EconS 503 - Advanced Microeconomics - II
Midterm Exam #2 - Answer key

1. **Selten's horse.** Consider the "Selten's Horse" game depicted in Figure 1. Player 1 is the first mover in the game, choosing between C and D. If he chooses C, player 2 is called on to move between C' and D'. If player 2 selects C' the game is over. If player 1 chooses D or player 2 chooses D', then player 3 is called to move without being informed whether player 1 chose D before him or whether it was player 2 who chose D'. Player 3 can respond choosing L or R, and then the game ends.

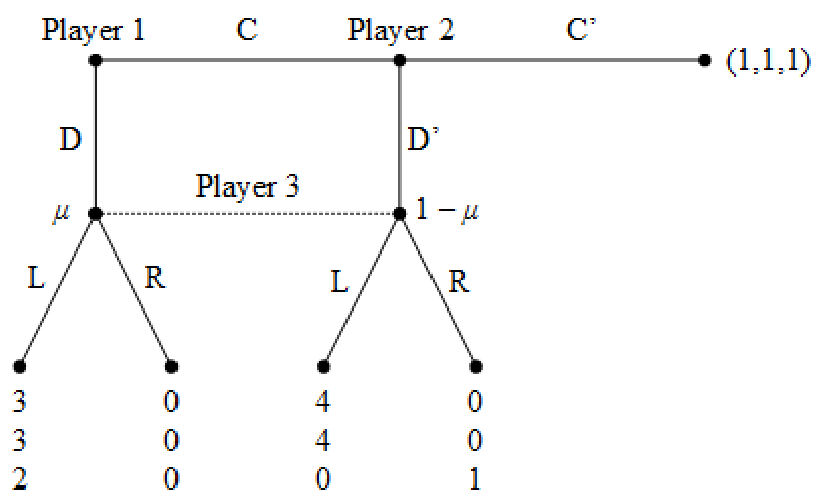

Figure 1. Selten's horse.

(a) Find all pure strategy Nash equilibria of the game.

- The strategy spaces of the players are as follows:

$$S_1 = \{C, D\}$$
$$S_2 = \{C', D'\}$$
$$S_3 = \{L, R\}$$

In Figure 2, we represent the strategies and payoffs of the three players in the following normal form representation of the game, where Player 1 chooses between the rows, Player 2 chooses between the columns, and Player 3 chooses between the matrixes.

Player 3 choosing L

Player 1 \ Player 2	C'	D'
C	1, 1, 1	4, 4, 0
D	3, 3, 2	3, 3, 2

Player 3 choosing R

Player 1 \ Player 2	C'	D'
C	1, 1, 1	0, 0, 1
D	0, 0, 0	0, 0, 0

Figure 2. Selten's horse - Matrix representation.

- We also underline the best responses of the three players in Figure 3, and identify that (C, C', R) and (D, C', L), are the two pure strategy Nash equilibria of this game.

Player 3 choosing L

Player 1 \ Player 2	C'	D'
C	1, 1, $\underline{1}$	$\underline{4}$, $\underline{4}$, 0
D	$\underline{3}$, $\underline{3}$, $\underline{2}$	3, $\underline{3}$, $\underline{2}$

Player 3 choosing R

Player 1 \ Player 2	C'	D'
C	$\underline{1}$, $\underline{1}$, $\underline{1}$	$\underline{0}$, 0, $\underline{1}$
D	0, $\underline{0}$, 0	$\underline{0}$, $\underline{0}$, 0

Figure 3. Selten's horse - Underlining best response payoffs.

(b) Argue that one of the pure strategy Nash equilibria found in part (a) is not sequentially rational.

- (D, C', L) is not sequentially rational. If Player 1 chooses D, then Player 3's belief is $\mu = 1$, responding with L (see left-hand side at the bottom of the tree). Anticipating that Player 3 choosing L, Player 2 compares his payoff from C', 1, against that from D' (which is followed by Player 3 responding with L), 4, and thus chooses D'. Therefore, Player 2 choosing C' is not sequentially rational.

(c) Show that there is only one Perfect Bayesian equilibrium (PBE) and it coincides with one of the pure strategy Nash equilibria you have identified in part (a).

- *Separating strategy profile* C, D'. First, we check the separating strategy profile, C, D', where Player 1 chooses C and Player 2 selects D.
  As depicted in Figure 4, since player 1 chooses C (as illustrated by the blue horizontal arrow) and player 2 chooses D' (as illustrated by the green vertical arrow), Player 3's belief is totally concentrated on Player 2 choosing D' (right-hand node of his information set), entailing that $\mu = 0$ (that is, $1 - \mu = 1$).

In this context, Player 3 is better off choosing R (as illustrated by the red arrows), which yields a payoff of 1, than choosing L, which yields a payoff of 0. However, given this response by Player 3, Player 2 maximizes his payoff by choosing *C'*, which provides him with a payoff of 1 (choosing D' gives him a payoff of 0). Therefore, the separating strategy profile C, D' cannot be supported as a PBE of this game.

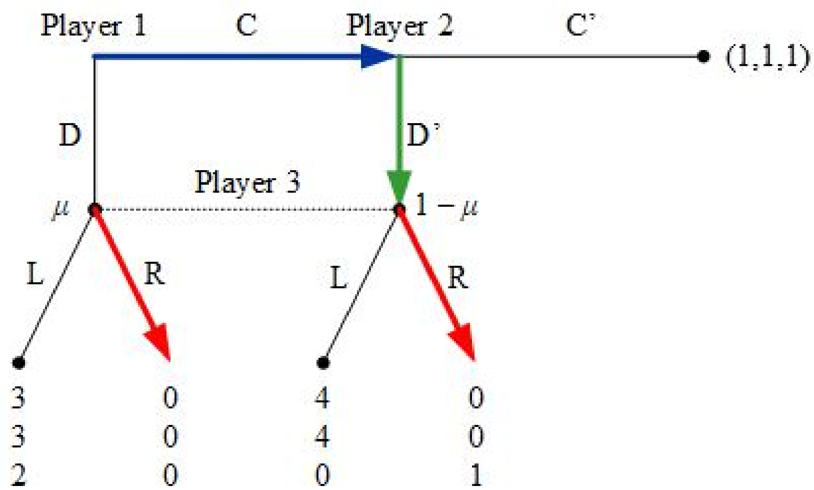

Figure 4. Separating Strategy Profile C, D'

