EconS 501 - Microeconomic Theory I
Homework #1 - Answer key

1. **Exercise from FMG.** Chapter 1, exercise 16. Check if the Cobb-Douglas utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$, where $\alpha, \beta > 0$, satisfies the following properties: (a) local non-satiation; (b) decreasing marginal utility for both goods 1 and 2; (c) quasiconcavity; and (d) homotheticity.

   (a) *Local non-satiation (LNS).* When working with a differentiable utility function we can check LNS by directly checking for monotonicity (since monotonicity implies LNS). In order to test for monotonicity, we just need to confirm that the marginal utility from additional amounts of goods 1 and 2 are non-negative,

   $$\frac{\partial u(x_1, x_2)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta} = \alpha \frac{x_1^{\alpha} x_2^{\beta}}{x_1} > 0 \text{ if and only if } \alpha > 0$$

   $$\frac{\partial u(x_1, x_2)}{\partial x_2} = \beta x_1^{\alpha} x_2^{\beta-1} = \beta \frac{x_1^{\alpha} x_2^{\beta}}{x_2} > 0 \text{ if and only if } \beta > 0$$

   In fact, since the marginal utility of increasing either good is strictly positive, the Cobb-Douglas utility function not only satisfies monotonicity, but also strong monotonicity.

   (b) *Decreasing marginal utility.* We need to show that the marginal utilities we found above are nonincreasing. That is,

   $$\frac{\partial^2 u(x_1, x_2)}{\partial x_1^2} = \alpha(\alpha - 1) x_1^{\alpha-2} x_2^{\beta} \leq 0 \text{ if and only if } \alpha \leq 1$$

   $$\frac{\partial^2 u(x_1, x_2)}{\partial x_2^2} = \beta(\beta - 1) x_1^{\alpha} x_2^{\beta-2} \leq 0 \text{ if and only if } \beta \leq 1$$

   Hence, while additional units of good 1 or good 2 increase this individual's utility, they do it at a decreasing rate.

   (c) *Quasiconcavity.* Let us first simplify the expression of the utility function by applying a monotonic transformation on $u(x_1, x_2)$, since any monotonic transformation of a utility function maintains the same preference ordering. In this case, we apply

   $$z_1 = \ln\ u(\cdot) = \alpha \ln x_1 + \beta \ln x_2$$

   We now need to find the bordered Hessian matrix, and then find its determinant. If this determinant is greater than (or equal to) zero, then this utility function is quasiconcave; otherwise it is quasiconvex.[1] The bordered Hessian matrix is

   $$\begin{vmatrix} 0 & z_1 & z_2 \\ z_1 & z_{11} & z_{12} \\ z_2 & z_{21} & z_{22} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\alpha}{x_1} & \frac{\beta}{x_2} \\ \frac{\alpha}{x_1} & -\frac{\alpha}{x_1^2} & 0 \\ \frac{\beta}{x_2} & 0 & -\frac{\beta}{x_2^2} \end{vmatrix}$$

[1] See Simon and Blume's *Mathematics for Economists* for references about the bordered Hessian matrix

and the determinant of this matrix is

$$0+\left[-\frac{\alpha}{x_1}\left(\frac{\alpha}{x_1}\cdot\left(\frac{-\beta}{x_2^2}\right)-0\right)\right]+\left[\frac{\beta}{x_2}\left(0-\left(-\frac{\alpha}{x_1^2}\right)\left(\frac{\beta}{x_2}\right)\right)\right]$$

$$0+\frac{\alpha^2\beta}{x_1^2x_2^2}+\frac{\alpha\beta^2}{x_1^2x_2^2}=\frac{\alpha\beta(\alpha+\beta)}{x_1^2x_2^2},$$

which is positive for all $x_1, x_2 \in \mathbb{R}_+$, ultimately implying that the Cobb-Douglas utility function $u(\cdot)$ is quasiconcave.

