EconS 503 - Microeconomic Theory II
Homework #8 - Answer key.

1. **Exercises from MWG:** Chapter 23 (mechanism design): Exercise 23.C.10.

    - See scanned pages at the end of this handout.

2. **Procurement auctions under complete information.** Consider a town mayor inviting N firms to bid in a procurement contract that will allocate to the selected firm the right of water distribution for town residents. The efficiency in implementing the project is $\theta_i \in [0,1]$, so bidders are regarded as more efficient when their efficiency parameter, θ_i, increases. In this exercise, we consider that all players can observe every bidder's efficiency while in the next exercise we relax this assumption, allowing bidder i to privately observe his efficiency parameter.

    The cost of bidder i to implement the contract is $C_i(q_i, \theta_i)$, which is increasing and convex in output q_i, decreasing and convex in bidder i's efficiency θ_i, and satisfies $\frac{\partial^2 C_i(q_i,\theta_i)}{\partial q_i \partial \theta_i} \leq 0$. Each bidder has a quasilinear utility function,

    $$U(q_i, \theta_i) = t_i(q_i) - C_i(q_i, \theta_i),$$

    where $t_i(q_i)$ represents the transfer that the bidder receives from the procurer when the bidder produces q_i units of output (e.g., gallons of water). For simplicity, assume that bidders earn a zero reservation utility if they choose to not participate in the auction.

    The procurer's welfare function is

    $$V(q_i) - (1+\lambda) t_i(q_i)$$

    where $V(q_i)$ denotes the value that the procurer assigns to q_i units of output, while λ captures the shadow cost of raising public funds (as the procurer needs to raise distortionary taxes in order to pay the transfer $t_i(q_i)$ to bidder i).

    (a) Interpret the sign of the cross partial derivative, $\frac{\partial^2 C_i(q_i,\theta_i)}{\partial q_i \partial \theta_i}$.

    - This negative cross partial derivative means that the more efficient is bidder i, the lower is his marginal cost of production. That is, $\frac{\partial C_i(q_i,\theta_i)}{\partial q_i}$ decreases in θ_i.

    (b) Setup the procurer's program that induces participation and revelation of the bidders.

    - The procurer chooses the output-transfer pair, (q_i, t_i), for each bidder i to maximize

      $$\max_{q_i, t_i(q_i)} V(q_i) - (1+\lambda) t_i(q_i)$$

      subject to the Individual Rationality condition

      $$U_i(q_i(\theta_i), \theta_i) \geq 0 \text{ for all } \theta_i \in [0,1]$$

      Intuitively, the procurer maximizes social welfare generated from all bidders, subject to the voluntary participation of bidders of all types θ_i.

(c) Solve for the socially optimal output of bidder i.

- Using bidder i's utility function, $U(q_i, \theta_i) = t_i(q_i) - C_i(q_i, \theta_i)$, we can solve for transfer $t_i(q_i)$ to obtain $t_i(q_i) = C_i(q_i, \theta_i)$ since the individual rationality condition must be binding, i.e., $U_i(q_i(\theta_i), \theta_i) = 0$. Inserting it into the procurer's objective function, yields

$$\max_{q_i \geq 0} V(q_i) - (1+\lambda) C_i(q_i, \theta_i)$$

which has only one choice variable, q_i. Differentiating with respect to q_i, we obtain

$$\frac{\partial V(q_i^{CI})}{\partial q_i} - (1+\lambda)\frac{\partial C_i(q_i^{CI}, \theta_i)}{\partial q_i} = 0$$

where q_i^{CI} denotes the socially optimal output under complete information. Rearranging this first-order condition, yields

$$\underbrace{\frac{\partial V(q_i^{CI})}{\partial q_i}}_{MB_i} = \underbrace{(1+\lambda)\frac{\partial C_i(q_i^{CI}, \theta_i)}{\partial q_i}}_{MC_i}$$

Intuitively, the procurer increases water production until its marginal benefit (MB_i, in the left-hand side of the above equality) coincides with its associated marginal cost (MC_i, in the right-hand side). Since benefit function $V(q_i)$ is increasing and concave, its derivative lies in the positive quadrant but decreases in q_i, as depicted in figure 10.1. Similarly, because the production cost $C_i(q_i, \theta_i)$ is increasing and convex, its derivative lies in the positive quadrant and increases in q_i. The crossing point between the marginal benefit and cost functions entails $MB_i = MC_i$, yielding a socially optimal output.

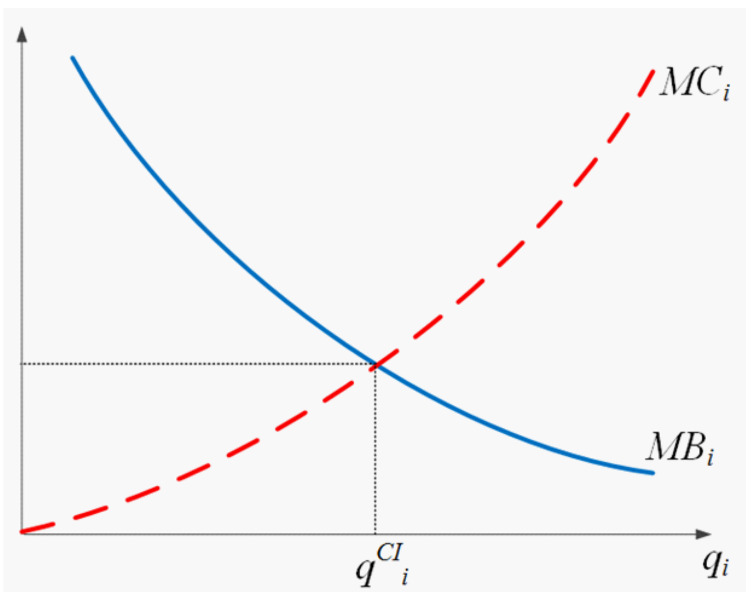

Figure 10.1. Optimal output under complete information.

If water produces a larger marginal benefit, the MB_i function shifts upward, increasing the socially optimal output q_i^{CI}. In contrast, an increase in the marginal cost of production, $\frac{\partial C_i(q_i,\theta_i)}{\partial q_i}$, or in the shadow cost of raising public funds, λ, yields an upward shift in the MC_i function, ultimately reducing the socially optimal output q_i^{CI} that the procurer implements.