- *Separating strategy profile* D, C'. Second, we check the separating strategy profile, D, C', where Players 1 chooses D and Player 2 selects *C'*.
As depicted in Figure 5 since player 1 chooses D (as illustrated by the blue vertical arrow) and player 2 chooses C' (as illustrated by the green horizontal arrow), Player 3's belief is totally concentrated on Player 1 choosing D (left-hand node of his information set), entailing that $\mu = 1$ (that is, $1 - \mu = 0$). In this context, Player 3 is better off responding with L (as illustrated by the red arrows), which yields a payoff of 2, than with R, which yields a payoff of 0. However, given this response, Player 2 is better off choosing *D'*, receiving a payoff of 4, than selecting *C'* (as prescribed in the above strategy profile). which gives him a payoff of only 1. Therefore, separating strategy profile D, C' cannot be supported as a PBE of this game when Player 3's belief is $\mu = 1$.

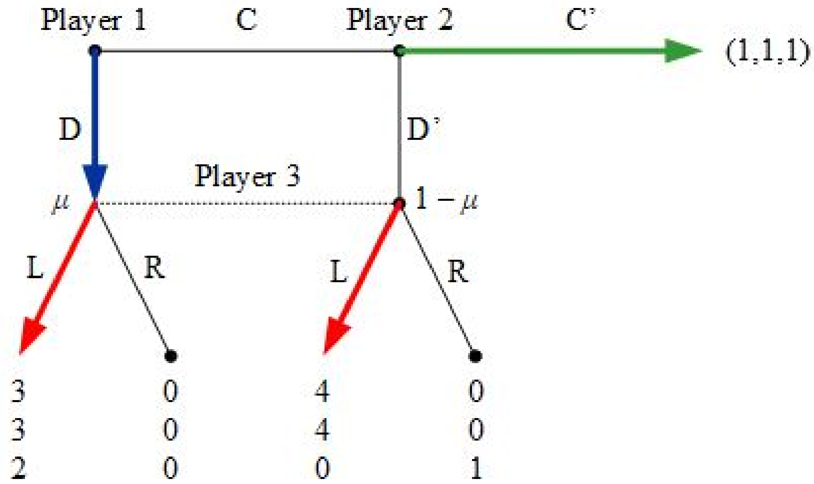

Figure 5. Separating Strategy Profile D, C'

- *Pooling strategy profile* C, C'. Third, we check the pooling strategy profile, C, C', where Player 1 chooses C and Player 2 selects C'.
  As depicted in Figure 6, since player 1 chooses C (as illustrated by the blue horizontal arrow) and player 2 chooses C' (as illustrated by the green horizontal arrow), messages D and D' are on the off-the-equilibrium path, leaving the beliefs of Player 3 unrestricted, that is, $\mu \in [0, 1]$. In other words, Player 3's information set should never be reached in this strategy profile.

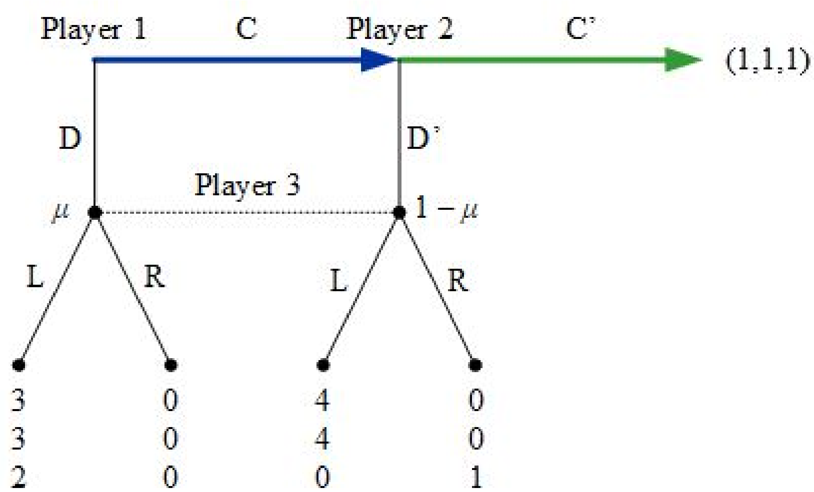

Figure 6. Pooling Strategy Profile C, C'

Therefore, if Player 3 is ever called out to move, he compares the expected payoff from responding with L and R, as follows:

$$EU_3(L) = 2 \times \mu + 0 \times (1 - \mu) = 2\mu$$
$$EU_3(R) = 0 \times \mu + 1 \times (1 - \mu) = 1 - \mu$$

Player 3 then responds with L if $2\mu > 1 - \mu$, which simplifies to $\mu > \frac{1}{3}$. Otherwise, he responds with R. This gives rise to two cases (one in which $\mu > \frac{1}{3}$, and Player 3 responds with L; and another in which $\mu \leq \frac{1}{3}$ and Player 3 responds with R), which we separately analyze below.

- *Case 1,* $\mu > \frac{1}{3}$. As depicted in Figure 7a, Player 3 responds with L (as illustrated by the red arrows) since $\mu > \frac{1}{3}$. In this context, Player 2 can improve his payoff by deviating from C', which only yields a payoff of 1, to D', which yields a payoff of 4. Therefore, the pooling strategy profile C, C' cannot be supported as a PBE of this game when Player 3's beliefs satisfy $\mu > \frac{1}{3}$.

