EconS 424 – Strategy and Game Theory
Midterm #2 – Answer Key

Exercise #1 – Simultaneous public good game

a. Parameter α measures the degree of publicness of the good. We can think about some extreme cases:

- When α=0 individual j does not benefit from the units of y_1 that individual i acquires (so y_1 becomes a private good, since its provision is only enjoyed by individual i).
- In contrast, when α=1 individual j benefits from every unit of y_1 that individual i acquires. In this case the good becomes completely public, as every unit of it is enjoyed by both individuals i and j.
- For intermediate values of α, only a share of every dollar spent on y_1 by individual i is also benefited by individual j.

b. Using the budget constraint (and assuming both goods have unit price) obtains

$$U^1 = \log(y_1^1 + \alpha y_1^2) + M - y_1^1$$

Taking first order conditions with respect to $y_1{}^1$, we obtain

$$\frac{1}{y_1^1 + \alpha y_1^2} - 1 = 0.$$

Solving for y_1^1 yields $y_1^1 = 1 - \alpha y_1^2$. This is the best response function of individual 1, telling us that when individual 2 does not purchase units of the y_1^2, individual 1 responds with $y_1^1 = 1$. When individual 2's purchases are positive, individual 1's purchases decrease at a rate α. Graphically, when the good is completely public, α=1, player 1's best response function has a slope of -1, but when the good is not-so-public, α<1, player 1's best response function becomes flatter.
That is, as the good becomes more private, player 1's best response function experiences an outward pivoting effect (with center at 1), indicating that player 1 does not see his own contribution to the public good as substitutable as that of player 2. In contrast, when the good is more public, player 1's best response function experiences an inward pivoting effect (with center at 1), suggesting that player 1 sees his contribution to the public good as strategically substitutable by that of player 2 (as player 1 can now free ride a larger share of every unit contributed by player 2).
The game is symmetric. So the solution is

$$y_1^1 = y_1^2 = y_1 = \frac{1}{1+\alpha}$$

Hence the consumption level in equilibrium is

$$x_1^1 = x_1^2 = x_1 = [1+\alpha]\, y_1 = 1$$

c. The level of social welfare is

$$W = \log(y_1^1 + \alpha y_1^2) + M - y_1^1 + \log(y_1^2 + \alpha y_1^2) + M - y_1^2$$

Applying symmetry obtains

$$W = 2\log([1+\alpha]\, y_1) + 2[M - y_1]$$

So

$$\frac{\partial W}{\partial y_1} = \frac{2}{y_1} - 2 = 0$$

Hence $y_1 = 1$ and $x_1 = 1+\alpha$. The two outcomes are the same if $\alpha = 0$.

d. Utility now becomes

$$U^1 = \log(y_1^1 + \alpha y_1^2) + M - y_1^1 + \alpha(M - y_1^2) .$$

The Nash equilibrium remains at $y_1^1 = y_1^2 = y_1 = \frac{1}{1+\alpha}$ with symmetry the level of welfare is

$$W = 2\log((1+\alpha)\,y_1) + 2(1+\alpha)(M - y_1), \text{ so}$$

$y_1 = \frac{1}{1+\alpha}$. The two outcomes are identical for all α.

e. In part (b) there is one private good and one public good when $\alpha \neq 0$. So free riding takes place when $\alpha \neq 0$. With $\alpha = 0$, there are two private goods, so the outcome is efficient. In part *d* both goods have an identical degree of publicness so the consumption externalities are balanced. It is not possible to free-ride on both goods, so efficiency results.

Exercise #2

Solving the game by backward induction, we first analyze the output decision of the last mover (firm 3).

