EconS 503 - Advanced Microeconomics - II
Final Exam - Answer key

1. **Signaling in the court.** A defendant in a court case appears before the judge. Suppose the actual harm to the plaintiff (the person or company accusing him) is equal to $1000x$. Parameter x denotes whether the defendant is innocent ($x = 0$) or guilty ($x = 1$) of the damage, and both realizations of x are equality likely (i.e., happen with probability 1/2). The defendant observes the realization of x and has evidence to provie it. The judge does not observe x, but knows the probability distribution of x. The next figure depicts the game and we describe its time structure below.

   Insert figure here

   The time structure of the game is the following:

   1 Nature determines the realization of x, $x = 1$ or $x = 0$, both happening with the same probability.

   2 The defendant privately observes the realization of x and chooses whether to provide evidence, E, or not providing evidence, N. Providing evidence to the court costs him \$1. If the defendant provides evidence, this decision is equivalent to revealing the realization of x to the judge.

   3. If evidence is provided, the judge observes x, on the left side of the figure, where he faces a complete information subgame. If evidence is not provided (on the right side of the figure), the judge must update its belief, q, about facing a guilty defendant ($x = 1$). Both after E and N, the judge must respond choosing the level of damages y (in thousands of dollars) that the defendant must pay. Payoffs in the figure indicate that the judge prefers to respond with a "fair" y: that is, she would like y to be as close as possible to x. The defendant's payoffs shows that he just seeks to minimize his monetary loss.

   Answer the following questions.

   (a) Find the Nash equilibrium in the subgames where the judge responds to E^1 and to E^0, both of them on the left side of the game tree.

   (b) Show that there is a unique Perfect Bayesian Equilibrium (PBE) of the game.

   (c) Verbally describe why the unique PBE that you found in part (b) is interesting from an economic standpoint.

   (d) Consider now a different version of the above game in which x is an integer between 0 and K, inclusive, with each of these values being equally likely (happening with probability $\frac{1}{K}$). Find a separating PBE of the game that resembles the PBE you found in part (b).

2. **First Welfare Theorem with External Effects**. Consider two individuals with utility functions $u^A = x^A y^A$ and $u^B = x^B y^B - 0.5x^A$, and endowments $e^A(e_x^A, e_y^A) = (15, 5)$ and $e^B(e_x^B, e_y^B) = (10, 15)$.

(a) Is individual A's utility affected by individual B's consumption? Is individual B's utility affected by A's consumption? Interpret.

- We can see that individual A is unaffected by individual B's consumption of either good. However, individual B is negatively impacted by individual A's consumption of good x. We can interpret this as individual A's consumption of good x creates a negative externality imposed on individual B.

(b) Find the equilibrium allocation.

- *Consumer A.* The tangency condition for consumer A, $MRS^A_{x,y} = \frac{p_x}{p_y}$, is

$$\frac{y^A}{x^A} = \frac{p_x}{p_y}.$$

Rearranging, we find $p_y y^A = p_x x^A$, which we can insert into consumer A's budget constraint $p_x x^A + p_y y^A = p_x 15 + p_y 5$ to find that

$$p_y y^A + p_y y^A = p_x 15 + p_y 5.$$

Simplifying, we first obtain $2p_y y^A = p_x 15 + p_y 5$, and solving for y^A, we have consumer A's demand for good y:

$$y^A = \frac{5}{2} + \frac{15}{2}\frac{p_x}{p_y}.$$

Plugging this expression back into the tangency condition $p_y y^A = p_x x^A$, we obtain

$$p_y \underbrace{\left(\frac{5}{2} + \frac{15}{2}\frac{p_x}{p_y}\right)}_{y^A} = p_x x^A.$$

