EconS 503 - Microeconomic Theory II
Homework #7 - Answer key

1. **Nonlinear pricing.** A monopolist faces demand function $Q(p) = 1 - p$, and constant marginal cost c, where $1 > c \geq 0$.

   (a) *Uniform pricing.* Suppose that the monopolist cannot discriminate in any way among the consumers and has to charge a uniform price, p^U. Calculate both the price that maximizes profits and the profits that correspond to this price.

   (b) *Single two-part tariff.* Suppose now that the monopolist can charge a two-part tariff (F, p), where F is the fixed fee and p is the price per unit. Expenditure then is $F + pq$. Calculate the two-part tariff that maximizes profits and the profits that the monopolist earns from this tariff. Compare p^U and F and comment briefly. Compare the situation with a uniform price and a two-part tariff in terms of welfare (a verbal argument is sufficient).

   (c) *Two two-part tariffs.* Assume now instead that there are two types of consumer. The consumers of type 1 have demand $Q_1(p) = 1 - p$ and the consumers of type 2 have demand $Q_2(p) = 1 - \frac{p}{2}$. The population is of size 1 and both consumer types are equally likely and assume, for simplicity, that $c = 1/2$. Calculate the two-part tariff that maximizes the profits of the monopolist. Compare the two-part tariffs found in part (b) and (c), both of them evaluated at $c = 1/2$, and comment briefly.

      - See scanned page at the end of this handout.

2. Exercises 12 and 13 from Bolton and Dewatripont.

   - See scanned pages at the end of this handout.

3. **Hidden information in financial contracts, based on Freixas and Laffont (1990).**[1] Asymmetric information significantly affects behavior in financial markets. In Freixas and Laffont (1990), the principal is a lender who provides a loan of size k to a borrower. The cost of capital is Rk to the lender since it could be invested elsewhere to earn the risk-free interest rate R. The lender has, thus, a utility function $V = t - Rk$. The borrower makes a profit $U = \theta f(k) - t$ where $\theta f(k)$ denotes the borrower's production with k units of capital (e.g., completing a project) and t represents the borrower's repayment to the lender. We assume that $f' > 0$ and $f'' < 0$. Parameter θ denotes a productivity shock, which is either high, θ_H, or low, θ_L, with probabilities p and $1 - p$, respectively.

   (a) *Complete information.* If the lender can observe the borrower's type (the realization of θ), which contract does he offer?

[1] Freixas, Xavier and Laffont, Jean-Jacques (1990) "Optimal Banking Contracts." In: *Essays in Honor of Edmond Malinvaud.* MIT Press. pp. 33-61.

- Upon observing θ_i, the principal solves

$$\max_{(U_i,k_i)} \ \theta_i f(k_i) - Rk_i - U_i$$
$$\text{subject to } \ U_i \geq 0$$

Since the participation constraint, $U_i \geq 0$, is binding, we have that $U_i = 0$, which simplifies the above problem to

$$\max_{k_i} \ \theta_i f(k_i) - Rk_i$$

Differentiating with respect to k_i, yields

$$\theta_i f'(k_i^*) = R.$$

Therefore, the borrower's return on capital, $\theta_i f'(k_i^*)$, is equal to the risk-free interest rate, R.

(b) *Incomplete information.* If the lender does not observe the borrower's type (the realization of θ), find the contract that he offers.

- Incentive and participation constraints can be written directly in terms of the borrower's information rents, that is, $U_L = \theta_L f(k_L) - t_L$ and $U_H = \theta_H f(k_H) - t_H$.

$$U_L \geq U_H - (\theta_H - \theta_L) f(k_H), \quad (1)$$
$$U_H \geq U_L - (\theta_H - \theta_L) f(k_L), \quad (2)$$
$$U_L \geq 0 \quad (3)$$
$$U_H \geq 0 \quad (4)$$

The principal's program takes now the following form:

$$\max_{\{(U_L,k_L);(U_H,k_H)\}} p\left[\theta_H f(k_H) - Rk_H\right] + (1-p)\left[\theta_L f(k_L) - Rk_L\right] - (pU_H + (1-p)U_L)$$

subject to (1) to (4)

We let the reader check that (2) and (3) are now the two binding constraints. As a result, there is no capital distortion with respect to the first-best outcome for the high-productivity type, implying that

$$U_L = 0, \text{ that is, } \theta_L f(k_L) = t_L, \text{ and}$$

$k_H^{SB} = k_H^*$, where k_H^* solves $\theta_H f'(k_H^*) = R$.

- However, there exists a downward distortion in the size of the loan given to a low-productivity borrower with respect to the first-best outcome. Indeed, we have that $k_L^{SB} = k_L^*$, where k_L^{SB} solves

$$\left(\theta_L - \frac{p}{1-p}(\theta_H - \theta_L)\right) f'(k_L^{SB}) = R$$

while k_L^* solves $\theta_L f'(k_L^*) = R$. Comparing the left-hand sides of both first-order conditions, we find that

$$\theta_L - \frac{p}{1-p}(\theta_H - \theta_L) < \theta_L$$

simplifies to $\frac{p}{1-p}(\theta_H - \theta_L) > 0$, which holds since $\theta_H > \theta_L$ by assumption.

4. **Moral hazard with multiple tasks.**[2] Let us consider a moral hazard problem between a principal and an agent. However, let us now allow the agent to take two effort levels e_1 and e_2. This represents, for instance, a salesman choosing how much effort to exert visiting potential customers, how much time to spend creating a more attractive website for online sales, investigating new sales strategies, etc. In this exercise we seek to understand how the multidimensionality in the agent's effort affects our results in the standard moral hazard problem analyzed in this chapter.

