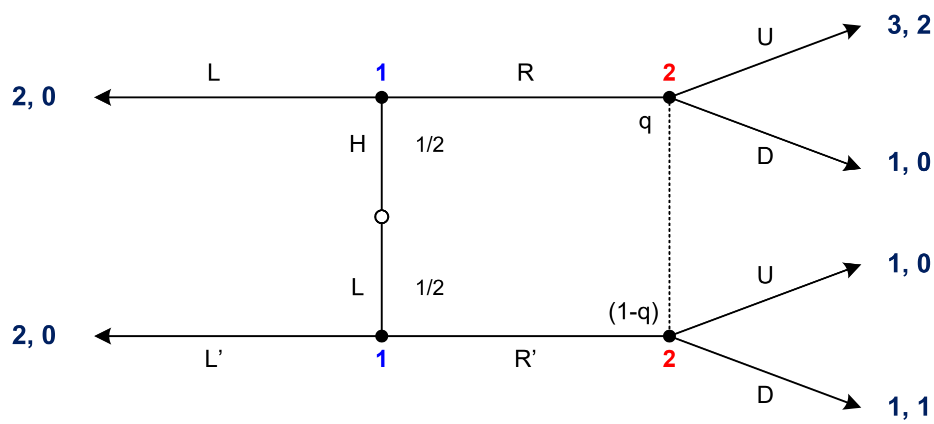
ECONS 424 – STRATEGY AND GAME THEORY

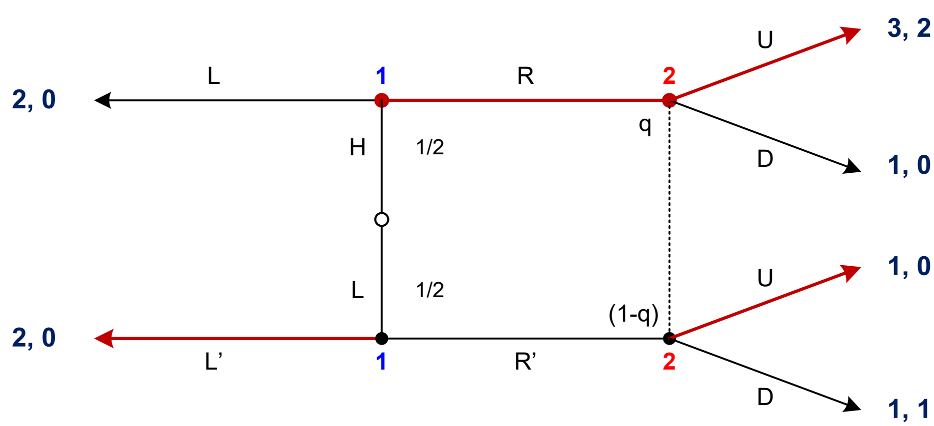
HOMEWORK #6 – ANSWER KEY

<u>Exercise 3-Chapter 28-Watson (*Checking the presence of separating and pooling equilibria*)</u>
Consider the following game of incomplete information:



a. Does this game have a Separating PBE? If so, fully describe it.

- **<u>First type of separating strategy profile: RL'</u>**



<u>First step (use Bayes' rule):</u>
After observing R, the second-mover's beliefs are

$$q = \frac{\frac{1}{2} \times 1}{\frac{1}{2} \times 1 + \frac{1}{2} \times 0} = 1$$

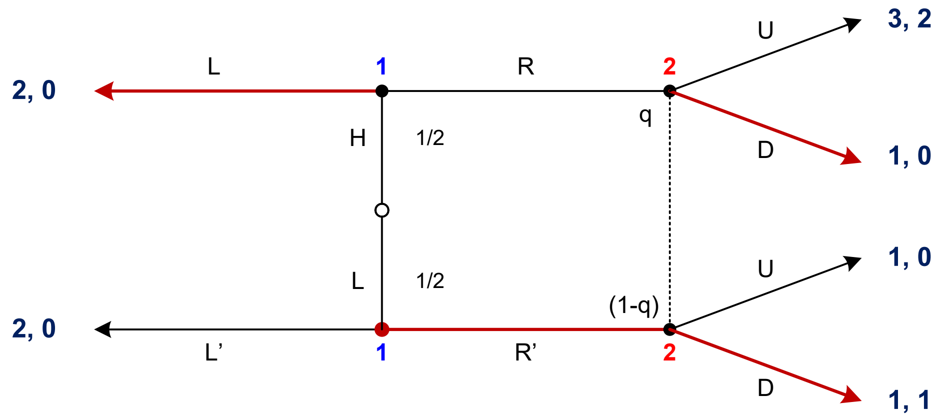
Intuitively, this implies that the second mover, after observing R, assigns full probability to R originating from an H-type of first mover, as indicated in the shaded R branch in the game tree.

Second step (focus on the second mover only):
After observing R, and given the second-mover's beliefs specified above, the second mover selects U, since 2>0.

Third step (we now analyze the first mover):

- When the first mover is H-type, he prefers to select R (as prescribed in this separating strategy profile) than deviate towards L since 3>2.
- When the first mover is L-type, he prefers to select L (as prescribed in this separating strategy profile) than deviate towards R since 2>1.
- Then, this separating strategy profile can be sustained as a PBE where (RL', U).

Second type of separating strategy profile, LR'



First step (use Bayes' rule):
After observing R, the second-mover's beliefs are

$$q = \frac{\frac{1}{2}\times 0}{\frac{1}{2}\times 1 + \frac{1}{2}\times 0} = 0$$

Intuitively, this implies that the second mover, after observing R, assigns full probability to R originating from an L-type of first mover, as indicated in the shaded R' branch in the game tree.