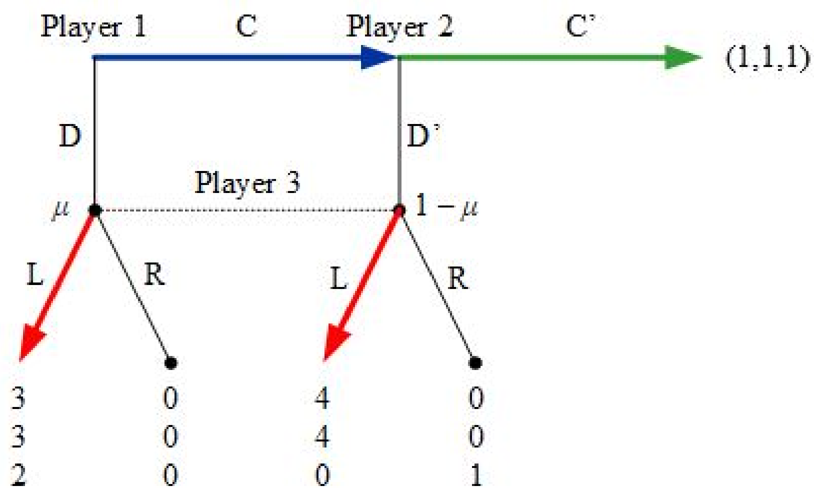

Figure 7a. Pooling Strategy Profile C, C' when $\mu > \frac{1}{3}$.

- *Case 2,* $\mu \leq \frac{1}{3}$. As depicted in Figure 7b, Player 3 responds with R (as illustrated by the red arrows) given that his beliefs are $\mu \leq \frac{1}{3}$. In this context, Player 2 does not deviate because his prescribed strategy, C', which gives him a payoff of 1, exceeds the payoff from deviating to D', which only gives him a payoff of 0. Similarly, Player 1 does not deviate because his prescribed strategy, C, which gives him a payoff of 1, exceeds his payoff from deviating to D, zero. Therefore, strategy profile C, C' can be supported as a PBE of this game when Player 3's beliefs satisfy $\mu \leq \frac{1}{3}$.

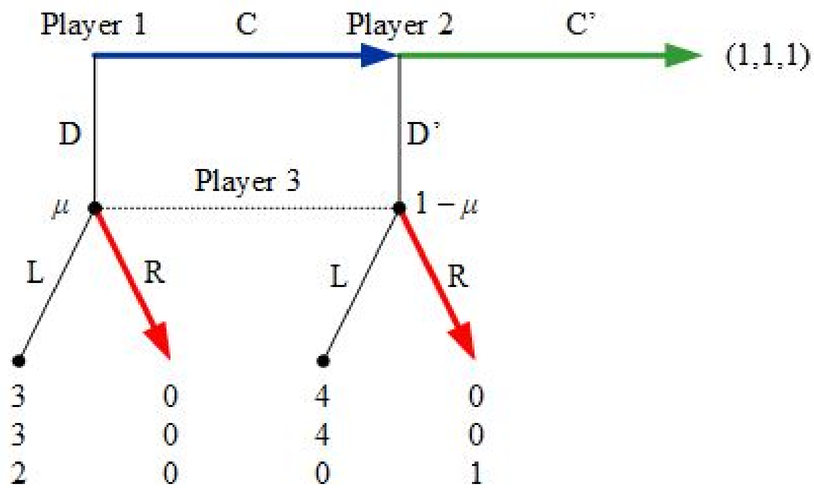

Figure 7b. Pooling Strategy Profile C, C' when $\mu \leq \frac{1}{3}$.

- *Pooling strategy profile* D, D'. Finally, we check the pooling strategy profile, D, D', when Player 1 chooses D and Player 2 selects *D'*.
  As depicted in Figure 8, since player 1 chooses D (as illustrated by the blue vertical arrow) and player 2 chooses D' (as illustrated by the green vertical arrow), messages D and D' are on the equilibrium path. In this setting, Player 3 being called out to move does not provide him with additional information about whether it is more likely that he is at the left- or right-hand side node on his information set. Therefore, if Player 3 is ever called out to move, he compares the expected payoff from responding with L and R, as follows:

$$EU_3(L) = 2 \times \mu + 0 \times (1 - \mu) = 2\mu$$
$$EU_3(R) = 0 \times \mu + 1 \times (1 - \mu) = 1 - \mu$$

  Player 3 then responds with L if $2\mu > 1 - \mu$, which simplifies to $\mu > \frac{1}{3}$. Otherwise, he responds with R. This gives rise to two cases (one in which $\mu > \frac{1}{3}$, and Player 3 responds with L; and another in which $\mu \leq \frac{1}{3}$ and Player 3 responds with R), which we separately analyze below.

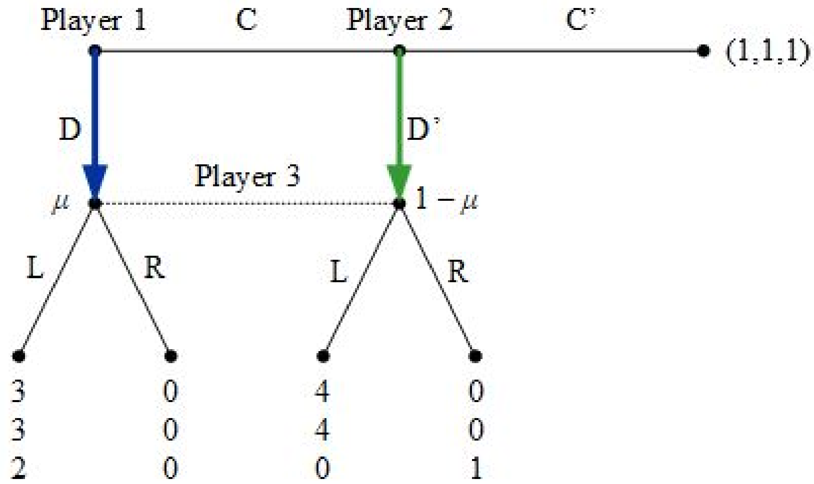

Figure 8. Pooling Strategy Profile D, D'.

- *Case 1,* $\mu > \frac{1}{3}$. As depicted in Figure 9a, Player 3 responds with L (as illustrated by the red arrows) given that his beliefs are $\mu > \frac{1}{3}$. Player 1 in this context can improve his payoff by deviating from the prescribed strategy of D, which yields a payoff of 3, to C, which yields a payoff of 4. Therefore, strategy profile $\{D, D', L\}$ cannot be supported as a PBE of this game when Player 3's beliefs satisfy $\mu > \frac{1}{3}$.