   Assume that the cost of exerting effort levels e_1 and e_2 is

$$c(e_1, e_2) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2$$

   These effort levels produce output y with output function

$$y = f_1 e_1 + f_2 e_2 + \varepsilon$$

   with performance $p = g_1 e_1 + g_2 e_2 + \phi$. Random shocks in output, ε, and performance, ϕ, follow distributions of $G(\phi)$ and $H(\varepsilon)$, respectively, with zero expectations, that is, $E(\varepsilon) = E(\phi) = 0$.

   For simplicity, assume that both principal and agent are risk neutral with payoff functions of $\pi = y - w$ for the principal (e.g., firm), where w denotes the salary she pays to the agent; and $U = w - c(e_1, e_2)$ for the agent (e.g., worker). Consider that the principal offers a salary $w = F + bp$ where F is fixed component of the contract and b is the bonus which provides a higher salary to the agent as his performance p increases. In particular, the timing of the game is as follows:

   - The principal and agent sign a contract $w = F + bp$.
   - The agent takes effort levels e_1 and e_2 which are unobservable to the principal.
   - Random shocks ε and ϕ, are realized, affecting the agent's output and performance, respectively.
   - Output y and performance p are observed by the principal and agent.
   - The agent receives wage $w = F + bp$.

   (a) Find the agent's optimal efforts and indirect utility as a function of the bonus parameter b.

[2] For a more general presentation, see Bolton and Dewatripont (2005), pp. 216-28.

- The agent solves the following expected utility maximization problem:

$$\begin{aligned}
\max_{e_1,e_2\geq 0} E_A\left[U\left(e_1,e_2\right)\right] &= E_A\left[w - c\left(e_1,e_2\right)\right] \\
&= \int \left[\underbrace{F+bp}_{w} - \underbrace{\left(\frac{1}{2}e_1^2+\frac{1}{2}e_2^2\right)}_{c(e_1,e_2)}\right] dG\left(\phi\right) \\
&= \int \left[F + b\underbrace{(g_1e_1+g_2e_2+\phi)}_{p} - \left(\frac{1}{2}e_1^2+\frac{1}{2}e_2^2\right)\right] dG\left(\phi\right) \\
&= F + b\left(g_1e_1+g_2e_2\right) - \frac{1}{2}e_1^2 - \frac{1}{2}e_2^2 + b\underbrace{\int \phi dG\left(\phi\right)}_{=0 \text{ since } E(\phi)=0} \\
&= F + bg_1e_1 + bg_2e_2 - \frac{1}{2}e_1^2 - \frac{1}{2}e_2^2
\end{aligned}$$

  Taking first-order conditions with respect to e_1 and e_2, we obtain

$$\begin{aligned}
\frac{\partial E_A\left[U\left(e_1,e_2\right)\right]}{\partial e_1} &= bg_1 - e_1 = 0 \\
\frac{\partial E_A\left[U\left(e_1,e_2\right)\right]}{\partial e_2} &= bg_2 - e_2 = 0
\end{aligned}$$

  Assuming interior solutions ($e_1 > 0$ and $e_2 > 0$), the agent's optimal efforts are

$$\begin{aligned}
e_1\left(b\right) &= bg_1 \\
e_2\left(b\right) &= bg_2
\end{aligned}$$

- *Indirect utility function.* Substituting the agent's optimal efforts back into his utility function, yields

$$\begin{aligned}
U\left(b,F\right) &= F + bg_1\cdot\underbrace{bg_1}_{e_1(b)} + bg_2\cdot\underbrace{bg_2}_{e_2(b)} - \frac{1}{2}\underbrace{[bg_1]}_{e_1(b)}^2 - \frac{1}{2}\underbrace{[bg_2]}_{e_2(b)}^2 \\
&= F + \left(bg_1\right)^2 + \left(bg_2\right)^2 - \frac{1}{2}\left(bg_1\right)^2 - \frac{1}{2}\left(bg_2\right)^2 \\
&= F + \frac{1}{2}\left(bg_1\right)^2 + \frac{1}{2}\left(bg_2\right)^2
\end{aligned}$$

(b) Find the principal's optimal contract w^* and his equilibrium profits.

- Operating by backwards induction, the principal anticipates the equilibrium effort levels that the agent chooses in the second stage of the game. Then, the principal solves the following expected profit maximization problem:

$$\begin{aligned}
\max_{b\geq 0} E_P\left[\pi\left(b\right)\right] &= E_P\left[y-w\right] \\
&= E_P\left[\underbrace{f_1 e_1\left(b\right)+f_2 e_2\left(b\right)+\varepsilon}_{y}-\underbrace{\left(F+bp\right)}_{w}\right] \\
&= \left(f_1-bg_1\right)\underbrace{e_1\left(b\right)}_{bg_1}+\left(f_2-bg_2\right)\underbrace{e_2\left(b\right)}_{bg_2}-F+\underbrace{\int \varepsilon dH\left(\varepsilon\right)}_{=0 \text{ since } E(\varepsilon)=0} \\
&= bg_1\left(f_1-bg_1\right)+bg_2\left(f_2-bg_2\right)-F
\end{aligned}$$

Taking first-order condition with respect to the bonus b, we find

$$\begin{aligned}
\frac{\partial E_P\left[\pi\left(b\right),F\right]}{\partial b} &= g_1\left(f_1-bg_1\right)-bg_1^2+g_2\left(f_2-bg_2\right)-bg_2^2 \\
&= g_1\left(f_1-2bg_1\right)+g_2\left(f_2-2bg_2\right)
\end{aligned}$$