Firm 3 (last stage). This firm maximizes the following profits:

$$\pi_3 = (18 - q_1 - q_2 - q_3)q_3 - (5 + 2q_3)$$

Differentiating with respect to q_3, we obtain

$$18 - q_1 - q_2 - 2q_3 - 2 = 0$$

So, firm 3's BRF is

$$q_3(q_1, q_2) = 8 - \frac{q_1 + q_2}{2}$$

Firm 2. Firm 2 anticipates firm 3's BRF. Inserting firm 3's BRF into firm 2's maximization problem yields,

$$\max_{q_2 \geq 0} \left[18 - q_1 - q_2 - \left(8 - \frac{q_1 + q_2}{2}\right)\right] q_2 - (5 + 2q_2)$$

Differentiating with respect to q_2, we obtain

$$18 - q_1 - 2q_2 - 8 + \frac{1}{2}(q_1 + 2q_2) - 2 = 0$$

Therefore, solving for q_2, we find that firm 2's BRF is

$$q_2(q_1) = 8 - \frac{q_1}{2}$$

Firm 1 (first mover). Inserting firm 3's and firm 2'as BRFs into firm 1's maximization problem yields,

$$\max_{q_1 \geq 0} \left[18 - q_1 - \left(8 - \frac{q_1}{2}\right) - \left(8 - \frac{q_1 + \left(8 - \frac{q_1}{2}\right)}{2}\right)\right] q_1 - (5 + 2q_1)$$

Differentiating with respect to q_1, we obtain

$$18 - 2q_1 - 8 + \frac{1}{2}2q_1 - 8 + \frac{1}{2}\left(2q_1 + 8 - \frac{1}{2}2q_1\right) - 2 = 0$$

Rearranging and solving for q1, we find firm 1's equilibrium output $q_1 = 8$ units. Therefore, in equilibrium, firm 2 produces $q_2(8) = 8 - \frac{8}{2} = 4$ units, and firm 3 produces $q_3(8{,}4) = 8 - \frac{8+4}{2} = 8 - 6 = 2$ units.

Exercise #3 – Cournot competition under incomplete information about market demand

a) Firm 1 (informed firm):

If *high* demand:

$$\pi_1^H = (20 - q_1^H - q_2) * q_1^H - 2q_1^H$$

Taking FOCs with respect to q_1^H,

$$20 - 2q_1^H - q_2 - 2 = 0 \ \rightarrow 18 - q_2 = 2q_1^H$$

Hence, solving for q_1^H, we obtain firm 1's best response function when facing a high demand

$$BRF_1^H \rightarrow q_1^H(q_2) = 9 - \frac{1}{2}q_2$$

If *low* demand:

$$\pi_1^L = (8 - q_1^L - q_2)q_1^L - 2q_1^L$$

Taking FOCs with respect to q_1^L,

$$8 - 2q_1^L - q_2 - 2 = 0 \rightarrow 6 - q_2 = 2q_1^L$$

Hence, solving for q_1^L, we obtain firm 1's best response function when facing a low demand

$$BRF_1^L \rightarrow q_1^L(q_2) = 3 - \frac{1}{2}q_2$$

b) Firm 2 (uninformed firm): Expected profit for Firm 2

$$E\pi_2 = \tfrac{2}{3}\left((20 - q_1^H - q_2)q_2 - 2q_2\right) + \tfrac{1}{3}\left((8 - q_1^L - q_2)q_2 - 2q_2\right)$$

$$= \frac{2}{3}(20q_2 - q_1^H q_2 - q_2^2) + \frac{1}{3}(8q_2 - q_1^L q_2 - q_2^2) - 2q_2$$

Taking FOCs with respect to q_2,

$$\frac{40}{3} - \frac{2}{3}q_1^H - \frac{4}{3}q_2 + \frac{8}{3} - \frac{1}{3}q_1^L - \frac{2}{3}q_2 - 2 = 0$$

And solving for q_2, we obtain firm 2's best response function.

$$BRF_2 \rightarrow q_2(q_1^H, q_1^L) = 7 - \frac{2}{6}q_1^H - \frac{1}{6}q_1^L$$

c) Plugging q_1^H and q_1^L into firm 2's best response function, we find

$$q_2 = 7 - \frac{1}{3}\left[9 - \frac{1}{2}q_2\right] - \frac{1}{6}\left[3 - \frac{1}{2}q_2\right]$$

and solving for q_2, we obtain firm 2's equilibrium output level, $q_2 = 4.66$.