(d) *Parametric example.* Let us now assume a parametric form for the value and cost functions in a setting with two bidders. In particular, assume that the cost function of bidder i is

$$C_i(q_i, \theta_i) = \frac{q_i^2}{1 + 2\theta_i}$$

Furthermore, the value that the procurer assigns to the output of bidder i is $V(q_i) = q_i$, and $\lambda = \frac{1}{10}$. Solve for the optimal output and transfer of bidder i.

- In this setting, the marginal benefit is $\frac{\partial V(q_i^*)}{\partial q_i} = 1$, the marginal cost of production is given by $\frac{\partial C_i}{\partial q_i} = \frac{2q_i}{1+2\theta_i}$, and the single-crossing property holds because

$$\frac{\partial^2 C_i}{\partial q_i \partial \theta_i} = -\frac{4q_i}{(1 + 2\theta_i)^2} < 0,$$

  that is, bidder i's marginal cost decreases in its efficiency parameter θ_i. The optimal output solves $MB_i = MC_i$, which in this parametric setting entails

$$1 = \left(1 + \frac{1}{10}\right) \frac{2q_i}{1 + 2\theta_i}$$

  Simplifying the above expression, yields

$$5 = \frac{11q_i}{1 + 2\theta_i}.$$

- Solving for output q_i, we obtain that the socially optimal output under complete information

$$q_i^{CI} = \frac{5\,(1 + 2\theta_i)}{11}$$

  which is increasing in bidder i's efficiency parameter, θ_i, as depicted in figure

10.2.

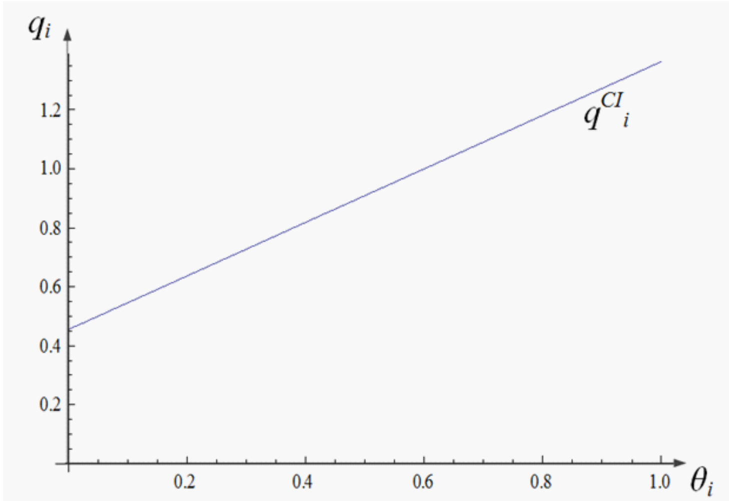

Figure 10.2. Socially optimal output, q_i^{CI}, as a function of θ_i.

- Substituting q_i^{CI} into the transfer function, we obtain

$$\begin{aligned} t_i^{CI} &= C_i(q_i, q_j, \theta_i) \\ &= \frac{1}{1+2\theta_i}\left[\frac{5\left(1+2\theta_i\right)}{11}\right]^2 \\ &= \frac{25\left(1+2\theta_i\right)}{121}. \end{aligned}$$

3. **Procurement auctions under incomplete information.** Consider the procurement auction in the previous exercise, but assume that every bidder i's efficiency of implementing the project, θ_i, is privately observable to bidder i. Efficiency θ_i is uniformly distributed, $U[0,1]$, which is common knowledge among all players.

   (a) Setup the procurer's program that induces participation and revelation of the bidders.

   - The procurer chooses the output-transfer pair, (q_i, t_i), for each bidder i to maximize

$$\max_{\{q_i, t_i(q_i)\}_{i=1}^N} \sum_{i=1}^N E_{\theta_i}\left[V\left(q_i\right) - \left(1+\lambda\right) t_i\left(q_i\right)\right]$$

subject to Incentive Compatibility:

$$U_i\left(q_i\left(\theta_i\right), \theta_i\right) \geq U_i\left(q_i(\widehat{\theta}_i), \widehat{\theta}_i\right) \quad \text{for every } \theta_i, \text{ where } \widehat{\theta}_i \neq \theta_i \qquad (IC_i)$$

and Individual Rationality:

$$U_i\left(q_i(0), 0\right) \geq 0 \qquad (IR_i)$$

Intuitively, the procurer maximizes expected social welfare generated from every bidder i, taking expectations over all possible realizations of efficiency parameter for this bidder, $\theta_i \in [0,1]$, and sums across all N bidders. This problem is subject to the truthful reporting of efficiency type by every bidder i, and to the voluntary participation of all bidders (including the least efficient one).

(b) Solve for the optimal output and transfer of bidder i. [*Hint*: Apply Myerson's Characterization Theorem to rewrite the incentive compatibility condition, and note that the individual rationality condition must hold with equality.]

- Using the Envelope Theorem, $\frac{\partial U_i(q_i(\widehat{\theta}_i),\widehat{\theta}_i)}{\partial q_i(\widehat{\theta}_i)} = 0$ evaluated at $\widehat{\theta}_i = \theta_i$, we obtain

$$dU_i(q_i, \theta_i) = -\frac{\partial C_i(q_i, \theta_i)}{\partial \theta_i} d\theta_i$$

Applying Myerson's Characterization Theorem (Myerson, 1981) to the Incentive Compatibility condition (IC_i), we obtain the following differential equation for all possible realizations of the efficiency parameter $\theta_i \in [0,1]$.

$$U_i(q_i, \theta_i) = U_i\left(q_i(0), 0\right) - \int_0^{\theta_i} \frac{\partial C_i(q_i, \widetilde{\theta}_i)}{\partial \theta_i} d\widetilde{\theta}_i$$

In addition, the Individual Rationality condition (IR_i) must hold with equality. Otherwise, the procurer could further reduce bidder i's residual utility and still induce his participation. When IR_i binds, $U_i\left(q_i\left(0\right), 0\right) = 0$, so that

$$U_i(q_i, \theta_i) = -\int_0^{\theta_i} \frac{\partial C_i(q_i, \widetilde{\theta}_i)}{\partial \theta_i} d\widetilde{\theta}_i \tag{1}$$