Assuming interior solutions, that is, $b>0$, the bonus b^* satisfies

$$b^*=\frac{f_1g_1+f_2g_2}{2\left(g_1^2+g_2^2\right)}$$

- *Equilibrium profits.* Substituting the principal's optimal contract back into the profit function, we obtain

$$\begin{aligned}
\pi\left(b^*,F\right) &= b^*g_1\left(f_1-b^*g_1\right)+b^*g_2\left(f_2-b^*g_2\right)-F \\
&= \frac{f_1g_1+f_2g_2}{2\left(g_1^2+g_2^2\right)}\left(f_1g_1+f_2g_2\right)-\left(\frac{f_1g_1+f_2g_2}{2\left(g_1^2+g_2^2\right)}\right)^2\left(g_1^2+g_2^2\right)-F \\
&= \frac{\left(f_1g_1+f_2g_2\right)^2}{2\left(g_1^2+g_2^2\right)}-\frac{\left(f_1g_1+f_2g_2\right)^2}{4\left(g_1^2+g_2^2\right)}-F \\
&= \frac{\left(f_1g_1+f_2g_2\right)^2}{4\left(g_1^2+g_2^2\right)}-F
\end{aligned}$$

  Inspecting the principal's equilibrium profit above, we notice that any fixed wage $F>0$ reduces her profit, without affecting the agent's optimal efforts; which do not depend on F, as shown in part (a) of the exercise. Therefore, the principal should set the optimal fixed wage at $F^*=0$.
- As a result, the optimal contract becomes

$$\begin{aligned}
w^* &= F^*+b^*p \\
&= \frac{f_1g_1+f_2g_2}{2\left(g_1^2+g_2^2\right)}p
\end{aligned}$$

  which depends on the random shock ϕ. Specifically, when a favorable shock $\phi>0$ is realized, the agent outperforms so that the principal pays a higher wage to him. On the other hand, when an unfavorable shock $\phi<0$ is realized, the agent underperforms so that the principal pays a lower wage to him. Therefore, in expectation (that is, at stage 1 of the game, before the shocks are realized), the expected wage is

$$E[w^*] = b\frac{f_1 g_1 + f_2 g_2}{2\left(g_1^2+g_2^2\right)}\left(g_1^2+g_2^2\right) + \frac{f_1 g_1 + f_2 g_2}{2\left(g_1^2+g_2^2\right)}\underbrace{E[p]}_{=0}$$
$$= \frac{\left(f_1 g_1 + f_2 g_2\right)^2}{4\left(g_1^2+g_2^2\right)}$$

(c) *Comparative Statics.* How is the optimal contract you found in part (b) affected by the output rates f_1 and f_2? How is it affected by the performance rates g_1 and g_2? Explain.

- Differentiating the optimal wage with respect to f_1 and f_2, we find

$$\frac{\partial w^*}{\partial f_1} = \frac{g_1\left(f_1 g_1 + f_2 g_2\right)}{2\left(g_1^2+g_2^2\right)} > 0$$
$$\frac{\partial w^*}{\partial f_2} = \frac{g_2\left(f_1 g_1 + f_2 g_2\right)}{2\left(g_1^2+g_2^2\right)} > 0$$

which means that as either effort level becomes more effective in producing output, wage payment increases.

- Differentiating the wage contract with respect to g_1 and g_2, yields

$$\frac{\partial w^*}{\partial g_1} = \frac{f_1\left(f_1 g_1 + f_2 g_2\right)\left(g_1^2+g_2^2\right) - g_1\left(f_1 g_1 + f_2 g_2\right)^2}{2\left(g_1^2+g_2^2\right)^2}$$
$$= \frac{g_2\left(f_1 g_1 + f_2 g_2\right)\left(f_1 g_2 - f_2 g_1\right)}{2\left(g_1^2+g_2^2\right)^2}$$
$$\frac{\partial w^*}{\partial g_2} = \frac{f_2\left(f_1 g_1 + f_2 g_2\right)\left(g_1^2+g_2^2\right) - g_2\left(f_1 g_1 + f_2 g_2\right)^2}{2\left(g_1^2+g_2^2\right)^2}$$
$$= \frac{g_1\left(f_1 g_1 + f_2 g_2\right)\left(f_2 g_1 - f_1 g_2\right)}{2\left(g_1^2+g_2^2\right)^2}$$

Therefore, if $\frac{f_1}{f_2} > \frac{g_1}{g_2}$, we obtain that $\frac{\partial w^*}{\partial g_1} > 0$ and $\frac{\partial w^*}{\partial g_2} < 0$. Intuitively, if effort 1 is more effective in generating output than in delivering performance, relatively to effort 2, the optimal wage increases in the performance rate of effort 1, g_1, but decrease in that of effort 2, g_2. The opposite holds if $\frac{f_1}{f_2} < \frac{g_1}{g_2}$ such that $\frac{\partial w^*}{\partial g_1} < 0$ and $\frac{\partial w^*}{\partial g_2} > 0$. In this case, the optimal wage increases in the performance rate of effort 2, g_2, but decreases in that of effort 1, g_1.

(d) Given the optimal contract found above, what are the principal's expected payoff, the agent's expected utility, and the expected social welfare in equilibrium?