We can now find firm 1's equilibrium output level. First, plugging q_2=4.66 into $q_1^H = 9 - \frac{1}{2}q_2$, we obtain:

$$q_1^H = 9 - \frac{1}{2}(4.66) = 6.7$$

Similarly, plugging q_2=4.66 into $q_1^L = 3 - \frac{1}{2}q_2$,we obtain.

$$q_1^L = 3 - \frac{1}{2}(4.66) = 0.66$$

Summarizing, the BNE of this incomplete information Cournot game is:

$$(q_1^H, q_1^L, q_2) = (6.7, 0.66, 4.66)$$

Exercise #4

a) First, we list all the strategies for both the worker and firm. For the worker, his strategy set is

$$\{(NE^H, NE^L), (NE^H, E^L), (E^H, NE^L), (E^H, E^L)\}$$

where the first element of each strategy regards to that when his type is *High productivity*, and the second element regards to *Low Productivity*.

The strategy set for the firm is

{(M', M), (M', C), (C', M), (C', C)}

where the first element in each strategy regards to *No Education* is observed, and the second regards to *Education.*

b) Then we draw and fill in the following payoff matrix with players' expected payoffs: Each cell in the payoff matrix is calculated by taking the expected payoff of each strategy combination. For example, in the cell where the worker chooses strategy (E^H, NE^L) and the firm chooses strategy (M', C), we can calculate the payoff for the worker by taking the linear combination of M' in the lower left-hand corner and C in the upper-right hand corner of the figure (Since they correspond with (E^H, NE^L) and (M', C)). The calculation for the worker is

$$EU_W = \frac{2}{3}(10) + \frac{1}{3}(0) = \frac{20}{3}$$

and that for the firm as

$$EU_F = \frac{2}{3}(0) + \frac{1}{3}(4) = \frac{4}{3}$$

Firm

Worker

	M', M	M', C	C', M	C', C
NE^H, NE^L	<u>10</u>, 10/3	<u>10</u>, 10/3	4, <u>4</u>	<u>4</u>, <u>4</u>
NE^H, E^L	16/3, 10/3	4/3, <u>6</u>	10/3, 4/3	-2/3, 4
E^H, NE^L	26/3, 10/3	20/3, 4/3	<u>14/3</u>, <u>6</u>	8/3, 4
E^H, E^L	4, 10/3	-2, <u>4</u>	4, 10/3	-2, <u>4</u>

c) As we can see from the above payoff matrix, there are two BNEs in this game:

- (*No Education if High Productivity, No Education if Low Productivity*), (*Cashier if No Education, Cashier if Education*)
- (*Education if High Productivity, No Education if Low Productivity*), (*Cashier if No Education, Manager if Education*)

d) Both of these BNEs are also the two PBEs we found in class. The first one corresponds to the pooling strategy profile (NE^H, NE^L) where it is PBE if off-the-equilibrium beliefs satisfy $\mu > \frac{2}{5}$. The second BNE corresponds to the separating strategy profile (E^H, NE^L). Therefore, in this case the BNE and PBE solution concepts yield the exact same equilibrium predictions.