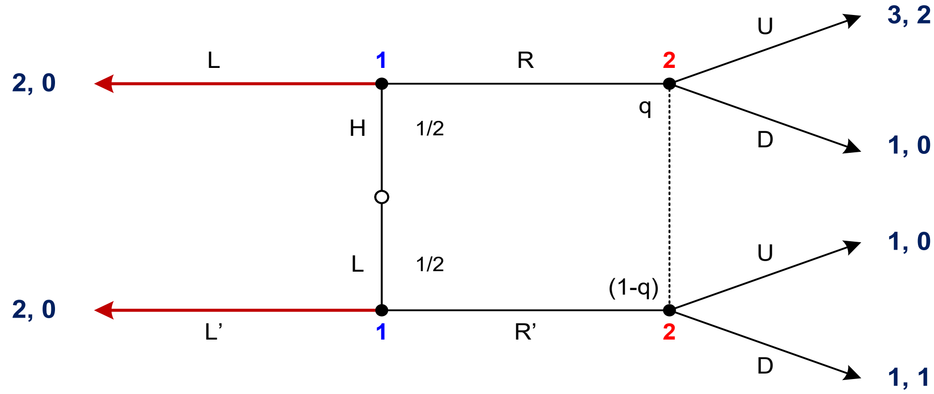
Second step (examine the second mover in the game):
After observing R, and given the second-mover's beliefs specified above, the second mover selects D, since 1>0.

Third step (analyze the first mover in the game):

- When the first mover is H-type, he prefers to select L (as prescribed in this separating strategy profile) than deviate towards R since 2>1.
- When the first mover is L-type, however, he prefers to deviate towards L than selecting R (the strategy prescribed for him in this separating strategy profile) since 2>1.
- Then, this separating strategy profile *cannot* be sustained as a PBE, since one of the players has incentives to deviate, namely, the L-type.

b. Does this game have a pooling PBE? If so, fully describe it.

- **First type of pooling strategy profile: LL'**



First step (use Bayes' rule):
After observing R (which occurs off-the-equilibrium, as indicated in the figure), the second-mover's beliefs are

$$q = \frac{0.5 * 0}{0.5 * 0 + 0.5 * 0} = \frac{0}{0}$$

Thus, off-the-equilibrium beliefs cannot be specified using Bayes' rule, and must be left undefined in the entire range of probabilities $q \in [0,1]$.

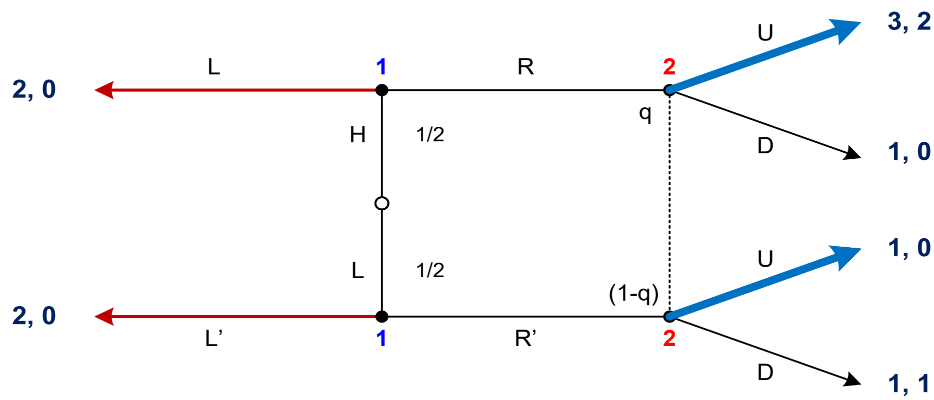
Second step (examine the second mover):
Let us analyze what the second-mover's optimal response is. Note that the second mover is only called on to move if the first mover chooses R, which occurs off-the-equilibrium. In such event, the second mover must compare his expected utility from choosing U versus that of selecting D, as follows

$$EU_2(U|R) = 2 \times q + 0 \times (1-q) = 2q$$
$$EU_2(D|R) = 0 \times q + 1 \times (1-q) = 1-q$$

Hence $EU_2(U|R) > EU_2(D|R)$ implies $2q > 1-q$, or simply $q > \frac{1}{3}$. Let us next divide our following analysis into two cases:

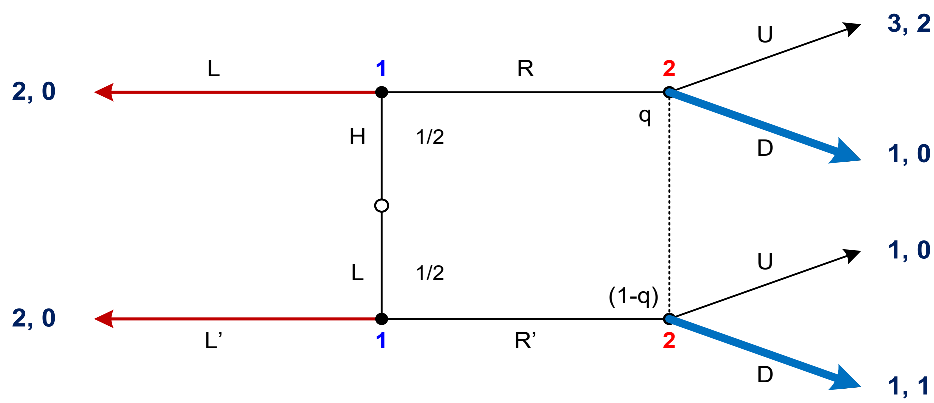
- ***Case 1:*** $q > \frac{1}{3}$, and thus the second mover responds selecting U; and
- ***Case 2:*** $q < \frac{1}{3}$, and thus the second mover responds selecting D;

CASE 1: $q > \frac{1}{3}$.



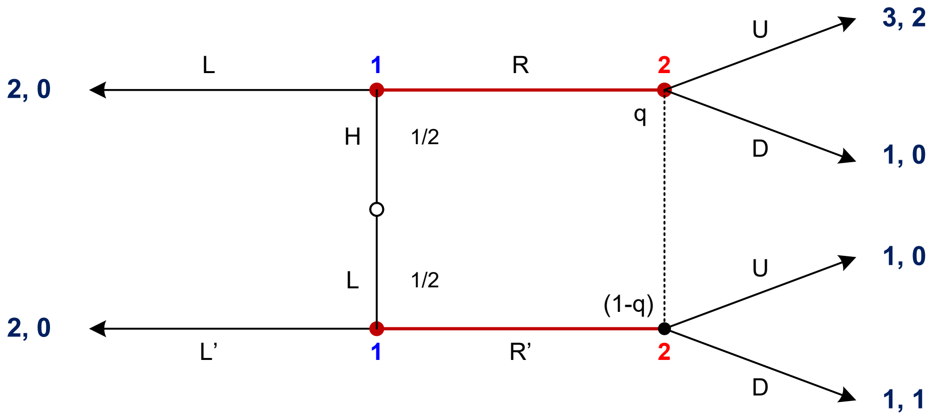
- When the first mover is H-type, he prefers to deviate towards R rather than selecting L (as prescribed in this strategy profile) since 3>2.
- This is already sufficient to conclude that this pooling strategy profile cannot be sustained as a PBE of the game.