- Substituting the optimal contract w^* into the agent's indirect utility function, we find

$$\begin{aligned}
U(w^*) &= F^* + \frac{1}{2}(b^* g_1)^2 + \frac{1}{2}(b^* g_2)^2 \\
&= 0 + \frac{1}{2}\left(\frac{f_1 g_1 + f_2 g_2}{2(g_1^2 + g_2^2)} g_1\right)^2 + \frac{1}{2}\left(\frac{f_1 g_1 + f_2 g_2}{2(g_1^2 + g_2^2)} g_2\right)^2 \\
&= \frac{1}{8}\left(\frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}\right)^2 (g_1^2 + g_2^2) \\
&= \frac{(f_1 g_1 + f_2 g_2)^2}{8(g_1^2 + g_2^2)}
\end{aligned}$$

- Similarly, substituting the optimal contract w^* into the principal's indirect utility function, we obtain

$$\pi(w^*) = \frac{(f_1 g_1 + f_2 g_2)^2}{4(g_1^2 + g_2^2)}$$

- Therefore, the expected social welfare is given by the sum of the principal's expected profit and the agent's expected utility in equilibrium.

$$\begin{aligned}
SW^* &= \pi(w^*) + U(w^*) \\
&= \frac{(f_1 g_1 + f_2 g_2)^2}{4(g_1^2 + g_2^2)} + \frac{(f_1 g_1 + f_2 g_2)^2}{8(g_1^2 + g_2^2)} \\
&= \frac{3(f_1 g_1 + f_2 g_2)^2}{8(g_1^2 + g_2^2)}
\end{aligned}$$

(e) What is the socially optimal contract? Compare it against the contract that emerges in the subgame perfect equilibrium of the game you found in part (b).

- The expected social welfare for both the principal and the agent is given by the sum

$$\begin{aligned}
SW &= E[\pi(b,F) + U(b,F)] \\
&= \underbrace{[bg_1(f_1 - bg_1) + bg_2(f_2 - bg_2) - F]}_{E[\pi(b,F)]} + \underbrace{\left[F + \frac{1}{2}(bg_1)^2 + \frac{1}{2}(bg_2)^2\right]}_{E[U(b,F)]} \\
&= bf_1 g_1 - b^2 g_1^2 + bf_2 g_2 - b^2 g_2^2 + \frac{1}{2}b^2 g_1^2 + \frac{1}{2}b^2 g_2^2 \\
&= b(f_1 g_1 + f_2 g_2) - \frac{1}{2}(bg_1)^2 - \frac{1}{2}(bg_2)^2
\end{aligned}$$

where the first line is operating by both parties maximizing joint payoffs *as if* efforts are observable; and because the fixed wage, F, which is an action-independent transfer from the principal to the agent, is cancelled out, we can set $F^{**} = 0$ without affecting the expected social welfare.

- Taking first-order condition with respect to b, yields

$$\frac{\partial SW}{\partial b} = f_1 g_1 + f_2 g_2 - bg_1^2 - bg_2^2$$

Assuming interior solutions, that is, $b > 0$, the socially optimal bonus b^{**} becomes

$$b^{**} = \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2} \tag{1}$$

- Comparing with the equilibrium bonus found part (b), $b^* = \frac{f_1g_1+f_2g_2}{2\left(g_1^2+g_2^2\right)}$, we see that $b^{**} = 2b^*$. In words, this result indicates that effort unobservability (in part b) reduces the bonus b by half. Intuitively, as the principal cannot observe the effort levels chosen by the agent, the principal believes that the observed performance can be a matter of luck (i.e., due to random shocks) other than the efforts exerted by the agent; and therefore, she reduces the variable wage (i.e., bonus) to the agent.

(f) What is the deadweight loss in this contractual setting?

- Substituting the socially optimal bonus found in part (e), b^{**}, into the social welfare function, we obtain

$$\begin{aligned}
SW^{**} &= E\left[\pi\left(b^{**}, F\right) + U\left(b^{**}, F\right)\right] \\
&= \underbrace{\left[b^{**} g_1\left(f_1 - b g_1\right) + b^{**} g_2\left(f_2 - b g_2\right) - F\right]}_{E[\pi(b,F)]} + \underbrace{\left[F + \frac{1}{2}\left(b^{**} g_1\right)^2 + \frac{1}{2}\left(b^{**} g_2\right)^2\right]}_{E[U(b,F)]} \\
&= b^{**} f_1 g_1 - \left(b^{**}\right)^2 g_1^2 + b^{**} f_2 g_2 - \left(b^{**}\right)^2 g_2^2 + \frac{1}{2}\left(b^{**}\right)^2 g_1^2 + \frac{1}{2}\left(b^{**}\right)^2 g_2^2 \\
&= b^{**}\left(f_1 g_1 + f_2 g_2\right) - \frac{1}{2}\left(b^{**} g_1\right)^2 - \frac{1}{2}\left(b^{**} g_2\right)^2 \\
&= \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}\left(f_1 g_1 + f_2 g_2\right) - \frac{1}{2}\left(\frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}\right)^2\left(g_1^2 + g_2^2\right)^2 \\
&= \frac{\left(f_1 g_1 + f_2 g_2\right)^2}{2\left(g_1^2 + g_2^2\right)}
\end{aligned}$$

Therefore, the deadweight loss is

$$\begin{aligned}
DWL &= SW^{**} - SW^* \\
&= \frac{\left(f_1 g_1 + f_2 g_2\right)^2}{2\left(g_1^2 + g_2^2\right)} - \frac{3\left(f_1 g_1 + f_2 g_2\right)^2}{8\left(g_1^2 + g_2^2\right)} \\
&= \frac{\left(f_1 g_1 + f_2 g_2\right)^2}{8\left(g_1^2 + g_2^2\right)}.
\end{aligned}$$

(g) *Numerical example.* Consider output rates $f_1 = \frac{1}{2}$ and $f_2 = \frac{1}{3}$, and performance rates $g_1 = \frac{2}{3}$ and $g_2 = \frac{1}{4}$. In this context, evaluate the equilibrium bonus b^*, efforts e_1^* and e_2^*, wage w^*, the agent's expected utility $U\left(w^*\right)$, the principal's expected profit $\pi\left(w^*\right)$, and social welfare SW^*. Then, evaluate the socially optimal bonus b^{**}, social welfare SW^{**} at b^{**}, and deadweight loss due to unobservability of effort.

