both prices are positive, the budget set is compact, so by Weirstrass' theorem the continuous utility function has a maximizer in the budget set, which is a solution of the consumer's problem.

   (b) If the preference relation is strictly convex then the consumer's problem has at most one solution.

   - By contradiction, assume that a preference relation is strictly convex but there are two bundlex $x$ and $y$, where $x \neq y$, that are both solutions to a consumer's UMP. Then, the linear combination between $x$ and $y$, $\alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$, also lies in the budget set (which is convex). Since $\alpha x + (1-\alpha)y$ is a more balanced bundle than $x$ and $y$ alone. Because preferences are strictly convex, the individual strictly prefers bundle $\alpha x + (1-\alpha)y$ to bundle $x$ and $y$. As a result, bundles $x$ and $y$ cannot be solutions to his UMP.

   (c) If the preference relation is monotone, then every solution of the consumer's problem must be on the budget line.

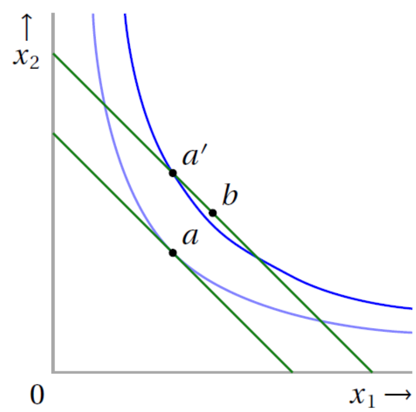
   - By contradiction, suppose that the bundle $x = (x_1, x_2)$ solves the consumer's UMP, but lies below the budget line, i.e., $p_1 x_1 + p_2 x_2 < w$. Then, there must be another bundle $y = (x_1 + \varepsilon, x_2 + \varepsilon)$, which is in the $\varepsilon$-ball around $x$, that also lies strictly below the budget line, that is,

$$p_1 (x_1 + \varepsilon) + p_2 (x_2 + \varepsilon) < w$$

for a small enough $\varepsilon > 0$. Because preferences satisfy monotonicity, bundle $y$ is strictly preferred to $x$, $y \succ x$, implying that bundle $x$ cannot be a solution to the UMP. (We found another bundle $y$ that was still affordable and strictly preferred to $x$.)

(d) The demand function of a rational consumer whose marginal rate of substitution, $MRS(x_1, x_2)$, is increasing in $x_2$ for every value of $x_1$ has the property that good 1 is normal.

- Consider the price pair $(p_1^0, p_2^0)$ and let wealth increase from $w$ to $w'$, where $w' > w$. Let bundle $a$ be a solution to the UMP when the consumer faces $(p_1^0, p_2^0, w)$, as depicted in the figure below. Consider now an increase in wealth from $w$ to $w'$, without changing any of the prices, and let $a'$ denote a bundle that lies on the new budget line, not necessarily a solution to the UMP, with the property that he consumes the same amount of good 1 in both bundles, $x_1 = x_1'$; also depicted in the figure.




Because the MRS is increasing in $x_2$ and $x_1 = x_1'$, we must have that $MRS(x_1', x_2') > MRS(x_1, x_2)$. Hence, the solution of the UMP when the consumer faces $(p_1^0, p_2^0, w')$, which we denote as bundle $b$ on the figure, must satisfy contain more units of good 1, that is, $b_1 > x_1 = x_1'$. Overall, an increase in wealth, lead to an increase in the consumer's demand for good 1.

3. **UMP in several utility functions.** Consider a consumer with budget line $p_1 x_1 + p_2 x_2 = w$, where $p_1$, $p_2$, and $w$ are all strictly positive. Find her Walrasian demand in the following utility functions, and explain how the consumer distributes her wealth, $w$, across both goods.

(a) Quasi-linear utility function: $u(x_1, x_2) = x_1 + \sqrt{x_2}$.