CASE 2: $q < \frac{1}{3}$.



- When the first mover is H-type, he prefers to select L (as prescribed in this strategy profile) rather than deviating towards R since 2>1.
- When the first mover is L-type, he prefers to select L' (as prescribed in this strategy profile) rather than deviating towards R' since 2>1.
- Hence, this pooling strategy profile can be sustained as a PBE of the game when $q < \frac{1}{3}$.

Second type of pooling strategy profile: RR'



First step (use Bayes' rule):
After observing R (which occurs in-equilibrium, as indicated in the figure), the second-mover's beliefs are

$$q = \frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2}} = \frac{1}{2}$$

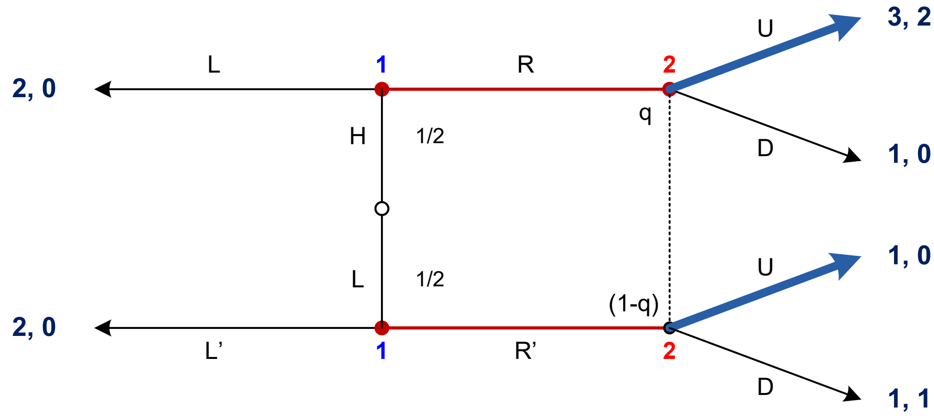
Therefore, the second-mover's updated beliefs after observing R coincide with the prior probability distribution over types.

Second step (examine the second mover in the game):
Let us analyze what the second-mover's optimal response is after observing R. Rhe second mover must compare his expected utility from choosing U versus that of selecting D, as follows

$$EU_2(U|R) = 2 \times \frac{1}{2} + 0 \times \left(1 - \frac{1}{2}\right) = 1$$
$$EU_2(D|R) = 0 \times \frac{1}{2} + 1 \times \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

Hence $EU_2(U|R) > EU_2(D|R)$ since 1>$\frac{1}{2}$, and the second mover chooses U.

Third step (analyze the first mover in the game):
Given that the second mover chooses U, we can shade the previous figure as follows:



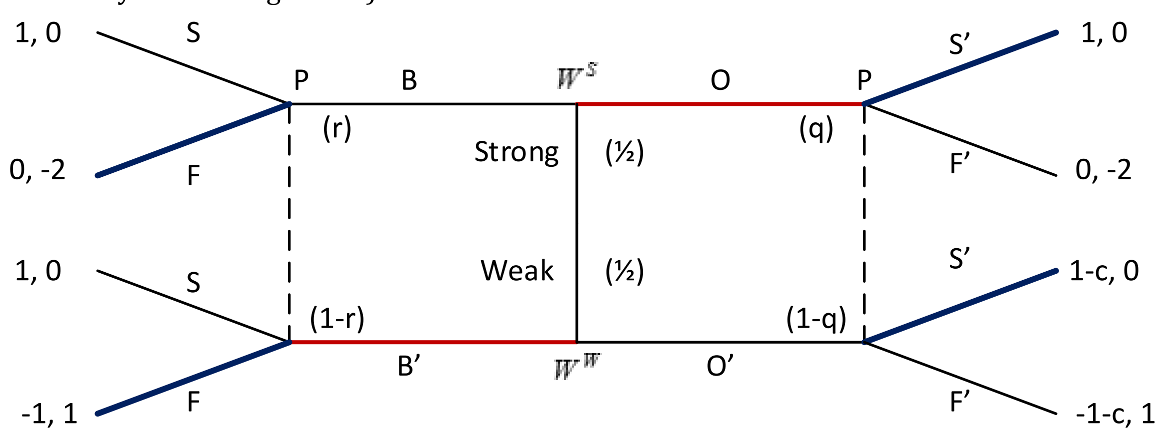
- When the first mover is H-type, he prefers to select R (as prescribed in this strategy profile) rather than deviating towards L since 3>2.
- When the first mover is L-type, he prefers to deviate towards L' rather than selecting R' (as prescribed in this strategy profile) since 2>1.
- Hence, this pooling strategy profile *cannot* be sustained as a PBE of the game since the L-type of first mover has incentives to deviate.

Exercise 6-Chapter 28-Watson

A. Let's first consider the separating PBE where Wesley gets out of bed when strong, and stays in bed when weak:

$$OB'$$

(Strategies for Wesley are given in the form of his action when strong, followed by his action when weak, $W^S W^W$. Strategies for Humperdinck are given in the form of his action given B, followed by his action given O.)



1. Prince Humperdinck's beliefs about Wesley's type in the separating strategy profile OB':

$r = 0$ and $q = 1$

2. Humperdinck's Best Responses given Wesley's action:

- After observing "Stay in Bed (B)" :

$$EU_{PH}(S|B) = 0 * 0 + 1 * 0 = 0$$
$$EU_{PH}(F|B) = 0 * -2 + 1 * 1 = 1$$

So, when Wesley stays in bed, Prince Humperdinck prefers to fight $[1 > 0]$.