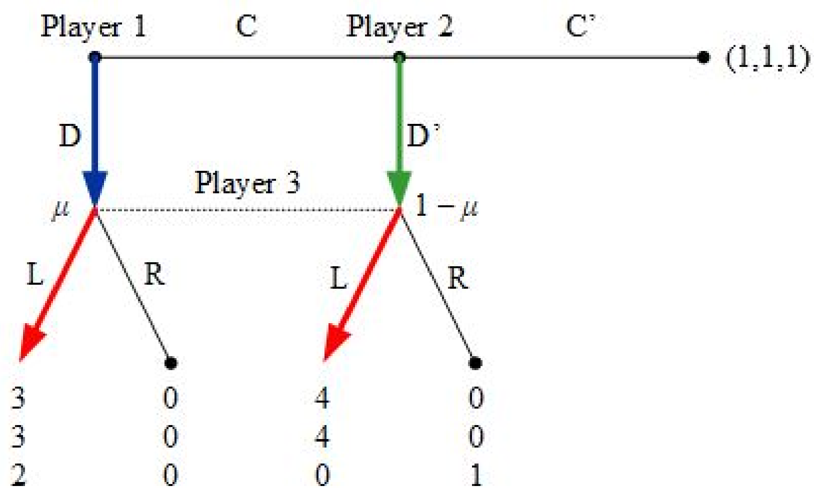

Figure 9a. Pooling Strategy Profile D, D' when $\mu > \frac{1}{3}$.

- *Case 2,* $\mu \leq \frac{1}{3}$. As depicted in Figure 9b, Player 3 responds with R (as illustrated by the red arrows) given that his beliefs are $\mu \leq \frac{1}{3}$. Player 2 in this context can improve his payoff by deviating from his prescribed strategy, D', which yields a payoff of 0, to C', which yields a higher payoff of 1. Therefore, strategy profile $\{D, D', R\}$ cannot be supported as a PBE of this game when Player 3's belief satisfies $\mu \leq \frac{1}{3}$. In other words, the pooling strategy profile D, D' cannot be supported as a PBE regardless of Player 3's beliefs since it couldn't be supported in cases 1 or 2.

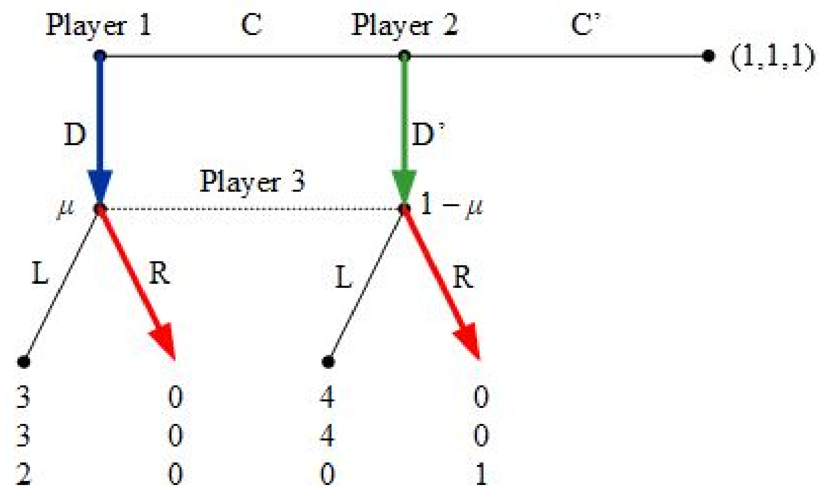

Figure 9b. Pooling Strategy Profile D, D' when $\mu \leq \frac{1}{3}$.

- In sum, $\{C, C', R\}$ is the unique PBE of this game, which can be sustained when Player 3's beliefs satisfy $\mu \leq \frac{1}{3}$.

2. **Signaling and Limit pricing.** Consider a market with inverse demand function $p(Q) = 1 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. Let us analyze an entry game with an incumbent monopolist (Firm 1) and an entrant (Firm 2) who analyzes whether or not to join the market. The incumbent's marginal costs are either high H or low L, i.e., $c_1^H = \frac{1}{2} > c_1^L = \frac{1}{3}$, while it is common knowlegde that the entrant's marginal costs are high, i.e., $c_2 = \frac{1}{2}$. To make the entry decision interesting, assume that when the incumbent's costs are low, entry is unprofitable; whereas when the incumbent's costs are high, entry is profitable. (Otherwise, the entrant would enter regardless of the incumbent's cost, or stay out regardless of the incumbent's cost.) For simplicity, assume no discounting of future payoffs throughout all the exercise.

   (a) *Complete information.* Let us first examine the case in which entrant and incumbent are informed about each others' marginal costs. Consider a two-stage game where, in the first stage, the incumbent has monopoly power and selects an output level, q. In the second stage, a potential entrant decides whether or not to enter. If entry occurs, agents compete as Cournot duopolists, simultaneously and independently selecting production levels, x_1 and x_2. If entry does not occur, the incumbent maintains its monopoly power during both periods (producing q in the first period and x in the second period). Find the subgame perfect equilibrium (SPNE) of this complete information game.

   - We next apply backward induction, starting from the second-period game.
   - *Second period.* When no entry occurs, the incumbent solves

$$\max_{x_1} \, (1 - x_1)x_1 - c_1^K x_1$$

thus selecting monopoly output $x_1^{K,m} = \frac{1-c_1^K}{2}$ for every incumbent type $K = \{H, L\}$. If entry occurs, every firm $i = \{1, 2\}$ solves

$$\max_{x_i} \ (1 - x_i - x_j)x_i - c_i^K x_i$$

which, after finding best response functions and simultaneously solving for incumbent and entrant's outputs, yields equilibrium output $x_1^{K,d} = \frac{1+c_2-2c_1^K}{3}$ for the incumbent and $x_2^{K,d} = \frac{1-2c_2+c_1^K}{3}$ for the entrant.

- *First period.* Regardless of the entrant's entry decision during the second period, the incumbent selects the standard monopoly output $q^{K,Info} = \frac{1-c_1^K}{2}$ in the first period. This is because the incumbent's output choice in this complete information setting does not affect the entrant's entry decision.