- The marginal rate of substitution is

$$MRS = \frac{MU_1}{MU_2} = \frac{1}{0.5x_2^{-0.5}} = 2\sqrt{x_2}.$$

As $x_2$ decreases, the $MRS = 2\sqrt{x_2}$ decreases from $2\sqrt{\frac{w}{p_2}}$ (where $x_2 = \frac{w}{p_2}$ indicates the vertical intercept of the indifference curve) to $2\sqrt{0} = 0$ when

$x_2 = 0$. This indicates that two corner solutions can arise, depending on the price ratio:

– If $2\sqrt{\frac{w}{p_2}} \geq \frac{p_1}{p_2}$ holds, there is a corner solution to the UMP, $(x_1^*, x_2^*)$, where $MRS = \frac{p_1}{p_2}$, or $2\sqrt{x_2^*} = \frac{p_1}{p_2}$. Solving for $x_2^*$, yields $x_2^* = \frac{p_1^2}{4p_2^2}$. His demand for good 1 can be found by inserting $x_2^*$ into the budget line, $p_1x_1 + p_2x_2 = w$, that is,

$$p_1x_1^* + p_2 \underbrace{\left(\frac{p_1^2}{4p_2^2}\right)}_{x_2^*} = w$$

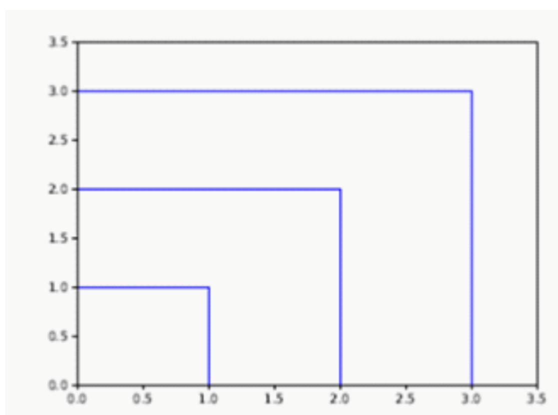
which simplifies to $p_1x_1^* + \frac{p_1^2}{4p_2} = w$. Solving for $x_1^*$, we obtain that $x_1^* = \frac{w}{p_1} - \frac{p_1}{4p_2}$.

– If, instead, $2\sqrt{\frac{w}{p_2}} < \frac{p_1}{p_2}$ holds, there is a corner solution where the individual spends all his wealth on the second good, $x_2^* = \frac{w}{p_2}$ and purchases zero units of good 1, $x_1^* = 0$.

Intuitively, the consumer purchases positive units of both goods when $2\sqrt{\frac{w}{p_2}} \geq \frac{p_1}{p_2}$, which implies that good 1 (that entering linearly in his utility function) is relatively inexpensive, i.e., price ratio $\frac{p_1}{p_2}$ must be low for condition $2\sqrt{\frac{w}{p_2}} \geq \frac{p_1}{p_2}$ to hold. If, instead, good 1 is relatively expensive, condition $2\sqrt{\frac{w}{p_2}} < \frac{p_1}{p_2}$ holds, inducing the consumer to spend all his income on the good that enters non-linearly. This is a common feature of quasi-linear utility functions.

(b) "Max" utility function: $u(x_1, x_2) = \max\{x_1, x_2\}$.

- First, we depict an indifference map.




- This utility function gives rise to three cases:

– If $\frac{p_1}{p_2} < 1$ $(p_1 < p_2)$, the solution will be at the lower right corner, where $x_1^* = \frac{w}{p_1}$ and $x_2^* = 0$. The consumer uses all her wealth on $x_1$.

– If $\frac{p_1}{p_2} = 1$ $(p_1 = p_2)$, both the lower right and upper left corners are solutions. The consumer evenly splits her wealth between both goods, that is, $x_1^* = x_2^* = \frac{w}{2p}$.

– If $\frac{p_1}{p_2} > 1$ ($p_1 > p_2$), the solution will be at the upper left corner, where $x_1^* = 0$ and $x_2^* = \frac{w}{p_2}$. The consumer uses all her wealth on $x_2$.

Intuitively, the consumer spends all her wealth on the cheaper good if prices differ, but evenly splits her wealth between both goods if their prices coincide. Other utility functions such as that in part (c), $u(x_1, x_2) = x_1^2 + x_2^2$, and $u(x_1, x_2) = x_1 + x_2$, generate similar Walrasian demands.