Substituting expression (1) into the utility function of bidder i, yields

$$-\int_0^{\theta_i} \frac{\partial C_i(q_i, \widetilde{\theta}_i)}{\partial \theta_i} d\widetilde{\theta}_i = t_i\left(q_i\right) - C_i\left(q_i, \theta_i\right)$$

Solving for the transfer of bidder i, we have

$$t_i\left(q_i\right) = C_i\left(q_i, \theta_i\right) - \int_0^{\theta_i} \frac{\partial C_i(q_i, \widetilde{\theta}_i)}{\partial \theta_i} d\widetilde{\theta}_i$$

- Inserting this transfer, the welfare maximization program of the procurer simplifies to the following unconstrained problem.

$$\max_{\{q_i\}_{i=1}^N} \sum_{i=1}^N E_{\theta_i}\left[V\left(q_i\right) - (1+\lambda)\underbrace{\left(C_i\left(q_i, \theta_i\right) - \int_0^{\theta_i} \frac{\partial C_i(q_i, \widetilde{\theta}_i)}{\partial \theta_i} d\widetilde{\theta}_i\right)}_{t_i(q_i)}\right]$$

Conducting integration by parts on the right-hand side of the integral yields

$$\begin{aligned}
&E_{\theta_i}\left[\int_0^{\theta_i}\frac{\partial C_i(q_i,\widetilde{\theta}_i)}{\partial\theta_i}d\widetilde{\theta}_i\right]\\
=&\int_0^1\int_0^{\theta_i}\frac{\partial C_i(q_i,\widetilde{\theta}_i)}{\partial\theta_i}d\widetilde{\theta}_i d\theta_i\\
=&\left[\theta_i\int_0^{\theta_i}\frac{\partial C_i(q_i,\widetilde{\theta}_i)}{\partial\theta_i}d\widetilde{\theta}_i\right]_0^1-\int_0^1\theta_i\frac{\partial C_i(q_i,\theta_i)}{\partial\theta_i}d\theta_i\\
=&\int_0^1(1-\theta_i)\frac{\partial C_i(q_i,\theta_i)}{\partial\theta_i}d\theta_i\\
=&E_{\theta_i}\left[(1-\theta_i)\frac{\partial C_i(q_i,\theta_i)}{\partial\theta_i}\right]
\end{aligned}$$

Then, the welfare maximization program of the procurer can be simplified to

$$\max_{\{q_i\}_{i=1}^N}\sum_{i=1}^N E_{\theta_i}\left[V(q_i)-(1+\lambda)\left(C_i(q_i,\theta_i)-(1-\theta_i)\frac{\partial C_i(q_i,\theta_i)}{\partial\theta_i}\right)\right]$$

The procurer maximizes welfare by taking first-order condition with respect to q_i.

$$\underbrace{\frac{\partial V(q_i^*)}{\partial q_i}}_{MB_i}=\underbrace{(1+\lambda)\left[\frac{\partial C_i(q_i^*,\theta_i)}{\partial q_i}-\overbrace{(1-\theta_i)\frac{\partial^2 C_i(q_i^*,\theta_i)}{\partial q_i\partial\theta_i}}^{\text{Information rent}}\right]}_{MVC_i}$$

In words, the procurer increases water production until the point at which its marginal benefit (MB_i, in the left-hand side of the above equality) coincides with its associated marginal virtual cost (MVC_i, in the right-hand side). This cost embodies not only firm i's marginal production cost (first term on the right-hand side) but also the information rent that the procurer needs to provide in order to induce bidder i report his type truthfully (last term on the right-hand side).

- From the welfare maximization program above, we can evaluate bidder i's transfer at the optimal output q_i^*, to obtain the optimal transfer to bidder i as follows,

$$t_i(q_i^*)=C_i(q_i^*,\theta_i)-(1-\theta_i)\frac{\partial C_i(q_i^*,\theta_i)}{\partial\theta_i}.$$

(c) *Comparison.* Compare your results against those in the complete information setting of the previous exercise. Interpret.

- Under a complete information setting, the last term in MVC_i (information rent) was absent, as shown in the previous exercise. Figure 10.3 superimposes MVC_i on figure 10.1 to facilitate our comparison. Since the cross-partial derivative $\frac{\partial^2 C_i(q_i^*,\theta_i)}{\partial q_i \partial \theta_i}$ is negative, we obtain that $MVC_i \geq MC_i$, as depicted in the figure.

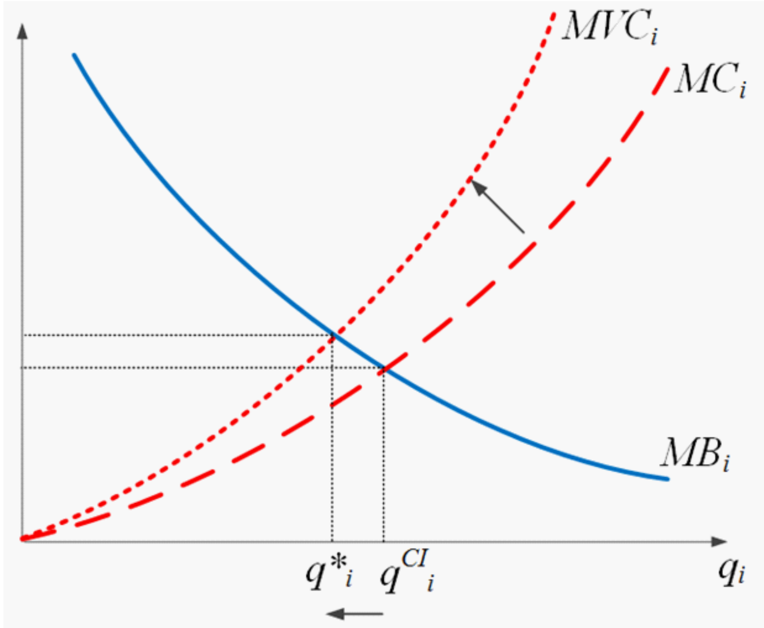

Figure 10.3. Socially optimal output under complete and incomplete information.