Solving for x^A, we first have

$$x^A = \frac{p_y}{p_x}\left(\frac{5}{2} + \frac{15}{2}\frac{p_x}{p_y}\right),$$

which simplifies to consumer A's demand for good x

$$x^A = \frac{15}{2} + \frac{5}{2}\frac{p_y}{p_x}.$$

- *Consumer B.* The tangency condition for consumer B, $MRS^B_{x,y} = \frac{p_x}{p_y}$ is

$$\frac{y^B}{x^B} = \frac{p_x}{p_y}.$$

Rearranging, we find $p_y y^B = p_x x^B$, which we can insert into consumer B's budget constraint $p_x x^B + p_y y^B = p_x 10 + p_y 15$ to find that

$$p_y y^B + p_y y^B = p_x 10 + p_y 15.$$

Simplifying, we first obtain $2p_y y^B = p_x 10 + p_y 15$, and solving for y^B, we have consumer B's demand for good y:

$$y^B = \frac{15}{2} + 5\frac{p_x}{p_y}.$$

Plugging this expression back into the tangency condition $p_y y^B = p_x x^B$, we obtain

$$\underbrace{p_y\left(\frac{15}{2} + 5\frac{p_x}{p_y}\right)}_{y^B} = p_x x^B.$$

Solving for x^B, we first have

$$x^B = \frac{p_y}{p_x}\left(\frac{15}{2} + 5\frac{p_x}{p_y}\right),$$

which simplifies to consumer B's demand for good x

$$x^B = 5 + \frac{15}{2}\frac{p_y}{p_x}.$$

- *Equilibrium prices.* Inserting the demands for good x from consumers A and B into the feasibility condition $x^A + x^B = 15 + 10$, we obtain

$$\underbrace{\frac{15}{2} + \frac{5}{2}\frac{p_y}{p_x}}_{x^A} + \underbrace{5 + \frac{15}{2}\frac{p_y}{p_x}}_{x^B} = 25,$$

which simplifies to $\frac{25}{2} + \frac{20}{2}\frac{p_y}{p_x} = 25$, and again to $\frac{20}{2}\frac{p_y}{p_x} = \frac{25}{2}$. Solving for $\frac{p_y}{p_x}$, we find an equilibrium price ratio of

$$\frac{p_y}{p_x} = \frac{25}{20} = \frac{5}{4}.$$

- *Equilibrium allocations.* We can plug the price ratio into each consumer's demands for each good to find that consumer A's allocation is

$$x^A = \frac{15}{2} + \frac{5}{2}\frac{p_y}{p_x} = \frac{15}{2} + \frac{5}{2}\frac{5}{4} = \frac{85}{8} = 10.625 \text{ units, and}$$
$$y^A = \frac{5}{2} + \frac{15}{2}\frac{p_x}{p_y} = \frac{5}{2} + \frac{15}{2}\frac{4}{5} = \frac{17}{2} = 8.5 \text{ units.}$$

Consumer B's allocation is

$$x^B = 5 + \frac{15}{2}\frac{p_y}{p_x} = 5 + \frac{15}{2}\frac{5}{4} = \frac{115}{8} = 14.375 \text{ units, and}$$
$$y^B = \frac{15}{2} + 5\frac{p_x}{p_y} = \frac{15}{2} + 5\frac{4}{5} = \frac{23}{2} = 11.5 \text{ units.}$$

(c) Show that the equilibrium allocation is not socially efficient. (*Hint:* Refer to the appendix in this chapter.)

- For an allocation to be efficient, it must solve

$$\max_{x^A, y^A, x^B, y^B \geq 0} \quad x^A y^A$$

subject to

$$\begin{aligned} x^B y^B - 0.5x^A &\geq \bar{u}^B \\ x^A + x^B &\leq 25 \\ y^A + y^B &\leq 20. \end{aligned}$$

The Lagrangian associated with this maximization problem is

$$\mathcal{L} = x^A y^A + \lambda \left[x^B y^B - 0.5x^A - \bar{u}^B \right] + \mu_1[25 - x^A - x^B] + \mu_2[20 - y^A - y^B].$$