- *Equilibrium outcomes.* Evaluating the equilibrium bonus, $b^* = \frac{f_1 g_1 + f_2 g_2}{2\left(g_1^2 + g_2^2\right)}$, found in part (b) of the exercise at the above parameter values, we obtain

$$b^* = \frac{\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}}{2\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)}$$
$$= \frac{30}{73}$$

Evaluating equilibrium efforts, $e_1\left(b^*\right) = b^* g_1$ and $e_2\left(b^*\right) = b^* g_2$, found in same part of the exercise, we obtain

$$e_1^* = \frac{20}{73}$$
$$e_2^* = \frac{15}{146}$$

Evaluating equilibrium wage, $w^* = F^* + b^* p = 0 + b^* p$, at the above parameter values, we obtain

$$w^* = \frac{30}{73} p$$

as a function of the performance p which in turn depends on the random shock ϕ, such that expected wage of the agent, $E\left[w^*\right] = \frac{\left(f_1 g_1 + f_2 g_2\right)^2}{4\left(g_1^2 + g_2^2\right)}$, at stage 1 of the game (i.e., before the shocks are realized) found in part (b) of the exercise becomes

$$E\left[w^*\right] = \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{4\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)}$$
$$= \frac{25}{292} \simeq 0.08$$

Therefore, expected utility of the agent, $U\left(w^*\right) = \frac{\left(f_1 g_1 + f_2 g_2\right)^2}{8\left(g_1^2 + g_2^2\right)}$, found in part (d) of the exercise becomes

$$U^* = \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{8\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)}$$
$$= \frac{25}{584} \simeq 0.04$$

and expected profit of the principal, $\pi\left(w^*\right) = \frac{\left(f_1 g_1 + f_2 g_2\right)^2}{4\left(g_1^2 + g_2^2\right)}$, which is also found in part (d) becomes

$$\pi^* = \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{4\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)}$$
$$= \frac{25}{292} \simeq 0.08$$

which implies a social welfare of $SW^* = \pi^* + U^* = \frac{75}{584} \simeq 0.12$.

- *Socially optimal outcomes.* Evaluating the socially optimal bonus $b^{**} = \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}$ we found in part (e) of the exercise at the above parameter values, we obtain

$$\begin{aligned} b^{**} &= \frac{\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}}{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2} \\ &= \frac{60}{73} \simeq 0.82 \end{aligned}$$

and social welfare, $SW^{**} = \frac{(f_1 g_1 + f_2 g_2)^2}{2\left(g_1^2 + g_2^2\right)}$, evaluated at bonus b^{**}, as found in part (e) of the exercise, is

$$\begin{aligned} SW^{**} &= \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{2\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)} \\ &= \frac{25}{146} \simeq 0.17 \end{aligned}$$

- *Comparison.* Deadweight loss, $DWL = \frac{(f_1 g_1 + f_2 g_2)^2}{8\left(g_1^2 + g_2^2\right)}$, as found in part (f) of the exercise, is

$$\begin{aligned} DWL &= \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{8\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)} \\ &= \frac{25}{584} \simeq 0.04 \end{aligned}$$

EconS 503 – Homework #8

Answer Key

Exercise #1 – Damaged good strategy (Menu pricing)

1. It is immediate that optimal price is $p^* = 3$ which yields profits of $\pi^* = 3/2$ (the alternative being a price of $p = 1$, yielding $\pi = 1$).
2. The damaged version is sold to the low-valuation consumers and the normal version to the high-valuation consumers. Hence, $p^{*damaged} = 5/10$. To satisfy the incentive constraint of the high type, $p^{*normal}$ must be chosen such that
$$\frac{6}{10} - p^{*damages} = \frac{1}{10} = 3 - p^{*normal}$$
or, equivalently, $p^{*normal} = 29/10$, which is less than 3. This results in profits of
$$(1/2)(5/10 - 1/10) + (1/2\,)(29/10) = 33/20 > 3/2.$$
3. Hence, the firm makes a higher profit by introducing the damages version in spite of the higher cost of production for this version. The high-valuation consumers are also better off when the damages version is introduced, as they face a lower price. Finally, the low-valuation consumers obtain zero utility in both cases. The introduction of the damaged version thus results in a Pareto improvement.

Exercise #2 - Nonlinear pricing

1. The monopoly choses p to maximize $\pi = (p-c)(1-p)$. The profit-maximizing price is easily found as $p^U = \frac{1+c}{2}$. The corresponding profit is $\pi^U = \frac{(1-c)^2}{4}$.
2. Facing a tariff (m, p), the participation constraint of a consumer is $CS(p) - m \geq 0$, where $CS(p)$ is the consumer surplus price p. With demand $Q(p) = 1 - p$, we have $CS(p) = \frac{(1-p)^2}{2}$. As the monopolist's best interest is to set $m = CS(p)$, its problem is thus to set p so as to maximize
$$\pi = CS(p) + (p-c)(1-p) = \left(\frac{1}{2}\right)(1-p)(1+p-2c).$$
The first-order condition yields $p - c = 0$. Hence, the optimal two-part tariff is $(m, p) = \left(\frac{(1-c)^2}{2}, c\right)$. The optimal two-part tariff consists of selling the good at marginal cost, so as to generate the largest consumer surplus, and in capturing this surplus fully through the fixed fee. The corresponding profit is $\pi^{TP} = m = \frac{(1-c)^2}{2}$. Welfare is maximized under the two-part tariff as it involves marginal cost pricing (whereas uniform pricing results in a deadweight loss).