- Therefore, the socially optimal output under complete information is larger than that under incomplete information, $q_i^{CI} \geq q_i^*$. Intuitively, the procurer must pay an information rent to all bidders to induce truthful revelation of their types, incurring more costs to implement the auction than under complete information, ultimately inducing lower output levels. This is commonly referred in the literature as "downward distortion" for all bidders with efficiency levels $\theta_i \neq 1$.
- However, the output of the bidder with the highest efficiency, $\theta_i = 1$, suffers no distortion when moving from a complete to an incomplete information context. Indeed, MVC_i simplifies to MC_i when evaluated at $\theta_i = 1$, so first-order conditions across information contexts coincide, and $q_i^{CI} = q_i^*$. Intuitively, the most efficient bidder has no incentives to underreport his valuation at $\theta_i = 1$. This result is known as "no distortion at the top."

(d) *Parametric example.* Consider the same parametric forms as in the previous exercise. Solve for the optimal output and transfer of bidder i. Compare your results with those in the previous exercise.

- The optimal output solves $MB_i = MVC_i$, which in this parametric setting entails

$$1 = \left(1 + \frac{1}{10}\right)\left[\frac{2q_i}{1+2\theta_i} + (1-\theta_i)\frac{4q_i}{(1+2\theta_i)^2}\right]$$

since $\frac{\partial V(q_i^*)}{\partial q_i} = 1$, $\lambda = \frac{1}{10}$, marginal cost is given by $\frac{\partial C_i}{\partial q_i} = \frac{2q_i}{1+2\theta_i}$, and the single-crossing property holds because

$$\frac{\partial^2 C_i}{\partial q_i \partial \theta_i} = -\frac{4q_i}{(1+2\theta_i)^2} < 0,$$

that is, bidder i's marginal cost decreases in its efficiency parameter θ_i. Simplifying the above expression, yields

$$1 = \frac{33q_i}{5(1+2\theta_i)^2}.$$

- Solving for output q_i, we obtain the optimal output

$$q_i^* = \frac{5(1+2\theta_i)^2}{33}$$

which is increasing in bidder i's efficiency parameter, θ_i, as depicted in figure 10.4. To facilitate our comparisons, the figure also includes the socially optimal output under complete information, $q_i^{CI} = \frac{5(1+2\theta_i)}{11}$, which clearly lies above q_i^* for all efficiency levels $\theta_i \neq 1$, but coincides at exactly $\theta_i = 1$ (no distortion at the top), since $q_i^{CI} \geq q_i^*$ entails

$$\frac{5(1+2\theta_i)}{11} \geq \frac{5(1+2\theta_i)^2}{33}$$

simplifies to

$$3 \geq 1 + 2\theta_i$$

that holds for all $\theta \in [0, 1]$.

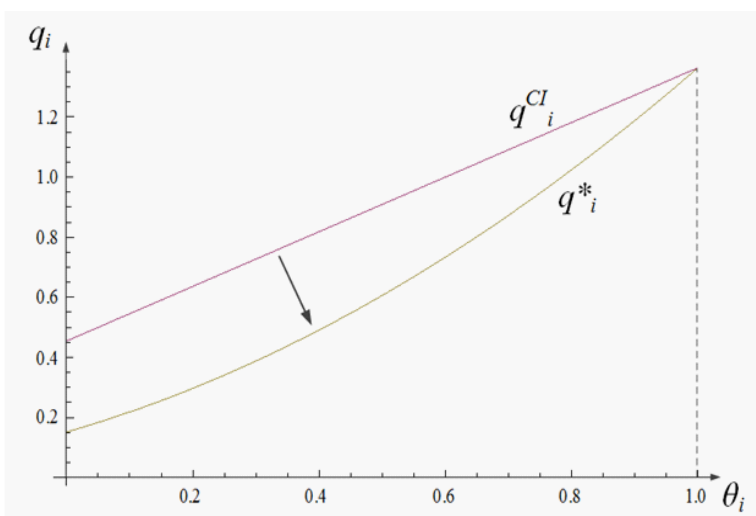

Figure 10.4. Socially optimal output under incomplete information as a function of θ_i.

- Inserting this optimal level into the transfer function $t_i(q_i)$ yields

$$\begin{aligned} t_i^* &= C_i(q_i, \theta_i) - (1-\theta_i)\frac{\partial C_i(q_i,\theta_i)}{\partial \theta_i} \\ &= \frac{q_i^2}{1+2\theta_i} + (1-\theta_i)\frac{2q_i^2}{(1+2\theta_i)^2} \\ &= \frac{3q_i^2}{(1+2\theta_i)^2} \\ &= \frac{25(1+2\theta_i)^2}{363} \end{aligned}$$

which is increasing in bidder i's efficiency parameter, θ_i, but falls below the transfer under complete information, t_i^{CI}, since

$$\frac{25(1+2\theta_i)^2}{363} < \frac{25(1+2\theta_i)}{121}$$

simplifies to $3 > 1+2\theta_i$ that is true for all $\theta_i \in [0,1]$. This happens since the procurer reduces the output of bidder i under incomplete information which reduces production cost more than the information rent that bidder i seeks to truthfully report his type, ultimately reducing the transfer that the procurer offers to bidder i.

4. **Stone-Geary utility function in pure exchange economy.** Consider a pure exchange economy with two individuals, A and B, whose utility functions are

$$\begin{aligned} u^A(x_1^A, x_2^A) &= (x_1^A - b_1)^{\frac{1}{2}}(x_2^A - b_2)^{\frac{1}{2}} \\ u^B(x_1^B, x_2^B) &= x_1^B x_2^B \end{aligned}$$

where $b_1, b_2 > 0$ represent the minimal amounts of goods 1 and 2 that individual A must consume in order to remain alive (such as water and shelter). Individuals A and B have endowments of $\omega^A = (\omega_1^A, \omega_2^A) = (4,2)$ and $\omega^B = (\omega_1^B, \omega_2^B) = (2,4)$, respectively.

(a) Set up the Lagrangian and find the individuals' Walrasian demand functions.