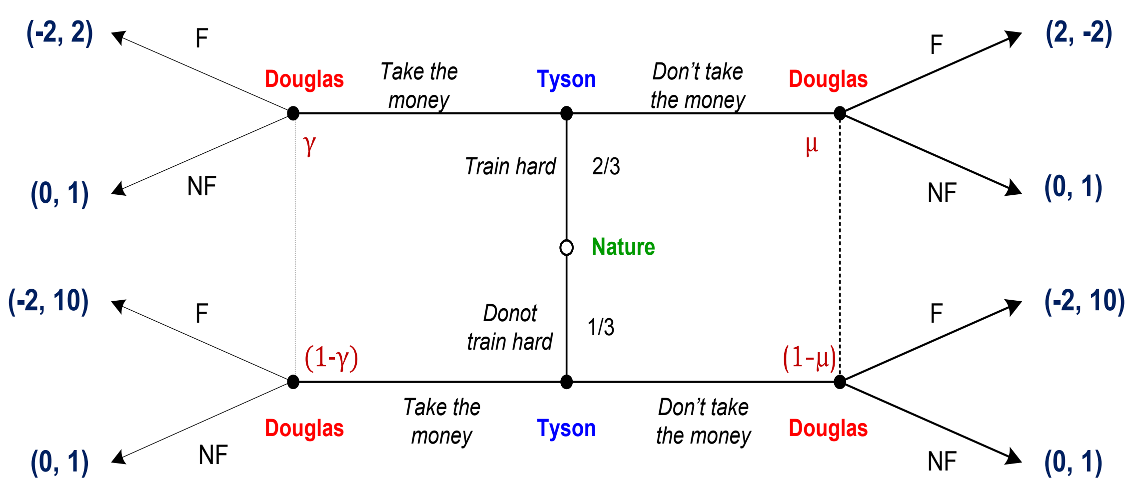
Exercise #5 – *Mike Tyson vs. Buster Douglas*

Consider the following sequential move game with incomplete information. The first player to move is Mike Tyson, who privately knows whether he trained hard, or he didn't. Let us assume that the type of training that Mike Tyson receives before a fight is not something he can strategically decide, but instead, it depends on his state of mind between the time he signed up the contract for a fight and the time of the fight.

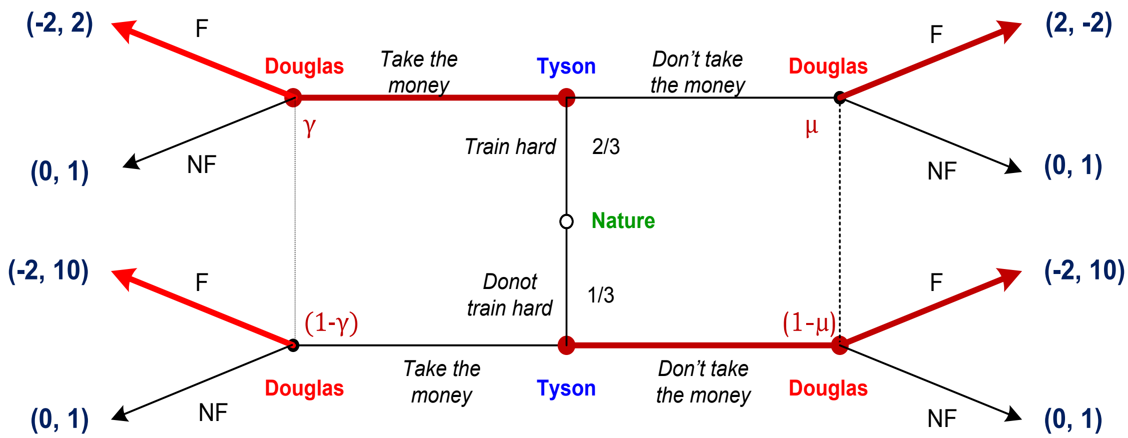
In particular, we will assume that the probability that Mike Tyson trains hard is given by Nature, and it is 2/3, as can be seen in the figure.

Knowing what kind of physical training he had, Mike Tyson decides whether to offer player 2 (Buster Douglas) $1 million dollars if he gives up his right to fight Mike Tyson. We will denote Mike Tyson's strategies as "Take the money" (that is, offering the bribe) or "Don't take the money" (don't offering any bribe to Douglas).

After observing whether Mike Tyson has offered him any money, Buster Douglas must decide whether to Fight (F) or Not Fight (NF), without knowing whether Mike Tyson has previously trained hard or not.



a) Show that there is no Separating PBE where Mike Tyson makes the offer "Take the money" when he has trained hard, but does not make such an offer (he chooses "Don't take the money") when he has not trained hard. To show this, follow the usual steps for finding PBE.