- After observing "Get out of Bed (O)" :

$$EU_{PH}(S|O) = 1 * 0 + 0 * 0 = 0$$
$$EU_{PH}(F|O) = 1 * -2 + 0 * 1) = -2$$

When Wesley gets out of bed, Prince Humperdinck prefers to surrender since 0>-2.

3. Given Humperdinck's responses and beliefs, is Wesley's strategy optimal?

- When Wesley is Strong, i.e., W^S, he should:

$$EU_{W^S}(B) = 0$$
$$EU_{W^S}(O) = 1$$

Because the expected utility from getting out of bed is greater than staying in bed, Wesley will get out of bed when he is the strong type.

- When Wesley is Weak, i.e., W^W, he should:

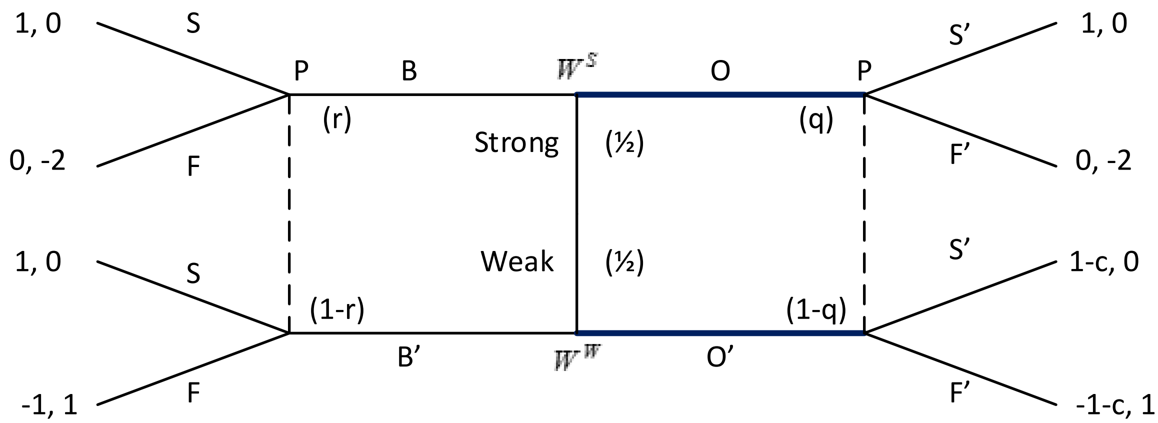
$$EU_{W^W}(B') = -1$$
$$EU_{W^W}(O') = 1 - c$$

In order for his strategy to stay in bed to be optimal when weak, we need $EU_{W^W}(B') > EU_{W^W}(O')$ to hold. Hence, we need

$$-1 > 1 - c, \quad \text{or} \quad c>2.$$

So, when **r=0, q=1, and c > 2, the strategy profile OB' FS' can be supported a separating PBE!**

B. Is it possible for OO' to be sustained as a pooling PBE?



1. First consider Prince Humperdinck's beliefs in this pooling strategy profile:

$$q = \frac{\frac{1}{2} * p^{strong}}{\frac{1}{2} * p^{strong} + \frac{1}{2} p^{weak}} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

$$r = \frac{\frac{1}{2} * (1 - p^{strong})}{\frac{1}{2} * (1 - p^{strong}) + \frac{1}{2} * (1 - p^{weak})} = \frac{\frac{1}{2} * 0}{\frac{1}{2} * 0 + \frac{1}{2} * 0} = \frac{0}{0}$$

After observing that Wesley stays in bed (B), which occurs off-the-equilibrium path, Prince Humperdinck's off-the-equilibrium beliefs cannot be restricted, i.e., $r \in [0,1]$.

2. Humperdinck's best responses:

   - After observing "Get out of Bed (O)",

   $$EU_{PH}(S'|O) = 0 * \frac{1}{2} + 0 * \frac{1}{2} = 0$$

   $$EU_{PH}(F'|O) = \frac{1}{2} * (-2) + \frac{1}{2} * 1 = -\frac{1}{2}$$

   Humperdinck selects S' after observing "Get out of Bed (O)" since $0 > -\frac{1}{2}$.

   - After observing "Stay in bed (B)",

   $$EU_{PH}(S|B) = 0 * r + 0 * (1 - r) = 0$$

   $$EU_{PH}(F|B) = -2 * r + 1 * (1 - r) = 1 - 3r$$

   Humperdinck selects F after observing B if and only if:

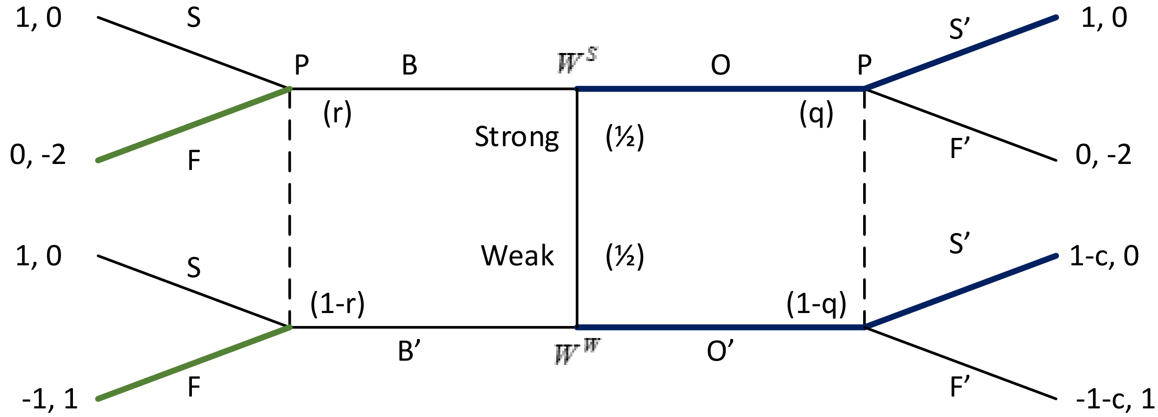
   $$1 - 3r > 0, \text{ or } \frac{1}{3} > r$$

   So we have two cases to separately consider in our following analysis:

   1. ***Case 1:*** $r < \frac{1}{3}$ entailing that Humperdinck plays F given B
   2. ***Case 2:*** $r > \frac{1}{3}$ entailing that Humperdinck plays S given B