(c) Utility function with both goods entering quadratically: $u(x_1, x_2) = x_1^2 + x_2^2$.

- As described above, this utility function generates a Walrasian demand like the utility function in part (b), where all the wealth is spend on the cheaper good. As a practice, we include some explanation below.
- First, note that this individual's indifference curves are strictly concave in the $(x_1, x_2)$-quadrant. Indeed, solving this individual's utility function for $x_2$, we obtain the equation of an indifference curve (evaluated at a generic utility level $u$) is $x_2 = \sqrt{u - x_1^2}$. Differentiating with respect to $x_1$, yields

$$x_2'(x_1) = \frac{1}{2}(u - x_2^2)^{-\frac{1}{2}}(-2x_1) < 0$$

and differentiating again with respect to $x_1$, we find that

$$x_2''(x_1) = -\frac{1}{4}(u - x_2^2)^{-\frac{3}{2}}(-2x_1)(-2x_1) + \frac{1}{2}(u - x_2^2)(-2) < 0$$

When indifference curves are not strictly convex, either strictly concave (like those in part b or in part c) or linear (perfect substitutes), we can anticipate that the consumer's Walrasian demands will be corner points.

- If we compare the utility levels associated to each of these bundles, we obtain that the consumer prefers $(\frac{w}{p_1}, 0)$ to $(0, \frac{w}{p_2})$ if and only if $u(\frac{w}{p_1}, 0) > u(0, \frac{w}{p_2})$, which implies $\frac{w^2}{p_1^2} > \frac{w^2}{p_2^2}$ or, after rearranging, $p_2 > p_1$. We can, then, summarize the Walrasian demand as follows:

$$(x_1(p, w), x_2(p, w)) = \begin{cases} (\frac{w}{p_1}, 0) & \text{if } p_2 > p_1 \\ (\frac{w}{2p}, \frac{w}{2p}) & \text{if } p_2 = p_1 \\ (0, \frac{w}{p_2}) & \text{if } p_2 < p_1 \end{cases}$$

(d) Utility function with both goods entering as a square root: $u(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$.

- As a practice, note that, after solving for $x_2$, we find the equation of an indifference curve (evaluated at a generic $u$), to be $x_2 = \left(u - \sqrt{x_1}\right)^2$. Differentiating this expression with respect to $x_1$, yields $1 - \frac{u}{\sqrt{x_1}}$, and differentiating again with respect to $x_1$, we find $\frac{u}{x_1^{3/2}}$, which is unambiguously positive (i.e., positive for any value of $x_1$ and $u$). As a result, we can claim that indifference curves are strictly convex (bowed in towards the origin). This property helps us anticipate interior solutions, as opposed to the previous part of the exercise.

- We now set the price ratio equal to the marginal rate of substitution, as follows,

$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{0.5x_1^{-0.5}}{0.5x_2^{-0.5}} = \frac{p_1}{p_2}$$

which, after rearranging, yields

$$x_2(p,w) = \left(\frac{p_1}{p_2}\right)^2 x_1.$$

Inserting this expression into the budget line, $p_1x_1 + p_2x_2 = w$, yields

$$p_1x_1 + \underbrace{p_2\left(\frac{p_1}{p_2}\right)^2 x_1}_{x_2} = w$$

After simplifying and solving for $x_1$, we obtain the Walrasian demand for good 1,

$$x_1(p,w) = \frac{wp_2}{p_1(p_2+p_1)}$$

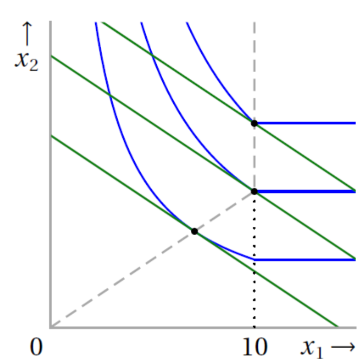
Similarly, the Walrasian demand function for good $x_2$ is