(d) *Homothetic preferences.* We know that the Cobb-Douglas utility function is homegeneous, and that all homogeneous functions are homothetic. Hence, the Cobb-Douglas utility function must be homothetic. For a more algebraic proof, however, let us first find the marginal rate of substitution

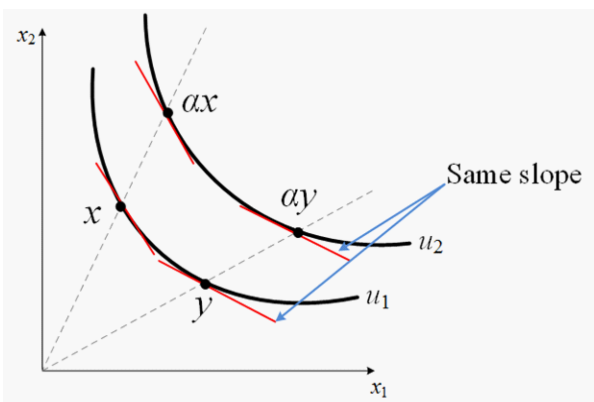
$$MRS_{1,2}=-\frac{\alpha x_1^{\alpha-1}x_2^{\beta}}{\beta x_1^{\alpha}x_2^{\beta-1}}.$$

Scaling up the amounts of all goods by a common factor t, we obtain

$$MRS_{1,2}=-\frac{\alpha\left(tx_1\right)^{\alpha-1}\left(tx_2\right)^{\beta}}{\beta\left(tx_1\right)^{\alpha}\left(tx_2\right)^{\beta-1}}=-\frac{t^{\alpha-1+\beta}}{t^{\alpha+\beta-1}}\frac{\alpha x_1^{\alpha-1}x_2^{\beta}}{\beta x_1^{\alpha}x_2^{\beta-1}}=-\frac{\alpha x_1^{\alpha-1}x_2^{\beta}}{\beta x_1^{\alpha}x_2^{\beta-1}}$$

which shows that the $MRS_{1,2}$ does not change when we scale up all goods by a common factor t, i.e., if we depict a ray from the origin (where the ratio between x_1 and x_2 is constant), indifference curves would have the same slope at the point where they are crossed by the ray.

- *A few remarks on Homothetic preferences.* When preferences are homothetic, the MRS between the two goods is just a function of the consumption ratio between the goods, $\frac{x_1}{x_2}$, but it does not depend on the absolute amounts consumed. As a consequence, if we double the amount of both goods, the MRS (i.e., the willingness of the individual to substitute one good for another) does not change; as depicted in the figure.



- Recall that this type of preferences induce wealth expansion paths that are straight lines from the origin, i.e., if we double the wealth level of the individual, then his wealth expansion path (the line connecting his demanded bundles for the initial and the new wealth level) are straight lines. A corollary of this property is that the demand function obtained from homothetic preferences must have an income-elasticity equal to 1, i.e., when the consumer's income increases by 1%, the amount he purchases of any good k must increase by 1% as well.

- Examples of preference relations that are homothetic: Cobb-Douglas (as in the previous example), preferences over goods that are considered perfect substitutes, preferences over goods that are considered perfect complements, and CES preferences. In contrast, quasilinear preference relations are not homothetic.

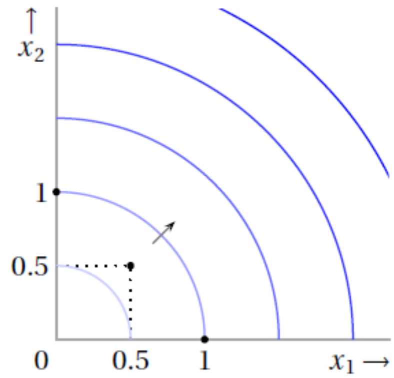
2. **Lexicographic preferences.** Show that lexicographic preferences are convex.

   - Assume that $x \succsim y$, where $x = (x_1, y_1)$ and $y = (y_1, y_2)$. We need to consider two cases:
     - If bundle x contains more units of good 1 than bundle y, $x_1 > y_1$, then the linear combination $\lambda x_1 + (1 - \lambda) y_1$ satisfies $\lambda x_1 + (1 - \lambda) y_1 > y_1$ for all $\lambda \in (0, 1)$, implying that $\lambda x + (1 - \lambda) y \succ y$. Therefore, convexity holds in this case.
     - If bundle x contains the same units of good 1 than bundle y, $x_1 = y_1$, then the linear combination $\lambda x_1 + (1 - \lambda) y_1$ satisfies $\lambda x_1 + (1 - \lambda) y_1 = y_1$ for all $\lambda \in (0, 1)$. In this case, bundle x must have more units of good 2 than bundle y, $x_2 \geq y_2$ (otherwise, we wouldn't have that $x \succsim y$). As a consequence, the linear combination $\lambda x_2 + (1 - \lambda) y_2$ satisfies $\lambda x_2 + (1 - \lambda) y_2 \geq y_2$, implying that $\lambda x + (1 - \lambda) y \succsim y$. Therefore, convexity holds in this case as well.