Differentiating with respect to x^A and x^B, we obtain

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x^A} &= y^A - 0.5\lambda - \mu_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x^B} &= \lambda y^B - \mu_1 = 0. \end{aligned}$$

Differentiating with respect to y^A and y^B, we obtain

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial y^A} &= x^A - \mu_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial y^B} &= \lambda x^B - \mu_2 = 0. \end{aligned}$$

Combining $\frac{\partial \mathcal{L}}{\partial x^A}$ and $\frac{\partial \mathcal{L}}{\partial y^A}$, we obtain

$$\frac{y^A - 0.5\lambda}{x^A} = \frac{\mu_1}{\mu_2},$$

and combining $\frac{\partial \mathcal{L}}{\partial x^B}$ and $\frac{\partial \mathcal{L}}{\partial y^B}$, we obtain

$$\frac{\lambda y^B}{\lambda x^B} = \frac{y^B}{x^B} = \frac{\mu_1}{\mu_2}.$$

These two equations tell us that optimality requires that

$$\frac{y^A - 0.5\lambda}{x^A} = \frac{y^B}{x^B}.$$

Rearranging, we find that the socially efficient allocation must satisfy $x^B(y^A - 0.5\lambda) = x^A y^B$. This does not coincide with the equilibrium allocation, which requires that $x^B y^A = x^A y^B$. The difference being that, when finding the socially optimal allocation, we consider the negative externality experienced by consumer B.

(d) Does the First Welfare Theorem hold? Interpret your results.

- As we can see in part (b), the First Welfare Theorem does not hold, as the equilibrium allocation and the efficient allocation do not coincide. In the equilibrium allocation, the consumers, who each choose their optimal allocation by setting their marginal rate of substitution equal to the price ratio, do not account for the negative externality. This negative externality is taken into consideration when maximizing joint utility in the efficient allocation, thus leading to a different allocation of the two goods.

3. **Emission fees and mechanisms, Duggan and Roberts (2003).**[1] Consider an industry with $N \geq 2$ polluting firms producing a homogenous good. Let the profit function of firm i be $\pi_i(q_i) = \ln q_i$, which is increasing and concave in its pollutants q_i. The social cost from pollution is

$$C(q_1, \cdots, q_n) = \sum_{i=1}^{n} \frac{\gamma_i}{2} q_i^2,$$

which is also increasing but convex in the pollutants q_i emitted by firm i. Finally, a regulator (e.g., government agency) considers the following welfare function

$$W(q_1, \cdots, q_n) = \sum_{i=1}^{n} \pi_i(q_i) - C(q_1, \cdots, q_n)$$

(a) *Complete information.* Assume that the regulator can observe pollution levels and sets an emission fee t_i per unit of emissions. Find the following: (i) firm i's profit-maximizing pollution level as a function of fee t_i, $q_i(t_i)$; (ii) the socially optimal pollution from firm i, q_i^{SO}; and (iii) the emission fee t_i that induces firm i to produce q_i^{SO}, i.e., the fee t_i that solves $q_i(t_i) = q_i^{SO}$.

- *Equilibrium pollution.* Firm i solves

$$\max_{q_i \geq 0} \pi_i(q_i) - t_i q_i = \ln q_i - t_i q_i$$

Differentiating with respect to q_i, we find

$$\frac{1}{q_i} = t_i$$

which, after solving for q_i, yields

$$q_i(t_i) = \frac{1}{t_i}.$$

- *Socially optimal pollution.* Differentiating with respect to q_i in the social welfare function, yields

$$\frac{1}{q_i} = \gamma_i q_i$$

which solving for q_i yields a socially optimal pollution of

$$q_i^{SO} = \frac{1}{\sqrt{\gamma_i}}.$$

[1] Duggan J. and Roberts J. (2003). Implementing the Efficient Allocation of Pollution. *American Economic Review,* 92(4), pp. 1070-78.