1. *Find Buster Douglas' beliefs in this Separating PBE (use Bayes' rule).*

   After observing the offer "Take the money" from Mike Tyson, Buster Douglas' beliefs are

$$\gamma = \frac{0.6 \times 1}{0.6 \times 1 + 0.4 \times 0} = 1 \quad \rightarrow \quad \gamma = 1$$

   - Graphically, Buster Douglas believes that if he observes "Take the money," he must be in the node at the upper left-hand corner of the game tree.

   After observing "Don't take the money" from Mike Tyson, Buster Douglas' beliefs are

$$\mu = \frac{0.6 \times 0}{0.6 \times 0 + 0.4 \times 1} = 0 \quad \rightarrow \quad \mu = 0$$

   - Graphically, Buster Douglas believes that if he observes "Don't take the money," he must be in the node at the lower right-hand corner of the game tree.

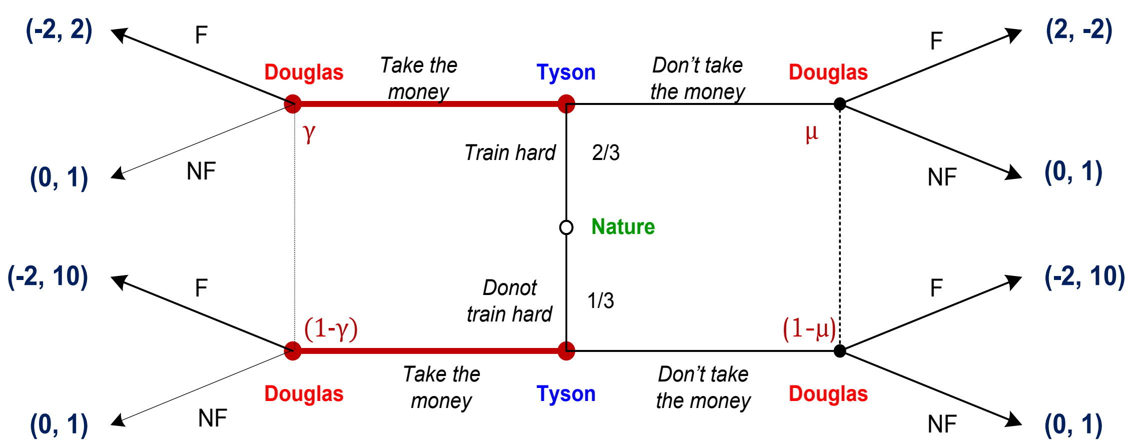
2. *Find Buster Douglas' optimal action (whether to Fight or Not Fight) after observing that Mike Tyson offers him "Take the money". In addition, find Buster Douglas' optimal action (whether to Fight or Not Fight) after observing that Mike Tyson does not offers him any bribe (Douglas observes the action "Don't take the money").*

   - After observing that Mike Tyson offers "Take the money," Buster Douglas responds with F, since he believes to be in the node at the upper left-hand corner of the game and 2>1.
   - After observing "Don't take the money," Buster Douglas responds with F, since he believes to be in the node at the lower right-hand corner of the game and 10>1.

3. *Find Mike Tyson's optimal action when he has trained hard, and when he has not trained hard.*
   - If he trained hard, Mike Tyson prefers to deviate towards "Don't take the money" than selecting "Take the money" (as prescribed in this strategy profile), since 2>-2.
   - We don't even need to check whether Mike Tyson chooses "Don't take the money" when he didn't trained hard (as prescribed in the separating strategy profile we are testing), since the above argument already shows that this strategy profile cannot be sustained as a PBE.

4. *Can this separating PBE be supported from your answer in c)? Obviously, you should obtain that it cannot be supported, but you have to show why from your answers in part c).*
   - No, since Mike Tyson prefers to deviate towards "Don't take the money" when he trained hard.

b) Find a Pooling PBE where Mike Tyson makes the offer "Take the money" when he has trained hard, and he also makes this offer "Take the money" when he has not trained hard. To show this, follow the usual steps for finding PBE.