(b) *Incomplete information.* In this section we investigate the case where the incumbent is privately informed about its marginal costs, while the entrant only observes the incumbent's first-period output which the entrant uses as a signal to infer the incumbent's cost. The time structure of this signaling game is as follows:

1. Nature decides the realization of the incumbent's marginal costs, either high or low, with probabilities $p \in (0, 1)$ and $1 - p$, respectively. The incumbent privately observes this realization but the entrant does not.
2. The incumbent chooses its first-period output level, q.
3. Observing the incumbent's output decision, the entrant forms beliefs about the incumbent's initial marginal costs. Let $\mu(c_1^H|q)$ denote the entrant's posterior belief about the initial costs being high after observing a particular first-period output from the incumbent q.
4. Given the above beliefs, the entrant decides whether or not to enter the industry.
5. If entry does not occur, the incumbent maintains its monopoly power; whereas if entry occurs, both agents compete as Cournot duopolists and the entrant observes the incumbent's type.

Write down the incentive compatibility conditions that must hold for a separating Perfect Bayesian Equilibrium (PBE) to be sustained. Then find the set of separating PBEs.

- In a separating equilibrium in which the high-cost firm selects q^H while the low-cost firm chooses q^L information about the incumbent's type is conveyed to the potential entrant, who responds entering after observing the incumbent producing q^H, and does not enter after observing q^L. For simplicity, we assume that all other output levels $q \neq q^H \neq q^L$ (i.e., off-the-equilibrium outputs) also lead the entrant to enter the industry. Let us next separately analyze each type of incumbent.
- *High-cost incumbent.* Since, by selecting q^H this type of incumbent attracts entry, this firm selects the output that maximizes its first-period (monopoly) profits, that is, q^H coincides with its output under complete information

$q^{H,Info} = \frac{1-c_1^H}{2}$. If, instead, the incumbent deviates towards the low-cost incumbent's output q^L, it conceals its type from the entrant and deters entry. Hence, the high-cost incumbent selects its equilibrium output q^H rather than deviating if $M_1^H(q^{H,Info}) + \delta D_1^H \geq M_1^H(q^L) + \delta \overline{M}_1^H$, where

$$M_1^H(q) = (1-q)q - c^H q \quad \text{for every output } q$$

denotes the incumbent's first-period monopoly profits, D_1^H represents second-period duopoly profits when the incumbent's costs are high, and $\overline{M}_1^H$ indicates the second-period monopoly profits for the incumbent (in the case of no entry) when its costs are high. We can now rewrite the above incentive compatibility condition as follows

$$M_1^H(q^{H,Info}) - M_1^H(q^L) \geq \delta \left[\overline{M}_1^H - D_1^H\right] \qquad (IC_H)$$

(where we grouped first-period profits on the left-hand side, and discounted second-period profits on the right-hand side). For our parameter values, we obtain profits of $M_1^H(q^{H,Info}) = \overline{M}_1^H = \frac{1}{16}$ since $c_1^H = 1/2$, and $D_1^H = \frac{1}{36}$ given that $c_1^H = c_2 = 1/2$. Hence, condition IC_H reduces to

$$\frac{1}{16} - \left[(1-q^L)q^L - \frac{1}{2}q^L\right] \geq \delta\left[\frac{1}{16} - \frac{1}{36}\right]$$

The difference in first-period profits, $M_1^H(q^{H,Info}) - M_1^H(q^L)$, becomes zero at $q^L = q^{H,Info}$ since at that point $M_1^H(q^{H,Info}) = M_1^H(q^L)$, but otherwise is positive since $M_1^H(q^{H,Info}) > M_1^H(q^L)$ for all $q^L \neq q^{H,Info}$. In contrast, the difference in discounted second-period profits, $\delta\left[\overline{M}_1^H - D_1^H\right]$, is constant in first-period output q^L. Hence, IC_H holds if output q^L lies in the range depicted in the horizontal axis of figure 2.

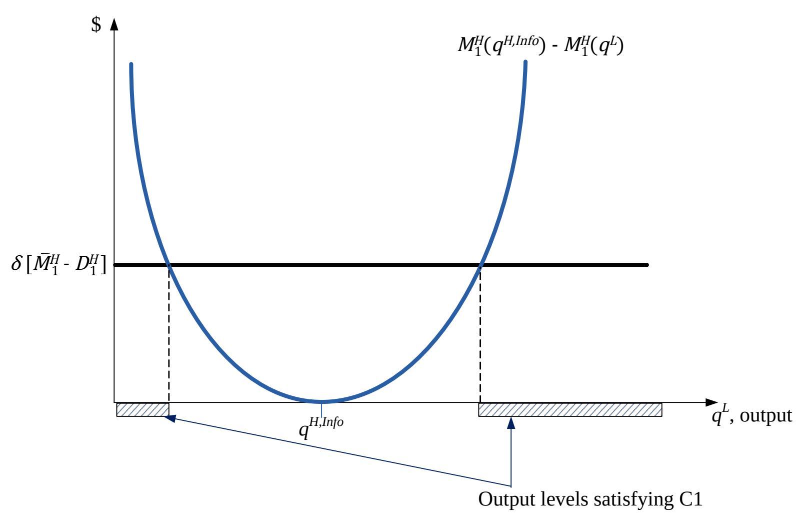

Fig 2. Incentive compatibility condition IC_H.

- *Low-cost incumbent.* If the low-cost incumbent chooses the equilibrium output q^L, it deters entry. If instead the incumbent deviates towards the high-cost incumbent's output, q^H, it attracts entry. Conditional on attracting entry, the low-cost incumbent would select output $q^{L,Info}$, since such output maximizes its first-period profits, yielding $M_1^L(q^{L,Info}) + \delta D_1^L$. Thus, the low-cost incumbent selects its equilibrium output of q^L if $M_1^L(q^{L,Info}) + \delta D_1^L \leq M_1^L(q^L) + \delta \overline{M}_1^L$, or equivalently,

$$M_1^L(q^{L,Info}) - M_1^L(q^L) \leq \delta \left[\overline{M}_1^L - D_1^L \right] \qquad (IC_L)$$

which, for our parameter values, yields $M_1^L(q^{L,Info}) = \overline{M}_1^L = 1/9$ and $D_1^L = \frac{25}{324}$ given that $c_1^L = 1/3$ and $c_2 = 1/2$. Hence, condition IC_L reduces to

$$\frac{1}{9} - \left[(1 - q^L)q^L - \frac{1}{3}q^L \right] \leq \delta \left[\frac{1}{9} - \frac{25}{324} \right]$$