$$x_2(p,w) = \frac{wp_1}{p_2(p_2+p_1)}.$$

If $p_1 > p_2$, Walrasian demands satisfy $x_2(p,w) > x_1(p,w)$ since $\frac{wp_1}{p_2(p_2+p_1)} > \frac{wp_2}{p_1(p_2+p_1)}$ simplifies to $\frac{p_1}{p_2} > \frac{p_2}{p_1}$, which holds true because $\frac{p_1}{p_2} > 1 > \frac{p_2}{p_1}$ given that $p_1 > p_2$. If, instead, $p_1 < p_2$, Walrasian demands satisfy $x_2(p,w) < x_1(p,w)$ using a similar argument. Finally, if prices coincide, $p_1 = p_2 = p$, these Walrasian demands coincide as well and simplify to $x_1(p,w) = x_2(p,w) = \frac{w}{2p}$.

(e) Utility function with intervals: $u(x_1,x_2) = x_1x_2$ if $x_1 \leq 10$ units, but $u(x_1,x_2) = 10x_2$ otherwise.

- The next figure depicts the consumer's problem. The consumer would never purchase more than 10 units of good 1, as doing so would not help him increase his utility. Graphically, we have well-behaved (Cobb-Douglas) indifference curves for all $x_1 \leq 10$ units, but a kink at $x_1 = 10$ and a flat segment for all $x_1 > 10$ (as with Leontieff preferences).



- **Case 1.** If $x_1^* < 10$, we set the price ratio equal to the marginal rate of substitution, obtaining
$$\frac{MU_{x_1}}{MU_{x_2}} = \frac{x_2}{x_1} = \frac{p_1}{p_2}$$
which, solving for $x_2$, yields
$$x_2(p, w) = \frac{p_1}{p_2} x_1.$$
Inserting this expression into the budget line, $p_1 x_1 + p_2 x_2 = w$, we find that
$$p_1 x_1 + p_2 \underbrace{\frac{p_1}{p_2} x_1}_{x_2} = w$$
which simplifies to $p_1 x_1 + p_1 x_1 = w$, and yields the Walrasian demand for good 1, as follows
$$x_1(p, w) = \frac{w}{2p_1}.$$
Note that, for the initial condition $x_1^* < 10$ to hold, we need that $\frac{w}{2p_1} < 10$, or $\frac{w}{20} < p_1$, intuitively indicating that good 1 is relatively expensive. Otherwise, this case cannot be supported, and Case 2 below is the only possible solution. Using the tangency condition in this setting, $x_2(p, w) = \frac{p_1}{p_2} x_1$, we can find the Walrasian demand for good 2, that is,
$$x_2(p, w) = \frac{p_1}{p_2} \frac{w}{2p_1} = \frac{w}{2p_2}.$$
Then, the indirect utility function of the consumer in this case is
$$v^{1st}(x_1, x_2) = x_1^* x_2^* = \frac{w}{2p_1} \frac{w}{2p_2} = \frac{w^2}{4p_1 p_2}.$$
- **Case 2.** If $x_1^* = 10$, the Walrasian demand function for good 2 can be found by inserting $x_1^* = 10$ into the budget line, that is
$$p_1 10 + p_2 x_2^* = w$$
Solving for $x_2^*$, yields $x_2(p, w) = \frac{w - 10p_1}{p_2}$. In this case, the consumer's indirect utility function becomes
$$v^{2nd}(x_1, x_2) = 10 x_2^* = 10 \frac{w - 10p_1}{p_2}.$$
- **Comparison.** If $\frac{w}{20} < p_1$ holds and, thus, Case 1 can be supported, we can compare the indirect utility functions in each case, $v^{1st}(x_1, x_2)$ and $v^{2nd}(x_1, x_2)$, as follows
$$\begin{aligned} v^{1st}(x_1, x_2) - v^{2nd}(x_1, x_2) &= \frac{w^2}{4p_1 p_2} - 10 \frac{w - 10p_1}{p_2} \\ &= \frac{(w - 20p_1)^2}{4p_1 p_2}, \end{aligned}$$

which is always positive. Therefore, the consumer chooses bundle $(x_1^*, x_2^*) = \left(\frac{w}{2p_1}, \frac{w}{2p_2}\right)$.