3. **Three examples of preference relations.** Describe each of the following three preference relations formally, giving a utility function that represents the preferences wherever possible, draw some representative indifference sets, and determine whether the preferences are monotone, continuous, and convex.

   (a) The consumer prefers the bundle (x_1, x_2) to the bundle (y_1, y_2) if and only if (x_1, x_2) is further from $(0, 0)$ than is (y_1, y_2), where the distance between two bundles is measured with the Euclidean distance.

   - The preference relation is represented by a utility function such as $u(x_1, x_2) = x_1^2 + x_2^2$, which exhibits indifference curves that are bowed out from the origin, as the figure below illustrates.



- These preferences are strongly monotone and continuous. However, they are not convex. To see this point, consider, for instance, bundles $(1, 0)$ and $(0, 1)$, which satisfy $(1, 0) \sim (0, 1)$. However, $(0, 1)$ is strictly preferred to the linear combination $(0.5, 0.5)$, which combines $(0, 1)$ and $(1, 0)$ with equal weights on each bundle, i.e., $\alpha = 1/2$.

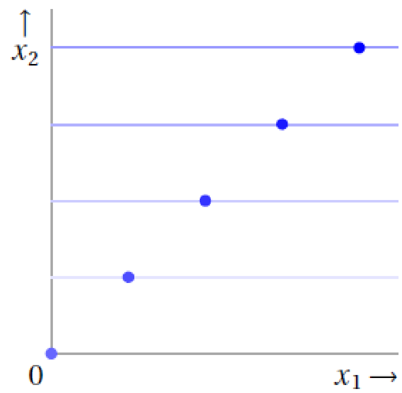
(b) The consumer prefers any balanced bundle, containing the same amount of each good, to any unbalanced bundle. Between balanced bundles, he prefers the one with the largest quantities. Between unbalanced bundles, he prefers the bundle with the largest quantity of good 2.

- A utility function that represents these preferences is given by

$$u(x) = \begin{cases} x_2 & \text{if } x_1 = x_2 \\ -\frac{1}{1+x_2} & \text{otherwise.} \end{cases}$$

  Other utility functions also represent these preferences. First, note that balanced bundles are preferred to unbalanced ones since $x_2 \geq 0 > -\frac{1}{1+x_2}$. Second, between balanced bundles, he prefers the one with the largest quantities, that is, $x \succsim y$ if and only if the utility of these bundles satisfy $u(x) = x_1 = x_2 \geq y_1 = y_2 = u(y)$. Third, between unbalanced bundles, he prefers the bundle with the largest quantity of good 2, that is, if $x_2 > y_2$, then the utilities of these bundles satisfy $u(x) = -\frac{1}{1+x_2} > -\frac{1}{1+y_2} = u(y)$, since $-\frac{1}{1+x_2} > -\frac{1}{1+y_2}$ simplifies to $\frac{1}{1+x_2} < \frac{1}{1+y_2}$, and subsequently to $1+y_2 < 1+x_2$, which holds because $x_2 > y_2$ in this case.

- The figure below illustrates some indifference sets. Each blue disk (on the 45-degree line where $x_1 = x_2$) is an indifference set. In addition, each horizontal line, excluding the disk on it, is another indifference set (corresponding to a different value of the utility function).