A similar argument as for IC_H applies to the graphical representation of IC_L. As figure 3 illustrates, the curve depicting the difference in first-period profits, $M_1^L(q^{L,Info}) - M_1^L(q^L)$, becomes zero at $q^L = q^{L,Info}$ since at that point $M_1^L(q^{L,Info}) = M_1^L(q^L)$, but otherwise is positive since $M_1^L(q^{L,Info}) > M_1^L(q^L)$ for all $q^L \neq q^{L,Info}$. In contrast, the difference in discounted second-period profits, $\delta \left[\overline{M}_1^L - D_1^L \right]$, is constant in first-period output q^L. Hence, IC_L holds if output q^L lies in the range depicted in the horizontal axis of figure 3.

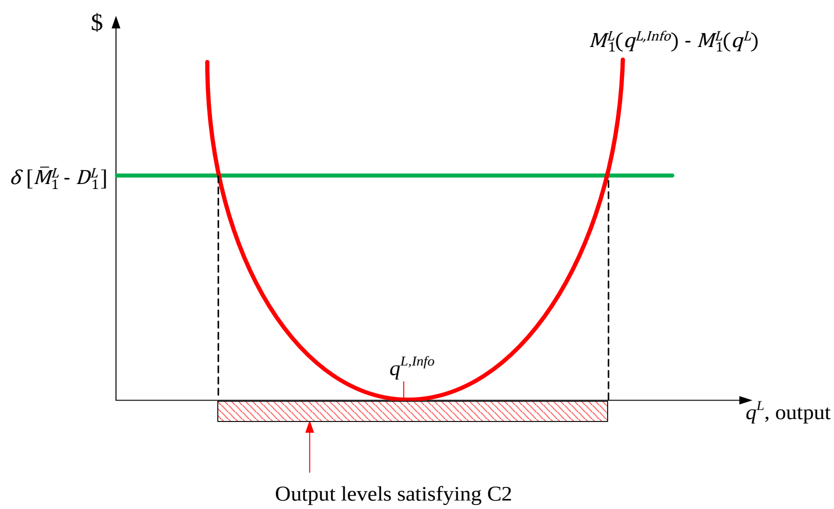

Fig 3. Incentive compatibility condition IC_L.

- *Combining both ICs.* Superimposing figures 2 and 3, we can examine the set of output levels that simultaneously satisfy condition IC_H and IC_L, as depicted in figure 4. In particular, the overlap between the range of outputs identified in figures 2 and 3 provides us with the set of output levels that constitute a separating PBE, $q^L \in [q^A, q^B]$. The low-cost incumbent increases its first-period output in order to communicate its efficient costs to the potential entrant, deterring entry as a result.

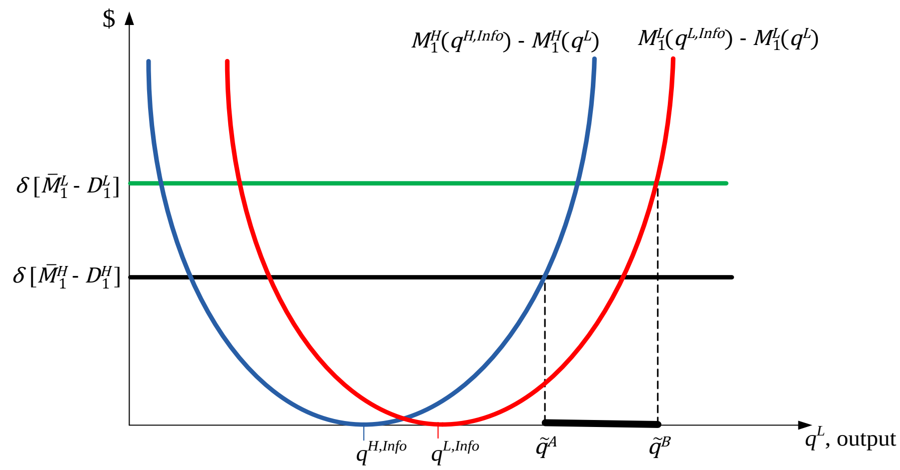

Fig 4. Separating equilibria in the limit pricing model.

In particular, the lower-bound output q^A solves condition IC_H with equality, and the upper-bound output q^B solves IC_L with equality. Rearranging condition IC_H, and assuming that there is no discounting, $\delta = 1$, we obtain

$$1 - 1 + \frac{1}{36} = (1 - q^L)q^L - \frac{1}{2}q^L$$

or

$$36\left(q^L\right)^2 - 18q^L + 1 = 0$$

and solving for output q^L yields two roots for the lower bound q^A, $q^A = 0.06$ and $q^A = 0.43$. Similarly operating with condition IC_L in order to obtain the upper bound q^B, and since there is no discounting, $\delta = 1$, IC_L simplifies to

$$324\left(q^L\right)^2 - 216q^L + 25 = 0.$$

Solving for output q^L yields two roots for the upper bound q^B, $q^B = 0.14$ and $q^B = 0.51$. Hence, the set of separating output levels for the low-cost firm must lie on the interval $q^L \in [0.43, 0.51]$.

(c) Which separating PBEs of those you found in part (b) survive the Cho and Kreps' Intuitive Criterion?

- Starting from the separating PBE in which the low-cost incumbent chooses the highest output level $q^L = q^B$, a deviation toward any output level in $q^L \in [q^A, q^B)$ can only be profitable for the low-cost incumbent (but not for the high-cost firm). Formally, deviating towards $q^L \in [q^A, q^B)$ is "equilibrium dominated" for the high-cost incumbent alone. Hence, the potential entrant would update its beliefs accordingly, making such a deviation profitable for the low-cost firm. A similar argument applies to all other separating PBEs in the interval $q^L \in (q^A, q^B)$ but not for $q^L = q^A$, the least-costly separating PBE (also known as the "Riley outcome").