- *Emission fee.* Hence, the emission fee t_i should be set to induce every firm i to produce the socially optimal amount of pollution, that is, $q_i(t_i) = q_i^{SO}$

$$\frac{1}{t_i} = \frac{1}{\sqrt{\gamma_i}}$$

which yields an emission fee

$$t_i = \sqrt{\gamma_i}.$$

Intuitively, the emission fee is set to make firm i's marginal profit from one more unit of pollution, t_i, to coincide with its marginal social cost, $\sqrt{\gamma_i}$.

(b) *Incomplete information.* Assume that the level of pollution is unobservable to the regulator but observable among all firms. Then, the regulator can devise a circular monitoring mechanism, in which firm i reports the observed pollution level of firm $i-1$, $\overline{q}_{i-1}$, firm $i-1$ reports the observed pollution of firm $i-2$, $\overline{q}_{i-2}$, and firm 1 reports that of firm n, $\overline{q}_n$. This allows the regulator to set an emission fee per unit of pollution

$$t_i = \frac{\partial C\left(\overline{q}_i, q_{-i}\right)}{\partial q_i},$$

where $\overline{q}_i$ denotes firm i's pollution (reported by firm $i+1$), and q_{-i} represents the true pollution level of all other firms. In addition, firm i faces a penalty of $\left(\overline{q}_{i-1} - q_{i-1}\right)^2$ for misporting his neighbor's pollution level not at q_{i-1}.

1. Will firm i misreport the output of firm $i-1$? Why or why not?
   - No, because firm i will face a penalty proportional to his misreporting, which is given by $\left(\overline{q}_{i-1} - q_{i-1}\right)^2$. As a consequence, firm i truly reports what it has observed from firm $i-1$ in order to avoid penalties. This applies to every firm $i \in \{1, \cdots, n\}$.
2. Write down firm i's profit-maximization problem and solve for its optimal output.
   - To find the pollution level that maximizes firm i's profit, we fix every firm j's report at truthtelling, $\overline{q}_j = q_j$, since firms have no incentives to misreport (see part a). Firm i chooses q_i to solve

$$\max_{q_i \geq 0} \quad \pi_i\left(q_i\right) - t_i q_i - \left(\overline{q}_{i-1} - q_{i-1}\right)^2$$

where the first term denotes firm i's profits, the second represents the emission fee payments, and the third captures the penalty from misreporting firm $i-1$'s pollution.
Since fee t_i is, by definition, $t_i = \frac{\partial C(\overline{q}_i, q_{-i})}{\partial q_i}$, and every firm i has no incentives to misreport (see part a), we can fix firms' reports at truthtelling, $\overline{q}_i = q_i$, the above problem becomes

$$\max_{q_i \geq 0} \quad \pi_i\left(q_i\right) - \frac{\partial C\left(q_i, q_{-i}\right)}{\partial q_i} q_i - \left(q_{i-1} - q_{i-1}\right)^2 = \ln q_i - \gamma_i q_i^2$$

Differentiating with respect to q_i, yields

$$\frac{1}{q_i} = 2\gamma_i q_i$$

Therefore, every firm i chooses a pollution level $q_i = \frac{1}{\sqrt{2\gamma_i}}$, which is lower than that under complete information, $q_i^{SO} = \frac{1}{\sqrt{\gamma_i}}$, as found in part (a). As a consequence, the difference

$$\frac{1}{\sqrt{\gamma_i}} - \frac{1}{\sqrt{2\gamma_i}} = \frac{2-\sqrt{2}}{2\sqrt{\gamma_i}} \simeq \frac{0.29}{\sqrt{\gamma_i}}$$

can be interpreted as the output inefficiency that arises due to regulator's incomplete information.