3. Is Wesley's strategy optimal?

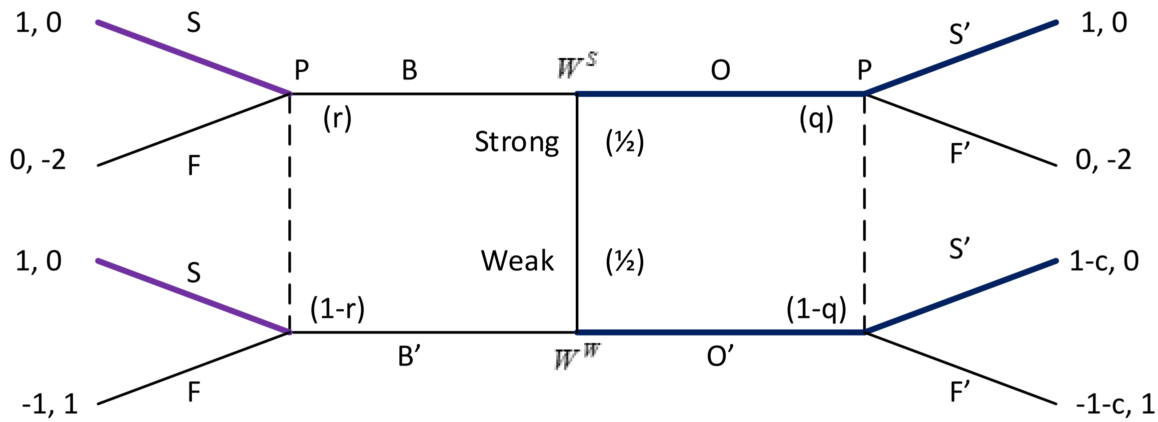
Case 1: $r<\frac{1}{3}$



- When strong, W^S, Wesley selects O (as prescribed) since 1>0.
- When Weak, W^W, Wesley selects O' (as prescribed) if and only if 1-c>-1, or c<2.

Hence, the strategy profile OO' FS' may be supported as a pooling PBE when beliefs satisfy q=1/2 and r<1/3, and parameter c is sufficiently low, i.e., c<2.

Case 2: $r>\frac{1}{3}$.



- When strong, W^S, Wesley selects O (as prescribed) since 1=1. Since he is indifferent between B and O, we can assume that he selects O.
- When Weak, W^W, Wesley selects O' (as prescribed) if and only if 1-c>1, or c<0.

So, the strategy profile OO' SS' may be supported as a pooling PBE when beliefs satisfy q=1/2 and r>1/3, and parameter c is negative, i.e., c<0.

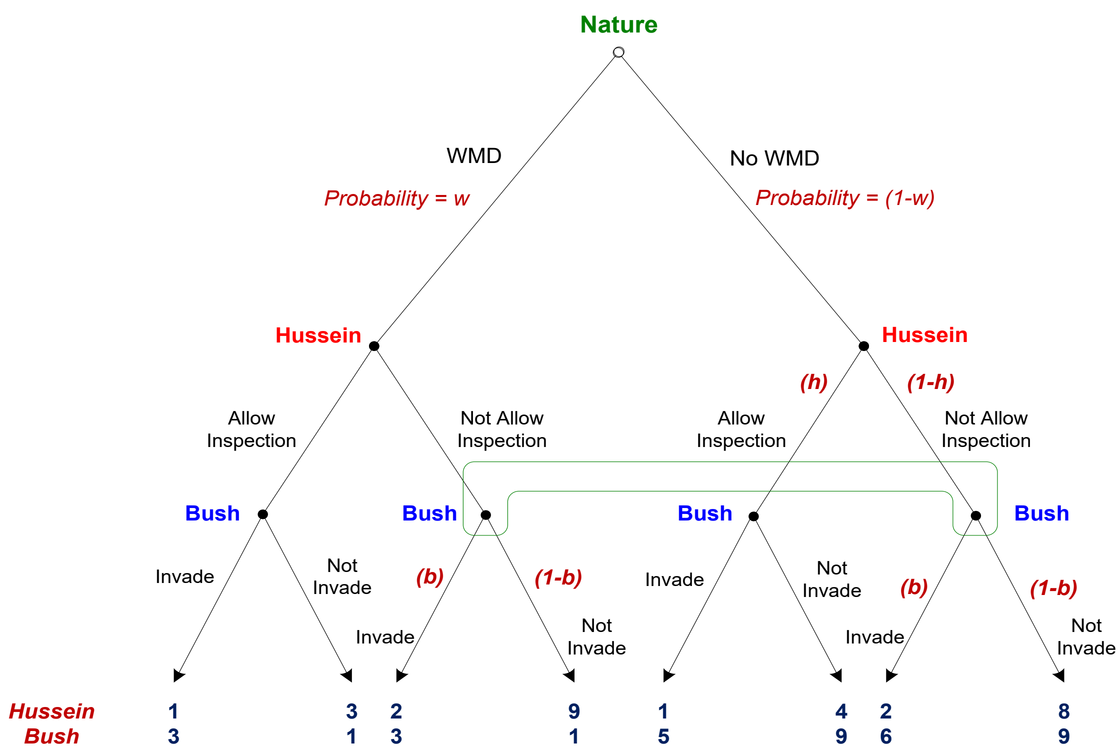
Summary:

- Overall, a pooling PBE with OO' may be sustained regardless of the precise value of r (both in case 1 and 2) so long as c<0.

- Note, however, that if the problem had specified that cost c must satisfy c>0 (which seems to be a sensible assumption for the cost of getting out of bed for the weak type of player) then a pooling PBE could only be sustained by the conditions specified under Case 1.

Exercise 4-Chapter 11-Harrington

The extensive form of the WMD game:



1. Nature moves first determining a presence of WMD:

- with probability w Hussein has WMD
- with probability $(1-w)$ he does not, where $0 < w < 1/3$

2. After observing his own type, Hussein's strategies are the following:

- when he <u>has</u> WMD then he does not allow inspections with probability 1.
- when he <u>does not have</u> WMD then he can choose either to allow inspection with probability h or do not allow – with probability $(1-h)$.