- Summarizing, the low-cost incumbent raises its first-period output from $q^{L,Info} = \frac{1-c_1^L}{2} = \frac{1-\frac{1}{3}}{2} = 0.33$, under complete information, to $q_1^A = 0.43$, under the separating equilibrium. Hence, the "separating effort" that this firm must exert in order to reveal its type to the potential entrant (and thus deter entry) is measured by the distance $q^A - q^{L,Info} = 0.43 - 0.33 = 0.10$.

3. **Contracting with input providers.** Consider a firm contracting inputs from a company that produces two types of inputs, A with probability α and B with probability $1-\alpha$. Intuitively, one can interpret these two types as if the firm purchases inputs from a company without observing the input's quality, but knowing that the frequency of A-types is α. The firm's profits from a known quality $i = \{A, B\}$ are $10q - p$, where q denotes the input units purchased and p represents the (lump-sum) price that the firm pays for the inputs.

   A company producing input i has total costs $c_i(q) = \gamma^i q^2$ from producing q units, which is increasing and convex in output, and where $\gamma^B < \gamma^A$. Therefore, input provider i earns profits $p - \gamma^i q^2$. For simplicity, assume that its reservation profit from rejecting a contract is zero.

   (a) *Symmetric information.* As a benchmark, let us first solve the principal's problem (firm buying inputs) when it can perfectly observe the input type i. Find the contract $\left(q^{SI}, p^{SI}\right)$, specifying the number of units ordered from input provider i and the lump-sum price that the principal pays to the input provider.

   - The principal observes the type of input provider it faces, and solves the following profit maximization problem

     $$\max_{q,p} \; 10q - p$$

     subject to

     $$p - \gamma q^2 \geq 0 \tag{PC}$$

     The participation constraint, PC, represents that the input provider is better off accepting the contract than rejecting it. This constraint must hold with equality. Otherwise, the principal could reduce price p and still induce participation. Therefore, we must have that $p = \gamma^i q^2$. Inserting this result into the principal's objective function, we obtain

     $$\max_{q} \; 10q - \underbrace{\gamma^i q^2}_{p}.$$

     Differentiating with respect to q, yields

     $$10 - 2\gamma^i q = 0$$

     and, rearranging, we find that the principals orders

     $$q^{SI} = \frac{5}{\gamma^i} \text{ units}$$

     from input provider i. Since $\gamma^B < \gamma^A$ by assumption, the principal order fewer units from the high-quality input provider (company A) than from the low-quality one (company B).

- *Price.* Therefore, equilibrium price is

$$p^{SI} = \gamma^i \left(\frac{5}{\gamma^i}\right)^2 = \frac{25}{\gamma^i}.$$

Since $\gamma^B < \gamma^A$ by assumption, the principal pays a lower lump-sum price, p^{SI}, when ordering units from the high-quality input provider (company A) than from the low-quality one (company B).

- *Profits.* In this symmetric information context, the input provider makes no profit, but the principal (firm ordering inputs) earns

$$\pi^{SI} = 10q^{SI} - p^{SI} = 10\frac{5}{\gamma^i} - \frac{25}{\gamma^i} = \frac{25}{\gamma^i}.$$

As above, since $\gamma^B < \gamma^A$, the principal earns a lower profit from the high-quality input provider (company A) than from the low-quality one (company B).

(b) *Asymmetric information.* Assume now that the principal cannot observe the input provider's type. Find the optimal contract pair, $\left(q^A, p^A\right)$ and $\left(q^B, p^B\right)$, in this context.

- The principal solves the following expected profit maximization problem

$$\max_{q^A, p^A, q^B, p^B} \quad \alpha\left(10q^A - p^A\right) + (1-\alpha)(10q^B - p^B)$$

subject to

$$p^A - \gamma^A(q^A)^2 \geq 0 \qquad (\text{PC}_A)$$

$$p^B - \gamma^B(q^B)^2 \geq 0 \qquad (\text{PC}_B)$$

$$p^A - \gamma^A(q^A)^2 \geq p^B - \gamma^A(q^B)^2 \qquad (\text{IC}_A)$$

$$p^B - \gamma^B(q^B)^2 \geq p^A - \gamma^B(q^A)^2 \qquad (\text{IC}_B)$$

The PC constraints, as usual, indicate that the input provider prefers to accept the contract offered by the principal than rejecting it (and earn a zero profit), which holds for every input provider i. The incentive compatibility conditions, IC, mean that each input provider i prefers its contract, (q^i, p^i), than that meant for the other type of input provider, (q^j, p^j), where $j \neq i$. As a remark, note that the input provider's type is fixed in both sides of every IC condition, i.e., γ^A in IC_A and γ^B in IC_B.

- If IC_B and PC_A hold, we have that

$$p^B - \gamma^B(q^B)^2 \underbrace{\geq}_{\text{From IC}_B} p^A - \gamma^B(q^A)^2 \underbrace{>}_{\text{From }\gamma^B<\gamma^A} p^A - \gamma^A(q^A)^2 \underbrace{\geq}_{\text{From PC}_A} 0$$

which, taking the first and last term, implies $p^B - \gamma^B(q^B)^2 > 0$. Therefore, PC_B must also hold, meaning that we can ignore it in our subsequent analysis.

The Lagrangean of the above problem is

$$\begin{aligned} L &= \alpha\left(10q^A - p^A\right) + (1-\alpha)(10q^B - p^B) \\ &\quad +\lambda\left[p^A - \gamma^A(q^A)^2\right] \\ &\quad +\mu_A\left[p^A - \gamma^A(q^A)^2 - p^B + \gamma^A(q^B)^2\right] \\ &\quad +\mu_B\left[p^B - \gamma^B(q^B)^2 - p^A + \gamma^B(q^A)^2\right]. \end{aligned}$$

Differentiating with respect to p_A and p_B, yields

$$\begin{aligned} \frac{\partial L}{\partial p_A} &= -\alpha + \lambda + \mu_A - \mu_B = 0, \text{ and} \\ \frac{\partial L}{\partial p_B} &= -(1-\alpha) - \mu_A + \mu_B = 0. \end{aligned}$$

Summing these two first-order conditions, we obtain

$$[-\alpha + \lambda + \mu_A - \mu_B] + [-(1-\alpha) - \mu_A + \mu_B] = \lambda - 1 = 0$$

or $\lambda = 1$, which implies that PC_A must hold with strict equality, $p^A - \gamma^A(q^A)^2 = 0$.