- The preferences are not monotonic, since a balanced bundle such as $(0, 0)$ is strictly preferred to an unbalanced bundle with more units of every good, such as $(1, 2)$, for instance.
- The preferences are not continuous. To proof this point, consider two balanced bundles such as $(2, 2)$ and $(1, 1)$, which satisfy $(2, 2) \succ (1, 1)$. Increasing the amount of either good in bundle $(2, 2)$, we obtain an unbalanced new bundle $(2 + \varepsilon, 2)$, where $\varepsilon > 0$, which satisfies $(1, 1) \succ (2 + \varepsilon, 2)$ for every $\varepsilon > 0$.
- Finally, the preferences are not convex either. Consider a balanced and an unbalanced bundle, such as $(2, 2)$ and $(0, 4)$, which satisfy $(2, 2) \succ (0, 4)$. Their linear combination $\frac{1}{2}(2, 2)+\frac{1}{2}(0, 4) = (1, 3)$ yields an unbalanced bundle with fewer units of good 2, thus not being prefered to $(0, 4)$, that is, we find a preference reversal since $(0, 4) \succ (1, 3)$.

(c) The consumer cares first about the sum of the amounts of the goods; if the sum is the same in two bundles, he prefers the bundle with more of good 1.

- This preference relation says that $x \succsim y$ if either: (i) $x_1 + x_2 > y_1 + y_2$; or (ii) $x_1 + x_2 = y_1 + y_2$ and $x_2 \geq y_2$. This preference resembles a lexicographic preference relation and it cannot be represented with a utility function. Each indifference set consists of a single bundle, as the lexicographic preferences.
- This preference is strongly monotonic: consider bundle (x_1, x_2) and increase the amount of all goods, yielding $(x_1+\delta, x_2+\delta)$, which implies that the sums satisfy $(x_1 + \delta) + (x_2 + \delta) = x_1 + x_2 + 2\delta > x_1 + x_2$, so that $(x_1 + \delta, x_2 + \delta) \succ (x_1, x_2)$.
- The preference is also convex: consider bundles (x_1, x_2) and (y_1, y_2) such that $(x_1, x_2) \succsim (y_1, y_2)$. Their linear combination $\alpha(x_1, x_2) + (1 - \alpha)(y_1, y_2)$ yields a sum $\alpha(x_1+y_1)+(1-\alpha)(x_2+y_2)$. We need to consider two cases: $(x_1, x_2) \succsim (y_1, y_2)$ holds because (i) $x_1 + x_2 > y_1 + y_2$; or because (ii) $x_1 + x_2 = y_1 + y_2$ and $x_2 \geq y_2$. In case (i), $x_1 + x_2 > y_1 + y_2$ implies that

$$\alpha(x_1 + y_1) + (1 - \alpha)(x_2 + y_2) > y_1 + y_2$$

holds since this inequality simplifies to

$$\underbrace{\alpha x_1 + (1-\alpha)x_2}_{<x_1+x_2} > \alpha y_2 + (1-\alpha)y_1$$

which we can compare with the initial condition $x_1 + x_2 > y_1 + y_2$ as follows

$$\begin{aligned} x_1 + x_2 &> \alpha x_1 + (1-\alpha)x_2 \text{ and} \\ y_1 + y_2 &> \alpha y_2 + (1-\alpha)y_1 \end{aligned}$$

ultimately implying that $\alpha(x_1 + y_1) + (1-\alpha)(x_2 + y_2) > y_1 + y_2$ holds. In case (ii), $x_1 + x_2 = y_1 + y_2$ implies that the linear combination $\alpha(x_1 + y_1) + (1-\alpha)(x_2 + y_2)$ simplifies to $[\alpha x_1 + (1-\alpha)x_2] + [\alpha y_1 + (1-\alpha)y_2]$, which further simplifies to $2[\alpha x_1 + (1-\alpha)x_2]$ since $x_1 + x_2 = y_1 + y_2$ entails that $\alpha x_1 + (1-\alpha)x_2 = \alpha y_1 + (1-\alpha)y_2$, as we place the weigh on each good. Therefore, $2[\alpha x_1 + (1-\alpha)x_2] > y_1 + y_2$ since $x_2 \geq y_2$.

- Finally, this preference is not continuous. To see this, consider two unbalanced bundles, such as $(1, 2)$ and $(2, 1)$, which satisfy $(1, 2) \succ (2, 1)$. Increasing the amount of good 2 in $(2, 1)$ by a small $\varepsilon > 0$, we obtain a preference reversal, that is, $(2, 1 + \varepsilon) \succ (1, 2)$.

4. **Ideal bundle.** The consumer has in mind an ideal bundle x^*. He prefers bundle x to y if and only if x is closer to x^* than y is, that is, $x \succsim y$ if and only if

$$|x_1 - x_1^*| + |x_2 - x_2^*| \leq |y_1 - y_1^*| + |y_2 - y_2^*| .$$

Show that this preference relation is continuous and convex.