3. Consumer surplus for agents of type 2 at any price $p \leq 2$ is computed as $CS_2(p) = \frac{(2-p)^2}{4}$. We recall from part (2) of the exercise that $CS_1(p) = \frac{(1-p)^2}{2}$ for any $p \leq 1$. For any price where the two types of consumer buy (i.e., $p \leq 1$), we check that $CS_2(p) \geq CS_1(p)$. One option for the monopolist is to make sure that consumers of both types buy; for this, $m = CS_1(p)$ and $p \leq 1$. We then have the same problem as in part (2), with $c = \frac{1}{2}$: $(m, p) = \left(\frac{1}{8}, \frac{1}{2}\right)$ and $\pi_1 = \frac{1}{8}$. The alternative option is to sell only to type-2 consumers with $m = CS_2(p)$ and $p \leq 2$.
The monopolist's problem is then to choose p to maximize

$$\pi_2 = (1/2)\frac{(2-p)^2}{4} + (p - 1/2)(1 - p/2).$$

The first-order condition yields $p = 1/2$. As $p = 1/2$ (i.e., marginal cost pricing violates the constraint), the monopolist chooses a price of $p = 1$, so that $m = 1/4$. The resulting profit is $\pi_2 = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$. Therefore, $\pi_2 > \pi_1$, implying that the monopolist prefers to sell to type-2 consumers only by setting a price above marginal cost.

Homework #3 – Moral Hazard

a) The contract specifies the payment w_S you receive when you deliver the beer, and the payment w_F you receive when you do not deliver the beer. The two inequalities are the incentive compatibility constraint and your friend's individual rationality constraint:

$$0.9(-e^{-0.2w_S}) + 0.1(-e^{-0.2w_F}) \geq -e^{-0.2(w_F+5)} \qquad \textbf{(IC)}$$

$$0.9(8 - w_S) + 0.1(-w_F) \geq 3 \qquad \textbf{(IR)}$$

The solution to these equations (when they hold with equality) is $w_S = 4.81$ and $w_F = -1.25$.

b) The two inequalities are the incentive compatibility constraint and your own individual rationality constraint:

$$0.9(-e^{-0.2w_S}) + 0.1(-e^{-0.2w_F}) \geq -e^{-0.2(w_F+5)} \qquad \textbf{(IC)}$$

$$0.9(-e^{-0.2w_S}) + 0.1(-e^{-0.2w_F}) \geq -1 \qquad \textbf{(IR)}$$

We make these equalities, and solve the two equations. The two imply that $-e^{-.2(w_F+5)} = -1$, and so (taking logs $0.2(w_F + 5) = 0$. Hence, $w_F = -5$. From (IR),

$$0.9(-e^{-0.2w_S}) + 0.1\left(-e^{-0.2(-5)}\right) = -1$$

Maximizing with respect to $R(x)$ for every positive x leads to the following first order conditions

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial R(x)} &= 0 \Leftrightarrow \frac{1}{u'(W_0 + R(x) - x)} = \frac{1}{\lambda} + \frac{\mu}{\lambda}\frac{p'(a)}{p(a)} \text{ for } \forall x > 0 \\ \frac{\partial \mathcal{L}}{\partial R(0)} &= 0 \Leftrightarrow \frac{1}{u'(W_0 + R(0))} = \frac{1}{\lambda} - \frac{\mu}{\lambda}\frac{p'(a)}{1 - p(a)}\end{aligned}$$

where we assumed that $p(a)$ never equals to 0 or 1. In other words, we assume that it is extremely costly for the agent to ensure there will be no loss whatsoever, but extremely cheap to make sure that the loss does not occur with probability 1. Given the positive Lagrange multipliers and $p'(a) < 0$, we find for $\forall x > 0$

$$\frac{1}{u'(W_0 + R(0))} > \frac{1}{\lambda} > \frac{1}{u'(W_0 + R(x) - x)}$$

$\Leftrightarrow$

$$R(0) > R(x) - x.$$

So the optimal contract is such that the insuree is punished when a loss occurs, but the punishment is not related to the size of the loss x since the probability of a larger loss is independent of the agent's effort. Therefore, the second-best contract will induce variation in the insuree's payoff only between 0 and $x > 0$. The exact shape of the contract will be fully determined by the first order condition with respect to effort and the zero-profit condition.

4.3 Question 12

Consider a principal-agent problem with three exogenous states of nature, θ_1, θ_2, and θ_3; two effort levels, a_L and a_H; and two output levels, distributed as follows as a function of the state of nature and the effort level:

State of nature	θ_1	θ_2	θ_3
Probability	0.25	0.5	0.25
Output under a_H	18	18	1
Output under a_L	18	1	1

The principal is risk neutral while the agent has utility function $\sqrt{w}$ when receiving monetary compensation w, minus the cost of effort, which is normalized to 0 for a_L and to 0.1 for a_H. The agent's reservation expected utility is 0.1.

1. Derive the first-best contract.
2. Derive the second-best contract when only output levels are observable.
3. Assume the principal can buy for a price of 0.1 an information system that allows the parties to verify whether state of nature θ_3 happened or not. Will the principal buy this information system? Discuss.