- *UMP of individual A.* Individual A chooses x_1^A and x_2^A to solve the following utility maximization problem,

$$\max_{x_1^A \geq b_1, x_2^A \geq b_2} u^A(x_1^A, x_2^A) = (x_1^A - b_1)^{\frac{1}{2}}(x_2^A - b_2)^{\frac{1}{2}}$$

$$\text{subject to } p_1 x_1^A + p_2 x_2^A = 4p_1 + 2p_2$$

Defining $\tilde{x}_1^A \equiv x_1^A - b_1$ and $\tilde{x}_2^A \equiv x_2^A - b_2$, which represent the above-subsistence consumption levels of individual A, we can rewrite his budget constraint as

$$p_1\tilde{x}_1^A + p_2\tilde{x}_2^A = p_1(4-b_1) + p_2(2-b_2)$$

Therefore, the Lagrangian function of individual A becomes

$$L_A = \left(\tilde{x}_1^A\right)^{\frac{1}{2}} \left(\tilde{x}_2^A\right)^{\frac{1}{2}} + \lambda_A \left[p_1 \left(4 - b_1\right) + p_2 \left(2 - b_2\right) - p_1 \tilde{x}_1^A - p_2 \tilde{x}_2^A\right]$$

The first order conditions of individual A's Lagrangian are

$$\begin{aligned} \frac{\partial L_A}{\partial \tilde{x}_1^A} &= \frac{1}{2} \left(\frac{\tilde{x}_2^A}{\tilde{x}_1^A}\right)^{\frac{1}{2}} - \lambda_A p_1 \leq 0 \\ \frac{\partial L_A}{\partial \tilde{x}_2^A} &= \frac{1}{2} \left(\frac{\tilde{x}_1^A}{\tilde{x}_2^A}\right)^{\frac{1}{2}} - \lambda_A p_2 \leq 0 \\ \frac{\partial L_A}{\partial \lambda_A} &= p_1 \left(4 - b_1\right) + p_2 \left(2 - b_2\right) - p_1 \tilde{x}_1^A - p_2 \tilde{x}_2^A \geq 0 \end{aligned}$$

with the associated Kuhn-Tucker conditions of

$$\begin{aligned} \tilde{x}_1^A \frac{\partial L_A}{\partial \tilde{x}_1^A} &= 0 \\ \tilde{x}_2^A \frac{\partial L_A}{\partial \tilde{x}_2^A} &= 0 \\ \lambda_A \frac{\partial L_A}{\partial \lambda_A} &= 0 \end{aligned}$$

Assuming interior solutions, the first order conditions hold with equality, so that by equating $\frac{\partial L_A}{\partial \tilde{x}_1^A} = \frac{\partial L_A}{\partial \tilde{x}_2^A} = 0$, we obtain

$$\frac{\frac{1}{2} \left(\frac{\tilde{x}_2^A}{\tilde{x}_1^A}\right)^{\frac{1}{2}}}{\frac{1}{2} \left(\frac{\tilde{x}_1^A}{\tilde{x}_2^A}\right)^{\frac{1}{2}}} = \frac{\lambda_A p_1}{\lambda_A p_2}$$

which, after rearranging, yields

$$p_1 \tilde{x}_1^A = p_2 \tilde{x}_2^A$$

Substituting $p_1 \tilde{x}_1^A = p_2 \tilde{x}_2^A$ into the budget constraint, we have

$$2 p_1 \tilde{x}_1^A = p_1 \left(4 - b_1\right) + p_2 \left(2 - b_2\right)$$

which is rearranged to give individual A's Walrasian demand of good 1,

$$\tilde{x}_1^A = \frac{4 - b_1}{2} + \frac{p_2 \left(2 - b_2\right)}{2 p_1}$$

and, similarly, we can obtain individual A's Walrasian demand of good 2,

$$\tilde{x}_2^A = \frac{p_1 \left(4 - b_1\right)}{2 p_2} + \frac{2 - b_2}{2}$$

- *UMP of individual B*. Individual B chooses x_1^B and x_2^B to solve the following utility maximization problem,

$$\max_{x_1^B, x_2^B \geq 0} u^B\left(x_1^B, x_2^B\right) = x_1^B x_2^B$$

subject to

$$p_1 x_1^B + p_2 x_2^B = 2p_1 + 4p_2$$

The Lagrangian function of individual B becomes

$$L_B = x_1^B x_2^B + \lambda_B \left[2p_1 + 4p_2 - p_1 x_1^B - p_2 x_2^B\right]$$

The first order conditions of individual B's Lagrangian are

$$\frac{\partial L_B}{\partial x_1^B} = x_2^B - \lambda_B p_1 \leq 0$$
$$\frac{\partial L_B}{\partial x_2^B} = x_1^B - \lambda_B p_2 \leq 0$$
$$\frac{\partial L_B}{\partial \lambda_B} = 2p_1 + 4p_2 - p_1 x_1^B - p_2 x_2^B \geq 0$$

with the associated Kuhn-Tucker conditions of

$$x_1^B \frac{\partial L_B}{\partial x_1^B} = 0$$
$$x_2^B \frac{\partial L_B}{\partial x_2^B} = 0$$
$$\lambda_B \frac{\partial L_B}{\partial \lambda_B} = 0$$

Assuming interior solutions, the first order conditions hold with equality, so that by equating $\frac{\partial L_B}{\partial x_1^B} = \frac{\partial L_B}{\partial x_2^B} = 0$, we obtain

$$\frac{x_2^B}{x_1^B} = \frac{\lambda_B p_1}{\lambda_B p_2}$$

which, after rearranging, yields

$$p_1 x_1^B = p_2 x_2^B$$

Substituting $p_1 x_1^B = p_2 x_2^B$ into the budget constraint, we have

$$2p_1 x_1^B = 2p_1 + 4p_2$$

which is rearranged to give individual B's Walrasian demand of good 1,

$$x_1^B = 1 + 2\frac{p_2}{p_1}$$

and, similarly, we can obtain individual B's Walrasian demand of good 2,

$$x_2^B = \frac{p_1}{p_2} + 2$$

(b) Find the set of Pareto efficient allocations (PEAs). (*Hint*: Your answer should be in terms of b_1 and b_2).