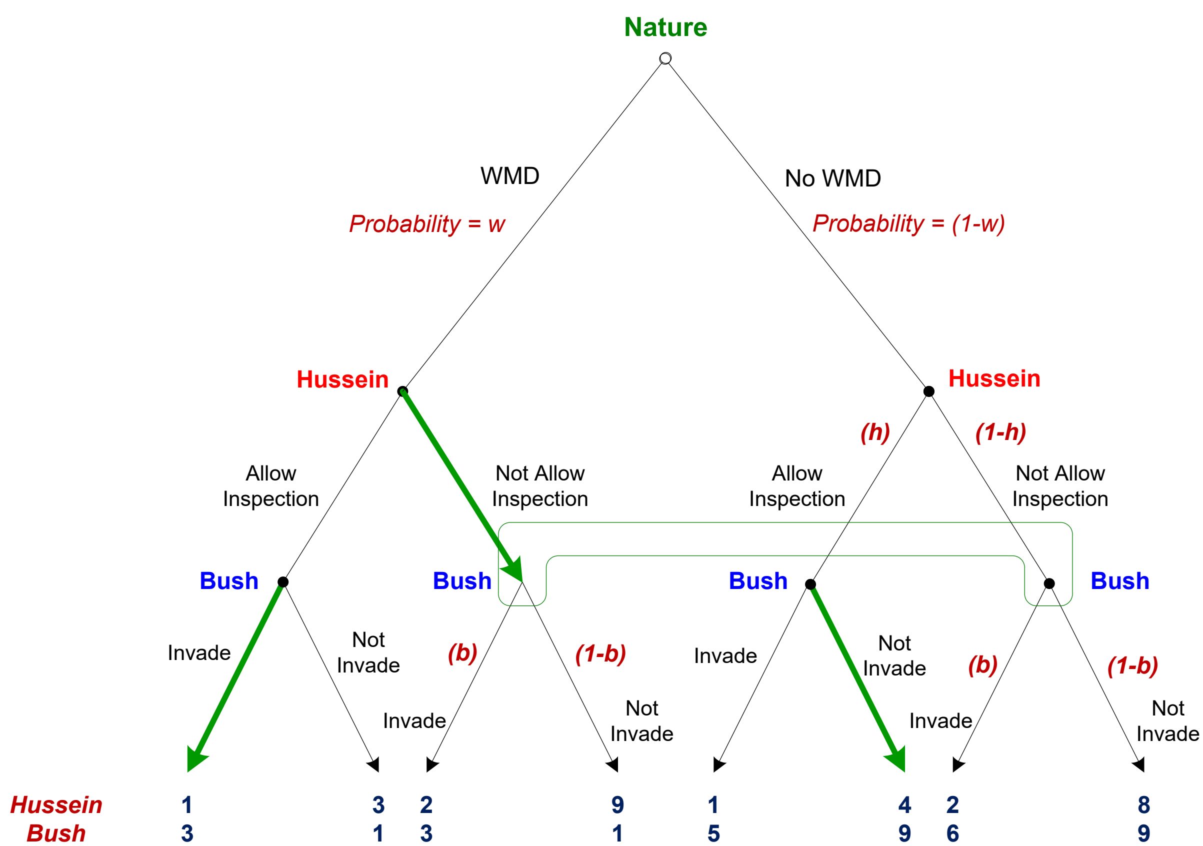
3. Assumptions:

- If Hussein has WMD, then Bush found out it and then Bush wants to invade;
- If Hussein does not have WMD, then Bush does not found it and he prefers not to invade.

After observing Hussein's decision about inspection, Bush strategies are:

- if Hussein <u>allows</u> inspections and WMD are <u>found</u>, then invade with probability 1.
- if Hussein <u>allows</u> inspections and WMD are <u>not</u> <u>found</u>, then do not invade with probability 1.
- if Hussein does not allow inspections, then Bush can guess that with some probability Hussein has WMD, so that Bush invade with probability *b*.

See graph 1 below:



Steps:

- **Step 1: Bush's beliefs**

  If Hussein does not allow inspections, then the probability of Hussein's having WMD is given by Bayes's rule:

$$P(WMD|Not\ Allow) = \frac{P(WMD) \times P(Not\ Allow, WMD)}{P(Not\ Allow)} = \frac{w \times 1}{w \times 1 + (1-w) \times (1-h)}$$

where Sadam has WMD with probability w, and in that event, he does not allow inspections with probability 1; and while with probability $(1-w)$, Sadam has WMD and, in that event, he does not allow inspections with probability $(1-h)$.

- **Step 2: Bush's optimal strategy given his beliefs**

  Its optimality is clear when there are inspections, whether WMD are found or not.

  - When inspections <u>are not allowed</u>, Bush is content to randomize (that is, $0 < b < 1$) if and only if:

$$E^{Bush}[INV|WMD\ or\ No\ WMD] = E^{Bush}[No\ INV|WMD\ or\ No\ WMD\ and\ NA]$$

$$3\left[\frac{w\times 1}{w\times 1+(1-w)\times(1-h)}\right]+6\left[\frac{w\times(1-h)}{w\times 1+(1-w)\times(1-h)}\right]$$
$$=\left[\frac{w\times 1}{w\times 1+(1-w)\times(1-h)}\right]+9\left[\frac{w\times(1-h)}{w\times 1+(1-w)\times(1-h)}\right]$$

The left-hand expression is the expected payoff from invading, and the right-hand expression is the expected payoff from not invading. Solving this equation for h yields:

$$h=\frac{3-5w}{3(1-w)}$$

Note that:

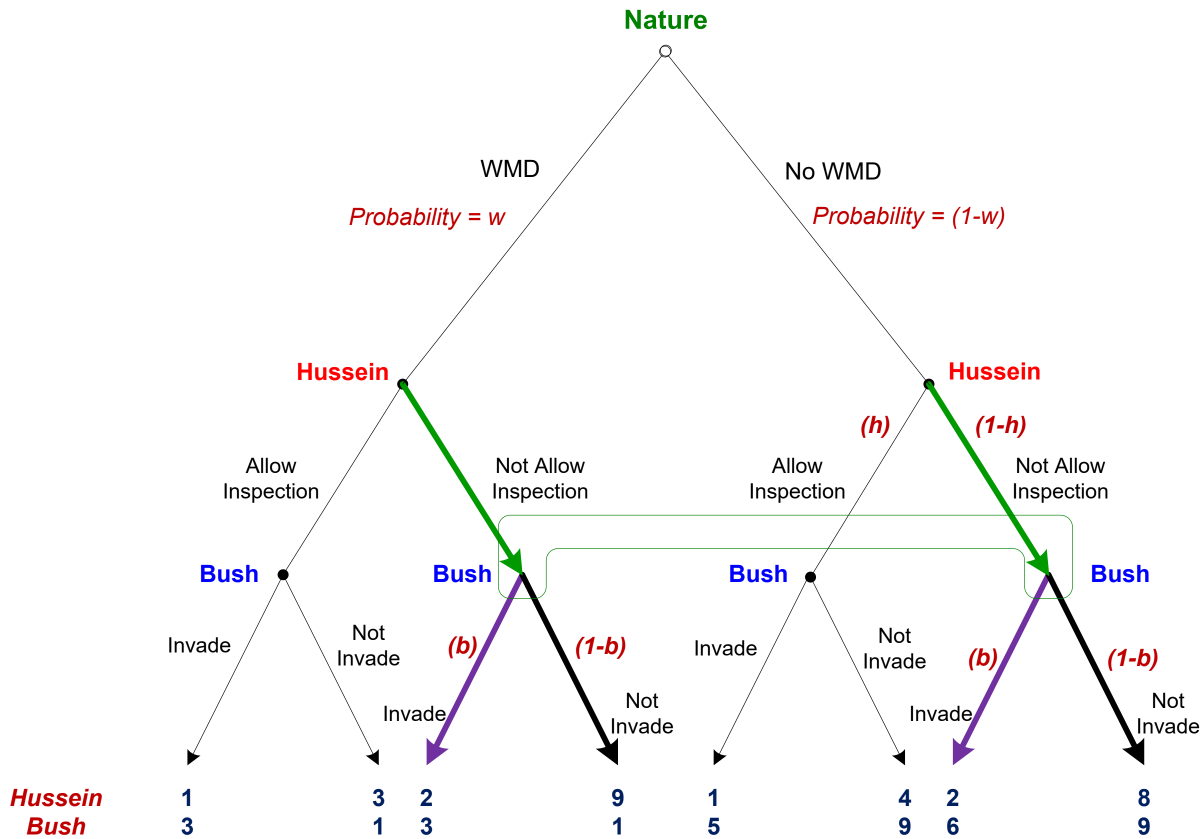
$$0<\frac{3-5w}{3(1-w)}<1\quad when\ \ 0<w<\frac{3}{5}$$

The latter condition was assumed. See graph 2 below.

  - When Sadam<u> has WMD</u>, it is clearly optimal for him to not allow inspections. When he does not have WMD, it is optimal to randomize if and only if:

$$2b+8(1-b)=4$$

where he earns a payoff of 4 by allowing inspections – in which case there is no invasion-- and gets an expected payoff of $2b+8(1-b)=4$ from not allowing inspections (where there is an invasion with probability b). Solving this equation, we can get $b=2/3$.



$$E^{Bush}[INV|WMD\ or\ No\ WMD] = P(NA|WMD) \times U(INV) + P(NA|No\ WMD) \times U(INV)$$
$$= 3 \times P(NA|WMD) + 6 \times P(NA|No\ WMD)$$
$$E^{Bush}[NINV|WMD\ or\ No\ WMD\ and\ NA] = 1 \times P(NA|WMD) + 9 \times P(NA|No\ WMD)$$