1. *Find Buster Douglas' beliefs in this Pooling PBE (use Bayes' rule).*
   - After observing that Mike Tyson offers "Take the money" (in equilibrium), Buster Douglas' beliefs cannot be updated, and simply coincide with the prior probability distribution, that is

$$\gamma = \frac{\frac{2}{3} \times p^{TH}}{\frac{2}{3} \times p^{TH} + \frac{1}{3} \times p^{NTH}}$$

where p^{TH} denotes the probability that Mike Tyson makes the offer "Take the money" after training hard (*TH*), and similarly p^{NTH} represents the probability that he makes this offer when he didn't train hard (*NTH*). Since in this pooling strategy profile $p^{TH} = p^{NTH} = 1$, Buster Douglas' beliefs become

$$\gamma = \frac{\frac{2}{3} \times 1}{\frac{2}{3} \times 1 + \frac{1}{3} \times 1} = \frac{2}{3}$$

Intuitively, Buster Douglas cannot infer any additional information from Mike Tyson's type after observing that he offers "Take the money."

- After observing that Mike Tyson chooses "Don't take the money" (which occurs off-the-equilibrium path), Buster Douglas' beliefs are

$$\mu = \frac{\frac{2}{3} \times 0}{\frac{2}{3} \times 0 + \frac{1}{3} \times 0} = \frac{0}{0}$$

and thus must be left undefined, i.e., $\mu \in [0, 1]$.

2. *Find Buster Douglas' optimal action (whether to Fight or Not Fight) after observing that Mike Tyson offers him "Take the money". In addition, find Buster Douglas' optimal action (whether to Fight or Not Fight) after observing that Mike Tyson does not offers him any bribe (Douglas observes the action "Don't take the money").*

- When Mike Tyson offers "Take the money", Douglas' expected payoff if fighting is:

$$EU_B(F|TM) = 2 \times \frac{2}{3} + 10 \times \frac{1}{3} = \frac{14}{3}$$

and if not fighting:

$$EU_B(NF|TM) = 1 \times \frac{2}{3} + 1 \times \frac{1}{3} = 1$$

Thus, Douglas will fight since:

$$EU_B(F|TM) > EU_B(NF|TM), \qquad i.e., \frac{14}{3} > 1$$

- When Mike Tyson offers "Don't take the money", Douglas' expected payoff if fighting is:

$$EU_B(F|DTM) = -2\mu + 10(1-\mu) = 10 - 12\mu$$

and if not fighting:

$$EU_B(NF|DTM) = \mu + (1-\mu) = 1$$

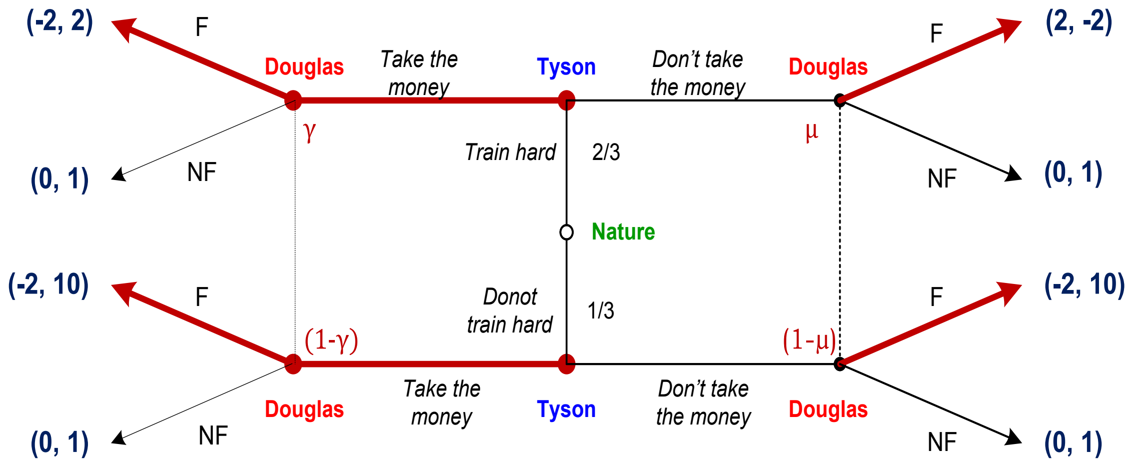
Thus, Douglas will fight only if:

$$EU_B(F|TM) > EU_B(NF|TM) \quad \leftrightarrow \quad 10 - 12\mu \geq 1 \quad \rightarrow \quad \mu \leq \frac{3}{4}$$

We then need to divide our following analysis into two cases:

1. **Case 1:** $\leq \frac{3}{4}$, and Buster Douglas chooses to fight after "Don't take the money."
2. **Case 2:** $\boldsymbol{\mu} > \frac{3}{4}$, and Buster Douglas chooses *not* to fight after "Don't take the money."

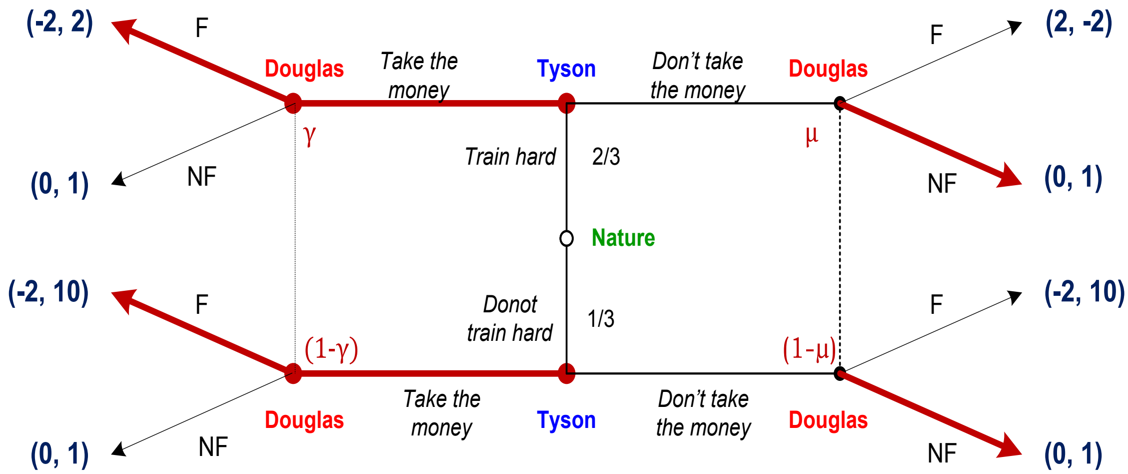
CASE 1: $\boldsymbol{\mu} \leq \frac{3}{4}$. (Buster Douglas chooses to fight after observing that Mike Tyson chooses "Don't take the money.")



Let us now check if this pooling strategy profile can be sustained as a PBE in this case ($\mu \leq \frac{3}{4}$):

- If Mike Tyson has trained hard, then he prefers to deviate towards "Don't take the money," where he obtains a payoff of 2, than selecting "Take the money" as prescribed in this pooling strategy profile, which only yields a payoff of -2.
- We don't even need to check whether Mike Tyson chooses "Take the money" when he didn't trained hard (as prescribed in the pooling strategy profile we are testing), since the above argument already shows that this strategy profile cannot be sustained as a PBE when $\mu \leq \frac{3}{4}$.

CASE 2: $\boldsymbol{\mu > \frac{3}{4}}$. (Buster Douglas chooses not to fight after observing that Mike Tyson chooses "Don't take the money.")