- The feasibility constraints in this pure exchange economy are

$$\underbrace{\left(\tilde{x}_1^A + b_1\right)}_{=x_1^A} + x_1^B = 4 + 2$$
$$\underbrace{\left(\tilde{x}_2^A + b_2\right)}_{=x_2^A} + x_2^B = 2 + 4$$

which are rearranged to give

$$\tilde{x}_1^A = 6 - b_1 - x_1^B$$
$$\tilde{x}_2^A = 6 - b_2 - x_2^B$$

The contract curve, which defines the set of Pareto efficient allocations, is the locus of tangency of indifference curves between individuals A and B, satisfying

$$MRS_{12}^A = \frac{MU_1^A}{MU_2^A} = \frac{MU_1^B}{MU_2^B} = MRS_{12}^B$$

which is rearranged to give

$$\frac{\tilde{x}_2^A}{\tilde{x}_1^A} = \frac{x_2^B}{x_1^B}$$

Substituting the feasibility constraints into the above expression, we obtain

$$\frac{6 - b_2 - x_2^B}{6 - b_1 - x_1^B} = \frac{x_2^B}{x_1^B}$$

which, after rearranging, yields the contract curve as follows,

$$x_2^B = \frac{6 - b_2}{6 - b_1} x_1^B$$

(c) Find the Walrasian equilibrium allocation (WEA). (*Hint*: Your answer should be in terms of b_1 and b_2).

- Substituting the Walrasian demands for good 1 into $\tilde{x}_1^A + x_1^B = 6 - b_1$, we obtain

$$\underbrace{\left(\frac{4 - b_1}{2} + \frac{p_2\left(2 - b_2\right)}{2p_1}\right)}_{=\tilde{x}_1^A} + \underbrace{\left(1 + 2\frac{p_2}{p_1}\right)}_{=x_1^B} = 6 - b_1$$

which is rearranged to yield the equilibrium price ratio, as follows.

$$\left(6 - b_2\right)\frac{p_2}{p_1} + \left(6 - b_1\right) = 2\left(6 - b_1\right)$$
$$\Rightarrow \frac{p_1}{p_2} = \frac{6 - b_2}{6 - b_1}$$

Substituting $\frac{p_1}{p_2} = \frac{6-b_2}{6-b_1}$ into the Walrasian demand functions of individual A, the Walrasian equilibrium allocation (WEA) of this individual becomes

$$\begin{aligned}
\tilde{x}_1^A &= \frac{4-b_1}{2} + \frac{(2-b_2)}{2} \cdot \frac{6-b_1}{6-b_2} \\
&= \frac{(4-b_1)(6-b_2) + (6-b_1)(2-b_2)}{2(6-b_2)} \\
&= \frac{18-4b_1-5b_2+b_1b_2}{6-b_2} \\
\tilde{x}_2^A &= \frac{(4-b_1)}{2} \cdot \frac{6-b_2}{6-b_1} + \frac{2-b_2}{2} \\
&= \frac{(4-b_1)(6-b_2) + (6-b_1)(2-b_2)}{2(6-b_1)} \\
&= \frac{18-4b_1-5b_2+b_1b_2}{6-b_1}
\end{aligned}$$

Given $\tilde{x}_1^A \equiv x_1^A - b_1$ and $\tilde{x}_2^A \equiv x_2^A - b_2$, we can rewrite the above expressions as

$$\begin{aligned}
x_1^A &= \tilde{x}_1^A + b_1 \\
&= \frac{18-4b_1-5b_2+b_1b_2}{6-b_2} + b_1 \\
&= \frac{18+2b_1-5b_2}{6-b_2} \\
x_2^A &= \tilde{x}_2^A + b_2 \\
&= \frac{18-4b_1-5b_2+b_1b_2}{6-b_1} + b_2 \\
&= \frac{18-4b_1+b_2}{6-b_1}
\end{aligned}$$

Similarly, the Walrasian equilibrium allocation (WEA) of individual B is

$$\begin{aligned}
x_1^B &= 1 + 2 \cdot \frac{6-b_1}{6-b_2} \\
&= \frac{18-2b_1-b_2}{6-b_2} \\
x_2^B &= \frac{6-b_2}{6-b_1} + 2 \\
&= \frac{18-2b_1-b_2}{6-b_1}
\end{aligned}$$

(d) Evaluate the contract curve and WEA at the following three different subsistence levels: (i) $(b_1, b_2) = (4, 2)$, (ii) $(b_1, b_2) = (3, 3)$, and (iii) $(b_1, b_2) = (2, 4)$. In which case(s) is individual A unable to survive?

- *First case.* Substituting $(b_1, b_2) = (4, 2)$ into the Walrasian equilibrium allocation,

$$\begin{aligned} x_1^{A*} &= \frac{18 + 2 \cdot 4 - 5 \cdot 2}{6 - 2} = 4 \\ x_2^{A*} &= \frac{18 - 4 \cdot 4 + 2}{6 - 4} = 2 \\ x_1^{B*} &= \frac{18 - 2 \cdot 4 - 2}{6 - 2} = 2 \\ x_2^{B*} &= \frac{18 - 2 \cdot 4 - 2}{6 - 4} = 4 \\ \frac{p_1}{p_2} &= \frac{6 - 2}{6 - 4} = 2 \end{aligned}$$

Summarizing, the WEA of

$$\left(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2} \right) = (4, 2; 2, 4; 2)$$

which means that individuals do not exchange their goods, and individual A can survive by consuming endowment ω^A. The contract curve in this context is

$$x_2^B = \frac{6 - 2}{6 - 4} x_1^B = 2x_1^B$$

- *Second case.* Substituting $(b_1, b_2) = (3, 3)$ into the Walrasian equilibrium allocation, we find

$$\begin{aligned} x_1^{A*} &= \frac{18 + 2 \cdot 3 - 5 \cdot 3}{6 - 3} = 3 \\ x_2^{A*} &= \frac{18 - 4 \cdot 3 + 3}{6 - 3} = 3 \\ x_1^{B*} &= \frac{18 - 2 \cdot 3 - 3}{6 - 3} = 3 \\ x_2^{B*} &= \frac{18 - 2 \cdot 3 - 3}{6 - 3} = 3 \\ \frac{p_1}{p_2} &= \frac{6 - 3}{6 - 3} = 1 \end{aligned}$$