3. Find the tax revenue generated by the mechanism, and the social cost of pollution.
   - Total tax revenue is

$$\begin{aligned}\sum_{i=1}^{n} t_i q_i &= \sum_{i=1}^{n} \frac{\sqrt{\gamma_i}}{\sqrt{2}} \frac{1}{\sqrt{2\gamma_i}} \\ &= \sum_{i=1}^{n} \frac{1}{2} \\ &= \frac{n}{2}.\end{aligned}$$

   - The social cost of pollution, evaluated at the equilibrium output, $(q_1, \cdots, q_n)$, is

$$\begin{aligned}C(q_1, \cdots, q_n) &= \sum_{i=1}^{n} \frac{\gamma_i}{2} \left(\frac{1}{\sqrt{2\gamma_i}}\right)^2 \\ &= \sum_{i=1}^{n} \frac{1}{4} \\ &= \frac{n}{4}.\end{aligned}$$

EconS 503 - Final exam

Answer key

Exercise 1- ***Signaling between a judge and a defendant***

a. Find the Nash equilibrium of the subgames after E^0 and E^1 (left side of the tree).

After E^1, the judge chooses $\bar{y}$ such that:

$$\underset{\bar{y}}{Max} \ -(\bar{y}-1)^2$$

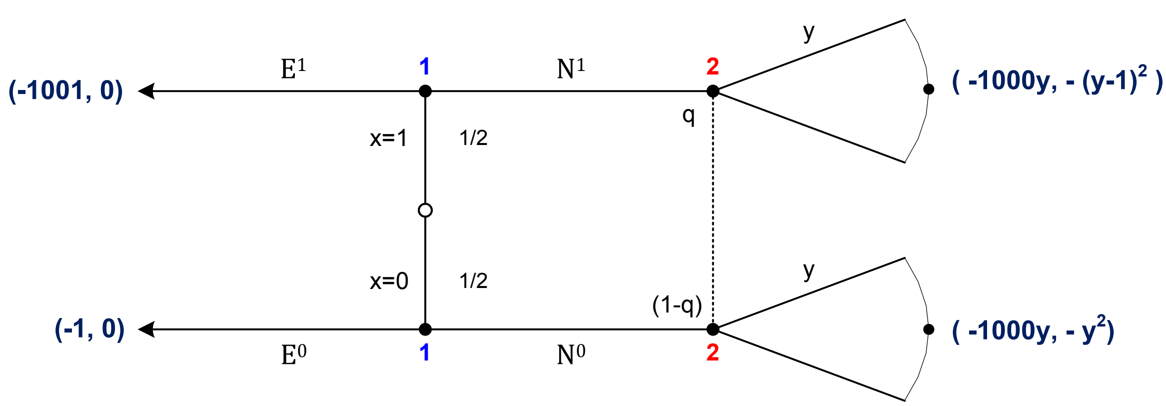
Taking FOCs with respect to $\bar{y}$, we obtain $-2(\bar{y}-1)=0$, which implies a corner solution where $\bar{y}=1$.

Similarly, after E^0, the judge chooses $\underline{y}$ such that:

$$\underset{\underline{y}}{Max} \ -\underline{y}^2$$

Taking FOCs with respect to $\underline{y}$, we obtain: $-2\underline{y} \leq 0$, which yields a corner solution $\underline{y}=0$.

Hence, the game tree simplifies as follows:



b. This game has a unique PBE. Find and report it.

- ***Let us first check for the existence of a separating PBE where E^0 and N^1:***

1. *Belief*: q=1 since N only comes from x=1

2. *Judge (second mover)*: After observing N, the judge selects y assigning full probability to being in the open node of his information set (see figure below)