Let us now check if this pooling strategy profile can be sustained as a PBE in this case ($\mu > \frac{3}{4}$):

- If Mike Tyson has trained hard, then he prefers to select "Take the money" (as prescribed in this strategy profile), where he obtains a payoff of 2, than deviating towards "Don't take the money", which only yields a payoff of 0.
- If Mike Tyson has not trained hard, then he prefers to deviate towards "Don't take the money," where he obtains a payoff of 0, than selecting "Take the money" (as prescribed in this strategy profile), which yields a lower payoff of -2.
- Hence, this pooling strategy profile cannot be sustained as a PBE when $\mu > \frac{3}{4}$.
- Concluding, the pooling strategy profile where Mike Tyson makes the offer "Take the money" cannot be supported as a PBE of the game, regardless of Buster Douglas' off-the-equilibrium beliefs, i.e., regardless of the precise value of μ.

Exercise #6
Harrington Ch. 14 Exercise 1

A.

ANSWER: This game has two Nash equilibria, (a, w) and (b, x). Thus, consider a strategy for player 1 in which, in period 1, he chooses c; in a future period, chooses c if the outcome was (c, y) in all past periods; and, in a future period, chooses b if the outcome was not (c, y) in some past period. For player 2, in period 1, she chooses y; in a future period, chooses y if the outcome was (c, y) in all past periods; and, in a future period, chooses x if the outcome was not (c, y) in some past period. The equilibrium condition is

$$\frac{7}{1-\delta} \geq 9 + \delta\left(\frac{4}{1-\delta}\right) \Rightarrow \delta \geq \frac{2}{5}.$$

B.

ANSWER: Suppose it is either period 1 or it is some future period in which the outcome was either (c, y) or (d, z) in the previous period. The prescribed action of c for player 1 is optimal if

$$7 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 9 + \delta \times 0 + \delta^2 \times 7 + \cdots \Rightarrow 7 + 7\delta \geq 9 \Rightarrow \delta \geq \frac{2}{7}.$$

Now consider a history in which in the previous period the outcome was neither (c, y) nor (d, z). The prescribed action of d for player 1 is optimal if

$$0 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 4 + \delta \times 0 + \delta^2 \times 7 + \cdots \Rightarrow 7\delta \geq 4 \Rightarrow \delta \geq \frac{4}{7}.$$

The conditions are the same for player 2. It is then a subgame perfect Nash equilibrium if

$$\delta \geq \frac{2}{7} \text{ and } \delta \geq \frac{4}{7}; \text{ or } \delta \geq \frac{4}{7}.$$

C.

ANSWER: Consider period 1 or a future period in which the previous period's outcome was either (c, y), (d, w), or (a, z). Player 1's prescribed action of c is optimal if

$$7 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 9 + \delta \times (-4) + \delta^2 \times 7 + \cdots$$
$$\Rightarrow 7 + 7\delta \geq 9 - 4\delta \Rightarrow \delta \geq \frac{2}{11}.$$

Next, consider a history in which, in the previous period, player 1 did not choose c and player 2 chose y. Player 1's prescribed action of d is optimal if

$$-4 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 2 + \delta \times 4 + \delta^2 \times 4 + \cdots$$
$$\Rightarrow -4 + 7\frac{\delta}{1-\delta} \geq 2 + 4\frac{\delta}{1-\delta} \Rightarrow \delta \geq \frac{2}{3}.$$

Next, consider a history in which, in the previous period, player 1 chose c and player 2 did not choose y. Player 1's prescribed action of a is optimal since any other action lowers the current period payoff and reduces the future payoff stream from

$$\delta \times 7 + \delta^2 \times 7 + \cdots$$

to

$$\delta \times 4 + \delta^2 \times 4 + \cdots.$$

Finally, for any other history, player 1 is to choose b. Note that b maximizes his current payoff given player 2 is to choose x. Furthermore, the future payoff is the same since, come next period, the previous period's history will involve player 2 choosing x and thus the outcome will be (b, x). This applies as well to all ensuing periods. The analysis is analogous for player 2. This strategy profile is then a subgame perfect Nash equilibrium if

$$\delta \geq \frac{2}{11} \text{ and } \delta \geq \frac{2}{3} \Rightarrow \delta \geq \frac{2}{3}.$$