Intuitively, individual A (B) exchanges 1 unit of good 1 (2) for 1 unit of good 2 (1) to yield the WEA

$$\left(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2} \right) = (3, 3; 3, 3; 1)\,,$$

such that individual A can remain alive with this trade. The contract curve in this setting is

$$x_2^B = \frac{6 - 3}{6 - 3} x_1^B = x_1^B$$

- *Third case.* Substituting $(b_1, b_2) = (2, 4)$ into the Walrasian equilibrium allocation, we obtain

$$x_1^{A*} = \frac{18 + 2 \cdot 2 - 5 \cdot 4}{6 - 4} = 1$$
$$x_2^{A*} = \frac{18 - 4 \cdot 2 + 4}{6 - 2} = \frac{7}{2}$$
$$x_1^{B*} = \frac{18 - 2 \cdot 2 - 4}{6 - 4} = 5$$
$$x_2^{B*} = \frac{18 - 2 \cdot 2 - 4}{6 - 2} = \frac{5}{2}$$
$$\frac{p_1}{p_2} = \frac{6 - 4}{6 - 2} = \frac{1}{2}$$

Summarizing, the WEA is

$$\left(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2} \right) = (1, 3.5; 5, 2.5; 0.5)$$

It is easy to check that, at this allocation, individual A's utility is negative, entailing that he cannot survive. In part (e) of the exercise, we examine a wealth redistribution program to keep this individual alive.
The contract curve in this context is

$$x_2^B = \frac{6 - 4}{6 - 2} x_1^B = \frac{1}{2} x_1^B$$

(e) Consider now a tax transfer so individual A survives in the case(s) you identify in part (b) where he suffers from a negative utility at the WEA. Identify the tax/transfer that the government can impose, and the resulting WEA. (For compactness, let us normalize $p_2 = 1$ so that $p \equiv p_1 = \frac{p_1}{p_2}$.)

- Suppose the government levies a tax t on individual B to provide it to individual A as a transfer. In this context, the budget constraint of individual A becomes

$$p\tilde{x}_1^A + \tilde{x}_2^A = p(4 - b_1) + (2 - b_2) + t$$

Substituting $p\tilde{x}_1^A = \tilde{x}_2^A$ and the price ratio $p = \frac{6-b_2}{6-b_1}$ into the budget constraint of individual A, we obtain

$$\begin{aligned}
\tilde{x}_1^A &= \frac{4 - b_1}{2} + \frac{2 - b_2 + t}{2p} \\
&= \frac{4 - b_1}{2} + \frac{2 - b_2 + t}{2} \cdot \frac{6 - b_1}{6 - b_2} \\
&= \frac{2(18 - 4b_1 - 5b_2 + b_1 b_2) + (6 - b_1)t}{2(6 - b_2)} \\
\tilde{x}_2^A &= \frac{p(4 - b_1)}{2} + \frac{2 - b_2 + t}{2} \\
&= \frac{4 - b_1}{2} \cdot \frac{6 - b_2}{6 - b_1} + \frac{2 - b_2 + t}{2} \\
&= \frac{2(18 - 4b_1 - 5b_2 + b_1 b_2) + (6 - b_1)t}{2(6 - b_1)}
\end{aligned}$$

Substituting the subsistence level of the third case we analyzed in part (d) of the exercise, $(b_1, b_2) = (2, 4)$, into the above expressions, yields

$$\tilde{x}_1^A = \frac{2 \cdot (-2) + 4t}{4} = t - 1$$
$$\tilde{x}_2^A = \frac{2 \cdot (-2) + 4t}{8} = \frac{t-1}{2}$$

Therefore, to ensure individual A can remain alive, we need

$$\tilde{x}_1^A \geq 0$$
$$\tilde{x}_2^A \geq 0$$

which is equivalent to

$$t = 1$$

Therefore, the equilibrium allocation of individual A is

$$x_1^{A*} = \tilde{x}_1^A + b_1 = 0 + 2 = 2$$
$$x_2^{A*} = \tilde{x}_2^A + b_2 = 0 + 4 = 4$$

- The budget constraint of individual B becomes now

$$px_1^B + x_2^B = 2p_1 + 4p_2 - t$$

Substituting $px_1^B = x_2^B$ into the budget constraint of individual B, we have

$$x_1^B = 1 + \frac{4-t}{2p}$$
$$x_2^B = p + \frac{4-t}{2}$$

Further substituting $p = \frac{1}{2}$ and $t = 1$ into the above expressions, we obtain the equilibrium allocation of individual B, as follows

$$x_1^{B*} = 1 + \frac{4-1}{2 \cdot \frac{1}{2}} = 4$$
$$x_2^{B*} = \frac{1}{2} + \frac{4-1}{2} = 2$$

Therefore, the Walrasian equilibrium allocation (WEA) becomes

$$\left(x_1^{A*}, x_2^{A*}; x_1^{B*}, x_2^{B*}; \frac{p_1}{p_2}\right) = (2, 4; 4, 2; 0.5)$$

which is supported by a tax-transfer, $t^* = 1$, from individual B to individual A.

Homework #8 - Answer Key

we know that:

$$\frac{\partial v(k(r,\theta_{-i}),r)}{\partial k} \geq \frac{\partial v(k(r,\theta_{-i}),\theta_i)}{\partial k} \quad \text{for all} \quad r \geq \theta_i . \qquad \text{(vi)}$$

using (v) and (vi) we get:

$$u(\theta_i,\hat{\theta}_i) - u(\theta_i,\theta_i) \leq \int_{\theta}^{\hat{\theta}} \left[\frac{\partial v(k(r,\theta_{-i}),r)}{\partial k} \cdot \frac{\partial k(r,\theta_{-i})}{\partial r} + \frac{\partial t(r,\theta_{-i})}{\partial r} \right] dr = 0$$

because the bracketed term equals zero for all r (see equation (23.C.12)). This, however, contradicts our negation assumption that $u(\theta_i,\hat{\theta}_i)-u(\theta_i,\theta_i)>0$ so $f(\cdot)$ must be truthfully implementable.

Case 2: Suppose $\hat{\theta}_i<\theta_i$. We can proceed as before, however the inequality in (vi) above will be reversed, and we will have a minus sign before the integral, so we will get the same contradiction.