Hence,

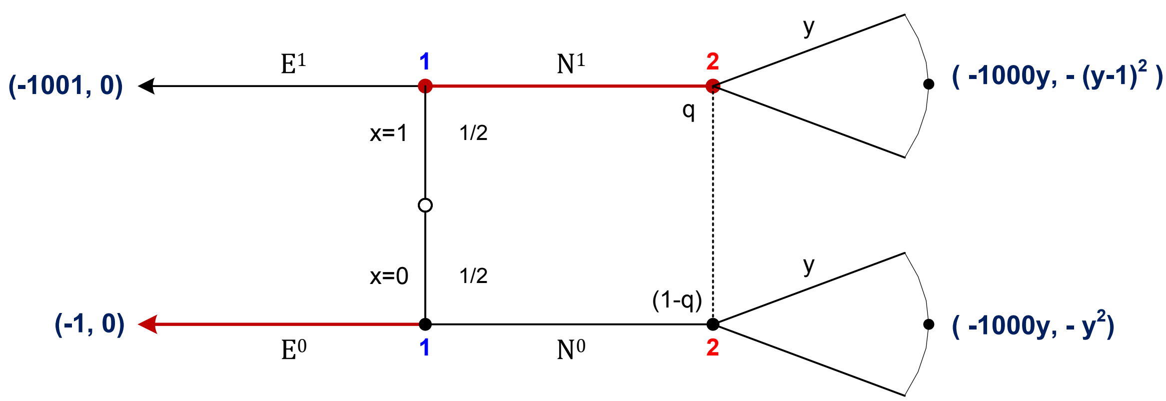
$$\underset{y}{Max} -(y-1)^2$$

Taking FOCs with respect to y, we obtain:

$$-2(y-1) = 0, \text{ which implies } y = 1$$

3. *Defendant (first mover)*:

- If $x = 1$, the defendant compares: -1001 if he chooses E^1, -1000 if he chooses N^1. So, N^1 is better.
- If $x = 0$, the defendant compares: -1 if he chooses E^0, -1000 if he chooses N^0. So, E^0 is better.



Hence, this separating PBE can be supported.

- ***Let us now check the separating N^0E^1***

1. *Beliefs*: $q = 0$ since N only comes from $x = 0$

2. *Judge*: After observing *N*, the judge assigns full probability to lower node of his information set. Then, he selects *y* such that:

$$\underset{y}{Max} -y^2$$

Taking FOCs with respect to y, we obtain $-2y \leq 0$, which implies $y = 0$.

3. *Defendant*:

- If $x = 1$, the defendant compares: -1001 if he chooses E^1, 0 if he chooses N^1, So, N^1 is better
- If $x = 0$, the defendant compares: -1 if he chooses E^0, 0 if he chooses N^0, So, N^0 is better.
- Hence, the separating N^0E^1 cannot be supported as PBE.

- ***Let us now check if a pooling PBE where $N^0 N^1$ can be sustained***

  1. *Beliefs*:

$$q = \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + \frac{1}{2} \times 1} = \frac{1}{2}$$

  2. *Judge*: After observing N, given his beliefs q=1/2, he must choose y to maximize his expected utility:

$$\underset{y}{Max} \ \frac{1}{2}[-(y-1)^2] + \frac{1}{2}[-y^2]$$

  Taking FOCs with respect to y, we obtain:

$$-\frac{1}{2} \times 2(y-1) - \frac{1}{2} \times 2y = 0, \text{which implies } y = \frac{1}{2}$$

  3. *Defendant*:

  - If $x = 1$, the defendant compares: -1001 if he chooses E^1, -1000 $\times \frac{1}{2} = -500$ if he chooses N^1. So, N^1 is better
  - If $x = 0$, the defendant compares: -1 if he chooses E^0, $-1000 \times \frac{1}{2} = -500$ if he chooses N^0, So, N^0 is better
  - Hence, the pooling $N^0 N^1$ cannot be sustained.