4.3.1 First-Best Contract

The probabilities are as follows:

$$\Pr(q = 18|a_H) = 0.75$$
$$\Pr(q = 18|a_L) = 0.25.$$

Thus, when the principal can contract on the level of effort, he will solve

$$\max_{a_i, w} 0.25q(a_i; \theta_1) + 0.5q(a_i; \theta_2) + 0.25q(a_i; \theta_3) - w$$

subject to

$$\sqrt{w} - \psi(a_i) \geq 0.1$$

Under action choice a_H,

$$\sqrt{w_H^*} - 0.1 = 0.1 \Rightarrow w_H = 0.04.$$

Therefore,

$$E\pi_P^*(a_H) = 18\,(0.75) + 1\,(0.25) - 0.04 = 13.71.$$

Under action choice a_L,

$$\sqrt{w_L^*} = 0.1 \Rightarrow w_L^* = 0.01.$$

Therefore,

$$E\pi_P^*(a_L) = 18\,(0.25) + 1\,(0.75) - 0.01 = 5.24.$$

Thus, the optimal contract specifies $a^* = a_H$ and $w^* = 0.04$.

4.3.2 Second-Best Contract

When only output levels are observable, the optimal contract to induce the effort level a_H solves a minimization of the wage bill,

$$\min_{w_{18}, w_1} 0.75w_{18} + 0.25w_1$$

subject to

$$0.75\sqrt{w_{18}} + 0.25\sqrt{w_1} - 0.1 \geq 0.1 \quad \text{(IR)}$$
$$0.75\sqrt{w_{18}} + 0.25\sqrt{w_1} - 0.1 \geq 0.25\sqrt{w_{18}} + 0.75\sqrt{w_1}. \quad \text{(IC)}$$

We can prove that in the two outcomes/two actions case both (IR) and (IC) must be binding at the optimum. If (IR) is not binding, the expected wage bill can be decreased by lowering w_1 whereas the (IC) will even be relaxed. If (IC) is not binding, but the (IR) is, then we can decrease w_{18} and increase w_1 such that the (IR) is still binding. In particular, given

$$\frac{dw_1}{dw_{18}} = -3\frac{\sqrt{w_1}}{\sqrt{w_{18}}},$$

this would change the expected wage bill by

$$0.75dw_{18} + 0.25(-3\frac{\sqrt{w_1}}{\sqrt{w_{18}}})dw_{18}.$$

By (IC), we have $w_1 < w_{18}$. Therefore, a decrease in w_{18} leads to a negative change in the wage bill,

$$0.75(1 - \sqrt{\frac{w_1}{w_{18}}})dw_{18} < 0$$

With the binding restrictions (IR) and (IC), it is easy to find the optimal wage schedule

$$\begin{aligned} w_1 &= 0.0025 \\ w_{18} &= 0.0625. \end{aligned}$$

Note that the limited liability constraint implied by the utility function $\sqrt{w}$ is not binding here. The expected utility of the principal is

$$E\pi_P^{**}(a_H) = 13.75 - 0.0625 \times 0.75 - 0.0025 \times 0.25 = 13.7025.$$

So the expected profit of inducing a_H is greater than the expected profit of inducing a_L, regardless of whether the effort level is observable or not.

4.3.3 Information System

For this specific price the answer is trivially no. The principal is not willing to buy the information system, because even with perfect information the highest profit he can attain is 13.71. Therefore, the maximal gain of using the information system will always be smaller than the cost of implementing the system,

$$13.71 - 13.7025 = 0.0075 < 0.1.$$

We now find the profits the principal can make when state θ_3 can be observed. Then we can find for which price the principal would be willing to pay for the information system. Let us denote by y the signal that takes the value 1 when θ_3 is realized and the value 0 otherwise.

If $y = 1$, the principal can still not identify the effort level being exerted. However, when $y = 0$, only the low effort level can lead to an output level of 1. So by penalizing the agent infinitely when $q = 1$ and $y = 0$, the first-best can be attained. Indeed, the agent will choose the high effort level to be sure to avoid the punishment. Given that no variation in received income is needed, the high effort can be implemented at the lowest cost. However, given the liability constraint implied by the utility function, the principal is restricted to positive levels of compensation. Hence, the principal will pay $w_1 = 0$ when the output is 1 and the signal is 0. In addition, w_{18} and w_{θ_3}, the wages paid when $q = 18$ and $y = 1, q = 1$ respectively, are set to solve

$$\min_{w_{18}, w_{\theta_3}} 0.75w_{18} + 0.25w_{\theta_3}$$

subject to

$$\begin{aligned}
0.75\sqrt{w_{18}} + 0.25\sqrt{w_{\theta_3}} - 0.1 &\geq 0.1 && \text{(IR)}\\
0.75\sqrt{w_{18}} + 0.25\sqrt{w_{\theta_3}} - 0.1 &\geq 0.25\sqrt{w_{18}} + 0.75\left(\frac{2}{3}w_1 + \frac{1}{3}\sqrt{w_{\theta_3}}\right). && \text{(IC)}
\end{aligned}$$

The incentive compatibility constraint is binding and simplifies to

$$0.75\sqrt{w_{18}} - 0.1 = 0.25\sqrt{w_{18}}.$$

Hence,

$$w_{18} = 0.04.$$

and

$$w_{\theta_3} = \left(\frac{0.2 - 0.75 \times 0.2}{0.25}\right)^2 = 0.04.$$

So the first-best can be achieved by the contract

$$w_{18} = w_{\theta_3} = 0.04 \text{ and } w_1 = 0.$$

Finally, from the calculation above, the principal will implement the information system if the cost falls below 0.0075.