- The utility function $-(|x_1 - x_1^*| + |x_2 - x_2^*|)$ that represents the preference relation is continuous, so the preference relation must also be continuous.
- Regarding convexity, assume that bundles y and z satisfy $y \succsim z$, which implies that bundle y is closer to the consumer's ideal point, x^*, than bundle z is. Therefore, a linear combination between bundles y and z, $\alpha y + (1-\alpha)z$, is closer to x than bundle z is, implying that $\alpha y + (1-\alpha)z$ is preferred to z; as required for convexity.

5. **Rationalizable choices.** Determine whether each of the following five choice functions over a set X is rationalizable. If the answer is positive, find a preference relation that rationalizes the choice function. Otherwise, prove that the choice function is not rationalizable.

(a) The set X consists of candidates for a job. An individual has a complete ranking of the candidates. When he has to choose from a set A, he first orders the candidates in A alphabetically, and then examines the list from the beginning. He goes down the list as long as the new candidate is better than the previous one. If the nth candidate is the first who is better than the $(n+1)$th candidate, he stops and chooses the nth candidate. If in his journey he never gets to a candidate who is inferior to the previous one, he chooses the last candidate.

- Assume that $X = \{a, b, c\}$ and the individual's preference relation is $c \succ a \succ b$. From the entire set he chooses a. From the set $\{a, c\}$ he chooses c. This choice function is not rationalizable

(b) The set X consists of n basketball teams, indexed 1 to n. The teams participate in a round robin tournament. That is, every team plays against every other team. An individual knows, for every pair of teams, which one wins. When he chooses a team from a set A, he chooses the one with the largest number of wins among the games between teams in A. If more than one team has the largest number of wins, he chooses the team with the lowest index among the tied teams.

- This choice function may violate rationality. Assume that $n = 5$ and that team 1 beats all teams except 2, and team 2 loses to all teams except 1. Then the individual chooses team 1 from the entire set of teams but team 2 from $\{1, 2\}$.

(c) The set X consists of pictures. An individual has in mind L binary criteria, each of which takes the value 0 (the criterion is not met) or 1 (the criterion is met). Examples of such criteria are whether the painting is modern, whether the painter is famous, and whether the price is above \$1,000. The criteria are ordered: criterion_1, criterion_2, . . . , criterion_L. When the individual chooses a picture from a subset of X, he rejects those that do not satisfy the first criterion. Then, from those that satisfy the first criterion, he rejects those that do not satisfy the second criterion. And so on, until only one picture remains. Assume that any two alternatives have a criterion by which they differ, so that the procedure always yields a unique choice.

- Let $\succsim_k$ be the preference relation that puts the pictures that satisfy criterion k at the top (with indifferences among them) and the pictures that do not satisfy the criterion at the bottom (with indifferences among them).
- The choice function is rationalized by lexicographic preferences with the priority ordering $\succsim_1$, $\succsim_2$, ... , $\succsim_k$. That is, first the alternatives ranked highest according to $\succsim_1$ are selected, then among these alternatives the ones ranked highest by $\succsim_2$ are selected, and so on.

(d) An individual has in mind two numerical functions, u and v, on the set X. For any set $A \subseteq X$, he first looks for the u-maximal alternative in A. If its u value is at least 10, he selects it. If not, he selects the v-maximal alternative in A.

- This choice function is rationalized by a preference relation that puts at the top the alternatives in set $\{x \in X : u(x) \geq 10\}$ with the preference induced by u. The remaining alternatives are put at the bottom with the preferences induced by v.

(e) An individual has in mind a preference relation on the set X. Each alternative is either red or blue. Given a set $A \subseteq X$, he chooses the best alternative among those with the color that is more common in A. In the case of a tie, he chooses among the red alternatives.

- This choice function may not be rationalizable. Let $X = \{a, b, c, d, e\}$. Assume that the preference relation coincides with alphabetical order, $Red =$

$\{a, b\}$ and $Blue = \{c, d, e\}$. From the entire set, the individual chooses c but from the subset $\{a, b, c\}$ he chooses a. Thus, the choice function is not rationalizable.