- ***Let us now check if a pooling PBE where $E^0 E^1$ can be sustained.***

1. *Beliefs*: $q \in [0, 1]$ since N is only observed off-the-equilibrium

2. *Judge*: From his beliefs, he chooses y in order to maximize his expected utility:

$$\underset{y}{Max} \ q[-(y-1)^2] + (1-q)[-y^2]$$

Taking FOCs with respect to y, we obtain:

$$-q \times 2(y-1) - (1-q) \times 2y = 0, \text{which implies } y = q$$

3. *Defendant*:

- If $x = 1$, the defendant compares: -1001 if he chooses E^1, $-1000q$ if he chooses N^1. So, N^1 is better for any $q < 1 \rightarrow$ Deviation from the prescribed pooling.
- If $x = 0$, the defendant compares: -1 if he chooses E^0, $-1000q$ if he chooses N^0. So, E^0 is better for any $q > 1/1000 \rightarrow$
- The pooling $E^0 E^1$ cannot be supported as PBE either.

c. Explain why the result of part (a) is interesting from an economic standpoint?

The only equilibrium that we can support in this game is the separating equilibrium in which the innocent defendant provides evidence of his innocence, whereas the guilty defendant does not provide such evidence. This is something desirable, since the judge can perfectly infer the true innocence of a defendant by simply observing whether he/she presented evidences.

d. When $x \in [0, K]$ with each value equally likely, find the PBE.

We are going to test the equilibrium where all but one type of defendant (those with types of x={0,…,K-1}) present evidence (E), but the last type x=K presents no evidence (N).

1) <u>Beliefs</u>

   After observing the evidence presented by the defendant, the judge can perfectly observe his type 0,1,2,…,K-1. In these cases we don't need to specify beliefs. When no evidence(N) is presented, the judge's beliefs are:

$$\mu(t_j|N) = 0 \ \ \forall \ j = \{0, \dots, K-1\}$$

$$\mu(t_K|N) = 1$$

   Which implies that after receiving no evidence, the judge assigns full probability to the K-type, and therefore no probability to any of the 0,1,2,…,K-1 types.

2) <u>Judge's Best Response:</u>

   <u>Given N:</u>

$$\max_y -(y-K)^2 \quad \rightarrow \quad y^N = K$$

   <u>Given E(where there is no information set and the judge knows what type has played E):</u>

$$\max_y[-(y-x)^2]$$

   where x is the specific type of the defendant that presented evidence (a type that is observed by the judge thanks to the presentation of evidence). Taking FOCs with respect to y, we obtain

$$-2y + 2x = 0$$

   Solving for y, we obtain

$$\rightarrow y^E = x$$

3) <u>Defendant's Best Response:</u>

   For types 0,…,K-1 : if he provides evidence, E, then they get y^E from the judge, providing:

$$-1000y^E - 1$$

   which must exceed his payoff from not presenting evidence: $-1000y^N = -1000K$

(in this case the judge interprets that the defendant is a K-type and chooses a sentence $y^N = K$)

- Note, type x=0 prefers the payoff he obtains by presenting evidence, $-1000y^E - 1 = -1$, than his payoff from not presenting evidence, $-1000K$ (since K > 2 given that there are more than two types of defendants).
- Similarly for type x=1, where $-1000y^E - 1 = -1001 > -1000 * K$; and for all other types x=2,3,...
- The defendant who obtains the lowest equilibrium payoff from providing evidence is x=K-1, who obtains $-1000y^E - 1 = -1000(K-1) - 1$. Let us check if his equilibrium payoff from providing evidence is larger than from deviating, that is:

  $$-1000K + 1000 - 1 > -1000K$$

  $$999 > 0$$

  This obviously holds, so the defendant behaves as prescribed when his type is x=0,...,K-1

For type K: if he doesn't provide evidence, N, (as initially prescribed) then he gets a sentence $y^N = K$ from the judge, providing a payoff of:

$$-1000K$$

This must exceed his alternative payoff from providing evidence (E):

$$-1000K - 1$$

The condition reduces to $-1000K > -1000K - 1$, which simplifies to 0>-1. This as well holds, showing the initially stated strategy, where types x=0,1,2,...,K-1 present evidence but type x=K does not, can be sustained as a PBE.