4.4 Question 13

Consider the modified linear managerial-incentive-scheme problem, where the manager's effort, a affects current profits, $q_1 = a + \varepsilon_{q_1}$, and future profits, $q_2 = a + \varepsilon_{q_2}$, where ε_{q_t} are i.i.d. with normal distribution $N(0, \sigma_q^2)$. The manager retires at the end of the first period, and the manager's compensation cannot be based on q_2. However, her compensation can depend on the stock price $P = 2a + \varepsilon_P$, where $\varepsilon_P \sim N(0, \sigma_P^2)$. Derive the optimal compensation contract $t = w + fq_1 + sP$. Discuss how it depends on σ_P^2 and on its relation with σ_q^2. Compare your solution with that in the Chapter.

4.4.1 CEO Compensation

Note that this question uses notation that is not consistent with the exposition in section 4.6.1. The program for this problem is the following,

$$\max_{a,w,f,s} E\left(q_1 + q_2 - t\right)$$

subject to

$$\begin{aligned}
E\left(-e^{-\eta[t-\psi(a)]}\right) &\geq -e^{-\eta\bar{t}} && \text{(IR)}\\
a &\in \arg\max_{\widehat{a}} E\left(-e^{-\eta[t-\psi(\widehat{a})]}\right) && \text{(IC)}\\
t &= w + fq_1 + sP
\end{aligned}$$

$\Leftrightarrow$

$$\max_{a,w,f,s} 2a - (w + fa + 2sa)$$

subject to

$$\begin{aligned} w + fa + 2sa - \frac{\eta}{2}\left(f^2\sigma_q^2 + s^2\sigma_P^2 + 2sf\sigma_{qP}\right) - \frac{c}{2}a^2 &\geq \bar{t} \\ a \in \arg\max_{\widehat{a}} w + f\widehat{a} + 2s\widehat{a} - \frac{\eta}{2}\left(f^2\sigma_q^2 + s^2\sigma_P^2 + 2sf\sigma_{qP}\right) - \frac{c}{2}\widehat{a}^2. \end{aligned}$$

We can apply the first-order approach and substitute the first order condition

$$a = \frac{f + 2s}{c}$$

for the incentive compatibility constraint. Introducing this efficient level of effort and substituting the outside opportunity level plus the risk premium plus the cost of effort for the wage, we obtain

$$\max_{f,s} 2\frac{f + 2s}{c} - \frac{\eta}{2}\left(f^2\sigma_q^2 + s^2\sigma_P^2 + 2sf\sigma_{qP}\right) - \frac{(f + 2s)^2}{2c} - \bar{t}.$$

The first order conditions with respect to f and s are respectively

$$\begin{aligned} \frac{2}{c} - \eta\left(f^*\sigma_q^2 + s^*\sigma_{qP}\right) - \frac{f^* + 2s^*}{c} &= 0 \\ \frac{4}{c} - \eta\left(s^*\sigma_P^2 + f^*\sigma_{qP}\right) - 2\frac{f^* + 2s^*}{c} &= 0. \end{aligned}$$

After some rewriting, the equations become

$$\begin{aligned} f^* &= \frac{2 - s^*(2 + \eta c\sigma_{qP})}{1 + \eta c\sigma_q^2} \\ f^* &= \frac{2 - s^*(2 + \frac{\eta c\sigma_P^2}{2})}{1 + \frac{\eta c\sigma_{qP}}{2}}, \end{aligned}$$

and we finally find

$$\begin{aligned} f^* &= \frac{\sigma_P^2 - 2\sigma_{qP}}{2\sigma_q^2 + \frac{\sigma_P^2}{2} - 2\sigma_{qP} + \frac{\eta c}{2}(\sigma_P^2\sigma_q^2 - \sigma_{qP}^2)} \\ s^* &= \frac{2\sigma_q^2 - \sigma_{qP}}{2\sigma_q^2 + \frac{\sigma_P^2}{2} - 2\sigma_{qP} + \frac{\eta c}{2}(\sigma_P^2\sigma_q^2 - \sigma_{qP}^2)}. \end{aligned}$$

4.4.2 Comparison

Compared to the example in Chapter 4, the importance of the agent's effort has doubled for the principal. Among the contractible variables, only the impact of a change in effort on the stock price has doubled. Although the agent's effort now determines the output in two periods, the output in the second period is not contractible. The stock price therefore becomes a more precise indicator of effort than the contractible output. If we consider the relative size of the incentives at the optimum

$$\frac{f^*}{s^*} = \frac{\sigma_P^2 - 2\sigma_{qP}}{2\sigma_q^2 - \sigma_{qP}}$$

and compare this to the equivalent ratio in Chapter 4[1]

$$\frac{f^*}{s^*} = \frac{\sigma_P^2 - \sigma_{qP}}{\sigma_q^2 - \sigma_{qP}},$$

we notice that the principal puts relatively more weight on the stock prices. This weight is exactly double when both indicators are independent, $\sigma_{qP} = 0$.

4.5 Question 14

Consider the following principal-agent problem. There is a project whose probability of success is a (a is also the effort made by the risk-neutral agent, at cost a^2). In case of success the return is R, and in case of failure the return is 0. The parameter R can take two values, X with probability λ and 1 with probability $1 - \lambda$. To undertake the project, the agent needs to borrow an amount I from the principal. The sequence of events is as follows:

- First, the principal offers the agent a debt contract, with face value D_0. The agent accepts or rejects this contract.
- Second, nature determines the value R that would occur in case of success. This value is observed by both principal and agent. The principal can then choose to lower the debt from D_0 to D_1.
- The agent chooses a level of effort a. This level is not observed by the principal.
- The project succeeds or not. If the project succeeds, the agent pays the minimum of R and the face value of debt D_1.

Answer the following questions:

1. Compute the subgame-perfect equilibrium of this game as a function of I, λ, and X.
2. When do we have $D_1 < D_0$? Discuss.

[1] In section 4.6.1 f and s and w and t are interchanged.