23.C.10 [First Printing Errata: At the end of the first paragraph insert: "Assume throughout that conditions are such that (23.C.8) holding is a necessary condition for $(k^*(\cdot),t_1(\cdot),\ldots,t(\cdot)_I)$ to be truthfully implementable in dominant strategies." Also, in the second line of part c) insert the word "implementable" before "ex post efficient social choice function".]

a) Sufficiency: Suppose that we can write $V^*(\theta) = \sum_i V_i(\theta_{-i})$. Consider the transfer functions of the form

$$t_i(\theta) = \left[\sum_{j\neq i} v_j(k^*(\theta),\theta_j)\right] + h_i(\theta_{-i}) .$$

where for all i,

$$h_i(\theta_{-i}) = -(I-1)V_i(\theta_{-i}) \quad \text{for all } \theta_{-i} .$$

By proposition 23.C.4, $(k^*(\cdot),t_1(\cdot),\ldots,t(\cdot)_I)$ is truthfully implementable in dominant strategies. Moreover, for all θ we have,

$$\sum_i t_i(\theta) = \sum_i \left[\sum_{j\neq i} v_j(k^*(\theta),\theta_j)\right] + \sum_i h_i(\theta_{-i})$$

$$= (I - 1)V^*(\theta) - (I - 1)\sum_i V_i(\theta_{-i}) = 0$$

Necessity: Suppose $(k^*(\cdot),t_1(\cdot),\ldots,t(\cdot)_I)$ is ex post efficient and is truthfully implementable in dominant strategies. Since (23.C.8) is necessary (by assumption) for truthful implementation, this means that there exist functions $(h_i(\theta_{-i}))_{i=1}^I$ such that

$$(I - 1)V^*(\theta) + \sum_i h_i(\theta_{-i}) = \sum_i \left[\sum_{j\neq i} v_j(k^*(\theta),\theta_j)\right] + \sum_i h_i(\theta_{-i})$$

$$= \sum_i t_i(\theta) = 0$$

But this implies that by defining

$$V_i(\theta_{-i}) = \left(\frac{-1}{I - 1}\right)h_i(\theta_{-i}) \ ,$$

we can then write $V^*(\theta) = \sum_i V_i(\theta_{-i})$.

b) If $v_i(k,\theta_i) = \theta_i k - \frac{1}{2} k^2$ for all i, then, $k^*(\theta) = \text{Argmax}_k(\sum_i\theta_i)k - \frac{3}{2} k^2$ for all θ, and so the FOC implies that $k^*(\theta) = \frac{\sum_i\theta_i}{3}$. Hence,

$$\begin{aligned} V^*(\theta) &= \sum_{i=1}^{3}\left[\theta_i\left(\frac{\sum_i\theta_i}{3} \right) - \frac{1}{2}\left(\frac{\sum_i\theta_i}{3} \right)^2 \right] \\ &= \left(\frac{\sum_i\theta_i}{3} \right)\sum_i\left[\theta_i - \frac{1}{2}\left(\frac{\sum_i\theta_i}{3} \right) \right] \\ &= (\theta_1+ \theta_2+ \theta_3)[\theta_1+ \theta_2+ \theta_3 - \tfrac{1}{2}(\theta_1+ \theta_2+ \theta_3)] \\ &= \tfrac{1}{2}(\textstyle\sum_i\theta_i)^2 \\ &= (\theta_1^2 + \theta_2^2 + \theta_3^2 + 2\theta_1\theta_2 + 2\theta_1\theta_3 + 2\theta_2\theta_3) \ . \end{aligned}$$

We now define,

$$V_1(\theta_2,\theta_3) = \frac{\theta_2^2 + \theta_3^2}{2} + 2\theta_2\theta_3 \ ,$$

$$V_2(\theta_1,\theta_3) = \frac{\theta_1^2 + \theta_3^2}{2} + 2\theta_1\theta_3 \ ,$$

$$V_3(\theta_1,\theta_2) = \frac{\theta_1^2 + \theta_2^2}{2} + 2\theta_1\theta_2 \ .$$

and the result then follows from part a) above since

$$V^*(\theta) = V_1(\theta_2,\theta_3) + V_2(\theta_1,\theta_3) + V_3(\theta_1,\theta_2) \ .$$

c) If $V^*(\theta) = \sum_i V_i(\theta_{-i})$ then clearly $\dfrac{\partial^I V^*(\theta)}{\partial\theta_1\cdots\partial\theta_I} = 0.$

d) In this case, $V^*(\theta_1,\theta_2) = v_1(k^*(\theta),\theta_1) + v_2(k^*(\theta),\theta_2)$, therefore,

$$\frac{\partial V^*}{\partial\theta_1} = \left(\frac{\partial v_1}{\partial k} + \frac{\partial v_2}{\partial k}\right)\frac{\partial k}{\partial\theta_1} + \frac{\partial v_1}{\partial\theta_1} \ ,$$

$$\frac{\partial^2 V^*}{\partial\theta_1\partial\theta_2} = \left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right)\left(\frac{\partial k}{\partial\theta_1}\right)\left(\frac{\partial k}{\partial\theta_2}\right) + \frac{\partial^2 v_2}{\partial k\partial\theta_2}\cdot\frac{\partial k}{\partial\theta_1} + \frac{\partial^2 v_1}{\partial k\partial\theta_1}\cdot\frac{\partial k}{\partial\theta_2}.$$

Since,

$$\frac{\partial v_1}{\partial k} + \frac{\partial v_2}{\partial k} = 0 \ ,$$

we have,

$$\frac{\partial^2 v_i}{\partial k\partial\theta_i} = -\frac{\partial k}{\partial\theta_i}\left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right),$$

which in turn implies that

$$\frac{\partial^2 V^*}{\partial\theta_1\partial\theta_2} = -\left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right)\left(\frac{\partial k}{\partial\theta_1}\right)\left(\frac{\partial k}{\partial\theta_2}\right) \neq 0 \ ,$$

thus proving the statement.

23.C.11 Let agent i's Bernoulli utility function be $u_i(v_i(k,\theta_i) + \bar{m}_i + t_i$ and assume in negation that Proposition 23.C.4 no longer holds. That is, there exists i, θ_i, $\hat{\theta}_i$, and θ_{-i} such that:

$$u_i(v_i(k^*(\hat{\theta}_i,\theta_{-i}),\theta_i) + \bar{m}_i + t_i(\hat{\theta}_i,\theta_{-i}) > u_i(v_i(k^*(\theta),\theta_i) + \bar{m}_i + t_i(\theta)$$

Substituting from (23.C.8) we get:

$$u_i(v_i(k^*(\hat{\theta}_i,\theta_{-i}),\theta_i)+\bar{m}_i+\sum_{j\neq i} v_j(k^*(\hat{\theta}_i,\theta_{-i}),\theta_j) + h_i(\theta_{-i})) >$$