- Differentiating with respect to q_A and q_B, we obtain

$$\begin{aligned} \frac{\partial L}{\partial q_A} &= 10\alpha - 2q^A\left(\lambda\gamma^A + \gamma^A\mu_A - \gamma^B\mu_B\right) = 0, \text{ and} \\ \frac{\partial L}{\partial q_B} &= 10(1-\alpha) + 2q^B\left(\gamma^A\mu_A - \gamma^B\mu_B\right) = 0. \end{aligned}$$

At this point, we recall that $\lambda = 1$, inserting it in the above first-order conditions. In addition, we can consider that IC_A holds with strict inequality, implying that its Lagrange multiplier is $\mu_A = 0$. (We confirm this property below.) Inserting $\mu_A = 0$ in the above first-order conditions, yields

$$\begin{aligned} \frac{\partial L}{\partial p_A} &= 1 - \alpha - \mu_B = 0, \\ \frac{\partial L}{\partial p_B} &= -(1-\alpha) + \mu_B = 0, \\ \frac{\partial L}{\partial q_A} &= 10\alpha - 2q^A\left(\gamma^A - \gamma^B\mu_B\right) = 0, \text{ and} \\ \frac{\partial L}{\partial q_B} &= 10(1-\alpha) - 2q^B\gamma^B\mu_B = 0. \end{aligned}$$

From $\frac{\partial L}{\partial p_A}$, we find that $\mu_B = 1 - \alpha$. Inserting this result in $\frac{\partial L}{\partial q_A}$, yields

$$\frac{\partial L}{\partial q_A} = 10\alpha - 2q^A\left(\gamma^A - \gamma^B(1-\alpha)\right) = 0$$

which, solving for q^A, yields

$$q^A = \frac{5\alpha}{\gamma^A - \gamma^B(1-\alpha)}.$$

Inserting $\mu_B = 1-\alpha$ into $\frac{\partial L}{\partial q_B}$, we find

$$\begin{aligned}\frac{\partial L}{\partial q_B} &= 10(1-\alpha) - 2q^B\gamma^B(1-\alpha) = 0 \\ &= 10 - 2q^B\gamma^B = 0\end{aligned}$$

which, solving for q^B, yields

$$q^B = \frac{5}{\gamma^B}.$$

Finally, we know that PC$_A$ holds with strict equality, $p^A - \gamma^A(q^A)^2 = 0$, or $p^A = \gamma^A(q^A)^2$, which helps us identify optimal price p^A, that is,

$$p^A = \gamma^A \left(\frac{5\alpha}{\gamma^A - \gamma^B(1-\alpha)}\right)^2.$$

In addition, since IC$_B$ holds with strict equality, $p^B - \gamma^B(q^B)^2 = p^A - \gamma^B(q^A)^2$, which entails

$$\begin{aligned}p^B &= \gamma^B(q^B)^2 - p^A + \gamma^B(q^A)^2 \\ &= \gamma^B(q^B)^2 - \underbrace{\gamma^A(q^A)^2}_{p^A} + \gamma^B(q^A)^2 \\ &= \gamma^B(q^B)^2 - (\gamma^A - \gamma^B)(q^A)^2 \\ &= \gamma^B\left(\frac{5}{\gamma^B}\right)^2 - (\gamma^A-\gamma^B)\left(\frac{5\alpha}{\gamma^A - \gamma^B(1-\alpha)}\right)^2.\end{aligned}$$

- *Constraint IC$_A$ holds with strict inequality.* We must now check that $p^A - \gamma^A(q^A)^2 > p^B - \gamma^A(q^B)^2$. Inserting our equilibrium results, we obtain that

$$\underbrace{\gamma^A(q^A)^2}_{p^A} - \gamma^A(q^A)^2 > \underbrace{\left[\gamma^B(q^B)^2 - (\gamma^A-\gamma^B)(q^A)^2\right]}_{p^B} - \gamma^A(q^B)^2$$

which simplifies to

$$\begin{aligned}0 &> \left(\gamma^B - \gamma^A\right)\left[(q^B)^2 + (q^A)^2\right] \\ &= \left(\gamma^B - \gamma^A\right)\left[\left(\frac{5}{\gamma^B}\right)^2 + \left(\frac{5\alpha}{\gamma^A - \gamma^B(1-\alpha)}\right)^2\right]\end{aligned}$$

which holds, since $\gamma^B < \gamma^A$ by assumption, and the term in brackets is positive. Therefore, IC$_A$ holds with strict inequality.

(c) Compare your equilibrium results under symmetric and asymmetric information.

- *Equilibrium output.* The equilibrium quantity under symmetric information, as found in part (a), is $q^{SI} = \frac{5}{\gamma^i}$. The equilibrium quantity under asymmetric information, as found in part (b), coincides with that under symmetric information for the low-quality input provider (company B), $q^B = \frac{5}{\gamma^B}$, but is larger than under symmetric information for the high-quality input provider (company A) since

$$q^A = \frac{5\alpha}{\gamma^A - \gamma^B(1-\alpha)} > \frac{5}{\gamma^A} = q^{SI}$$

given that $\gamma^A - \gamma^B(1-\alpha) < \gamma^A$. We can now compare equilibrium prices.

- *Equilibrium price.* Under symmetric information, equilibrium prices are $p^{SI} = \gamma^i(q^{SI})^2$ for every input provider i, which is lower than the price paid to this input provider under asymmetric information, where $p^A = \gamma^A(q^A)^2$, but $q^A > q^{SI}$, as shown above. For company B, we found that

$$p^B = \gamma^B(q^B)^2 - (\gamma^A - \gamma^B)(q^A)^2$$

in part (b). Therefore, despite output levels coincide across information settings for this firm, $q^B = q^{SI}$, it receives a lower price under asymmetric than under symmetric information since

$$\gamma^B(q^B)^2 - (\gamma^A - \gamma^B)(q^A)^2 < \gamma^B(q^{SI})^2.$$