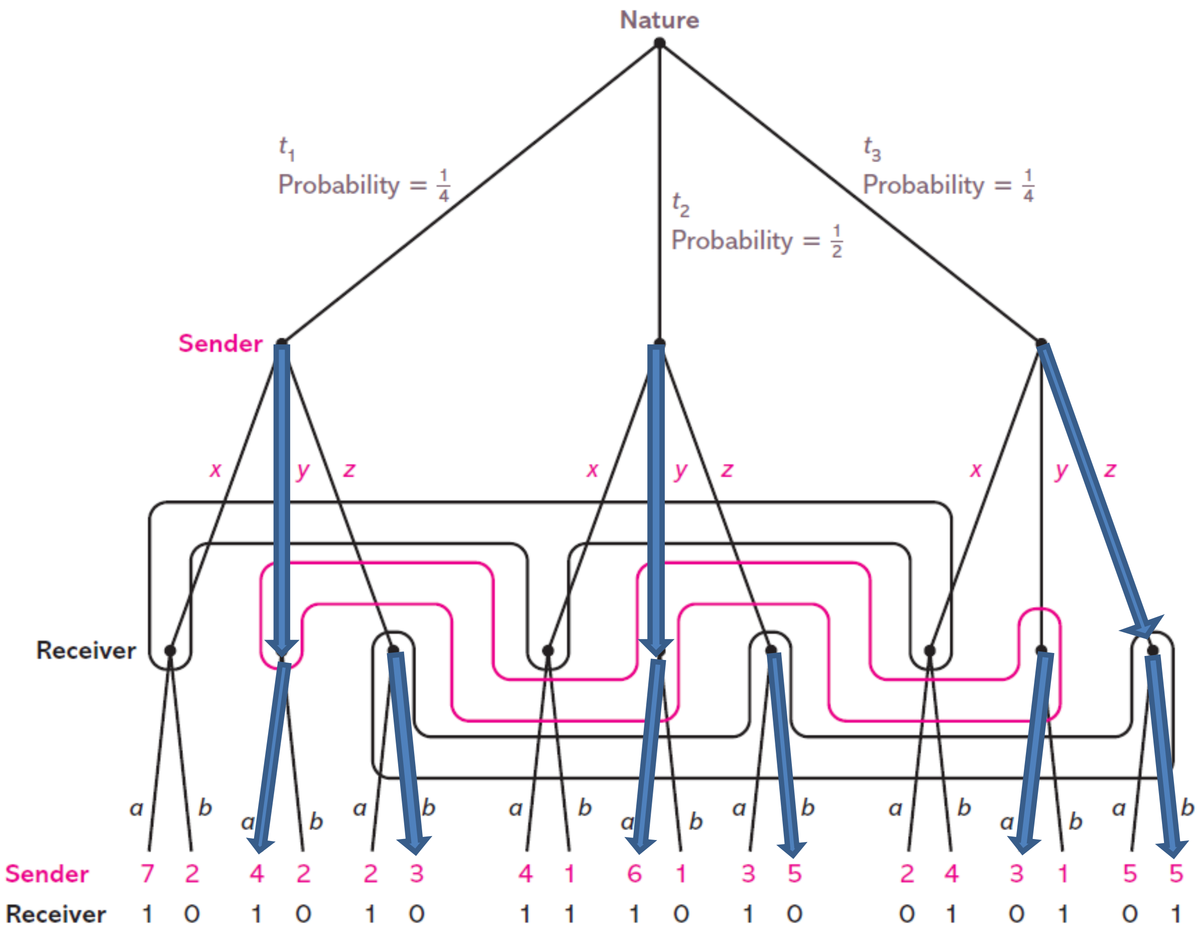
Exercise 7, Chapter 11 - HARRINGTON

Looking at the payoffs for the sender when she is type **t3.**
In this case, sender's optimal actions is **z** because it gives a highest payoff of 5 regardless how the receiver responds (a or b) and this payoff is higher than the payoff of choosing x or y.

While considering a semi-separating strategy profile in which type t1 and t2 choose an action distinct from t3, the receiver will be able to infer from action z that the sender is type t3. Hence, the receiver's optimal response to **z** is then action **b (**shade) since 1>0.

Consider **Semi-Separating Profile where types t1 and t2 choosing action y and type t3 chooses z (shaded).**

FIGURE PR11.7



A. <u>Beliefs</u>

- After observing message *y*, the probability that such action originates from sender type t1, t2 and t3, can be computed using Bayes' rule, as follows

$$prob(t_1|y) = \frac{\frac{1}{4}*1}{\frac{1}{4}*1+\frac{1}{2}*1+\frac{1}{4}*0} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

$$prob(t_2|y) = \frac{\frac{1}{2}*1}{\frac{1}{4}*1+\frac{1}{2}*1+\frac{1}{4}*0} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$$

$$prob(t_3|y) = 0$$

- After observing message z, the probability that such action originates from sender type t1, t2 and t3, can be computed using Bayes' rule, as follows

$$prob(t_1|z) = \frac{\frac{1}{4}*0}{\frac{1}{4}*0+\frac{1}{2}*0+\frac{1}{4}*1} = 0$$

$$prob(t_2|z) = \frac{\frac{1}{2} * 0}{\frac{1}{4} * 0 + \frac{1}{2} * 0 + \frac{1}{4} * 1} = 0$$

$$prob(t_3|z) = \frac{\frac{1}{4} * 1}{\frac{1}{4} * 0 + \frac{1}{2} * 0 + \frac{1}{4} * 1} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$$

- Regarding message *x*, we know that this can only occur off-the-equilibrium path, since no type of sender selects this message in the strategy profile we are testing. Hence, the receiver's off-the-equilibrium beliefs are

$$prob(t_1|x) = \gamma_1 \epsilon [0,1]$$

(Recall that, as described in class, the use of Bayes' rule doesn't provide a precise value for γ_1, and we must leave the receiver's beliefs unrestricted in the interval $\gamma_1 \epsilon [0,1]$).
Similarly, the conditional probability that such message of x originates from a type t2 sender is

$$prob(t_2|x) = \gamma_2 \epsilon [0,1],$$

And therefore,

$$and\ prob(t_3|x) = 1 - \gamma_1 - \gamma_2$$

B. Receiver

- After observing y, he responds with either a or b depending on which action yields him the highest expected utility. In particular,

$$EU_2(a|y) = \tfrac{1}{3} * 1 + \tfrac{2}{3} * 1 = 1$$

$$EU_2(b|y) = \tfrac{1}{3} * 0 + \tfrac{2}{3} * 0 = 0$$

} Hence, the receiver selects a after observing y

- After observing z, the receiver similarly compares his utility from a and b, as follows. (Note that in this case, the receiver doesn't need to compute expected utilities, since he is convinced to be dealing with a t3-type of sender, i.e., in the node at the right-hand side of the game tree)

$$EU_2(a|z) = 0$$

$$EU_2(b|z) = 1$$

} Thus, the receiver selects b after observing message z

- After observing *x* (off-the-equilibrium path), the receiver compares his expected utility from selects a and b, as follows

$$EU_2(a|x) = \gamma_1 * 1 + \gamma_2 * 1 + (1 - \gamma_1 - \gamma_2) * 0 = \gamma_1 + \gamma_2$$

$$EU_2(b|x) = \gamma_1 * 0 + \gamma_2 * 1 + (1 - \gamma_1 - \gamma_2) * 1 = 1 - \gamma_1$$

- Hence, after observing *x*, the receiver chooses a iff $\gamma_1 + \gamma_2 > 1 - \gamma_1$, or $\gamma_2 > 1 - 2\gamma_1$

1. **Sender's optimal actions given previous points:**

   - If he is type **t1**, sender send message **Y** since
     4>3>2 or payoff if plays (a given Y)> payoff if (b given Z)> payoff if (b given X)

   - If he is type **t2**, sender send message **Y** since
     6>5>1 or payoff if plays (a given Y)> payoff if (b given Z)> payoff if (b given X)

   - If he is type **t3**, sender send message **Z** since
     5>4>3 or payoff if plays (b given Z)> payoff if (b given X)>payoff if (a given Y)

So, proved that given semi-separating equilibrium can be supported.

Summary:

Sender's strategy:

If my type is **t1 or t2**, then choose action **y**.
In my type if **t3**, then choose action **z**.

Receiver's strategy and beliefs:

If action is x, then choose **b** if and only if $1 \geq 2\gamma_1 + \gamma_2$. Then the sender is type t1 with probability γ_1, type t2 with probability γ_2, and type t3 with prob. $1-\gamma_1-\gamma_2$

If action is **z** , then choose **b** with probability 1. Then the sender is type t3.

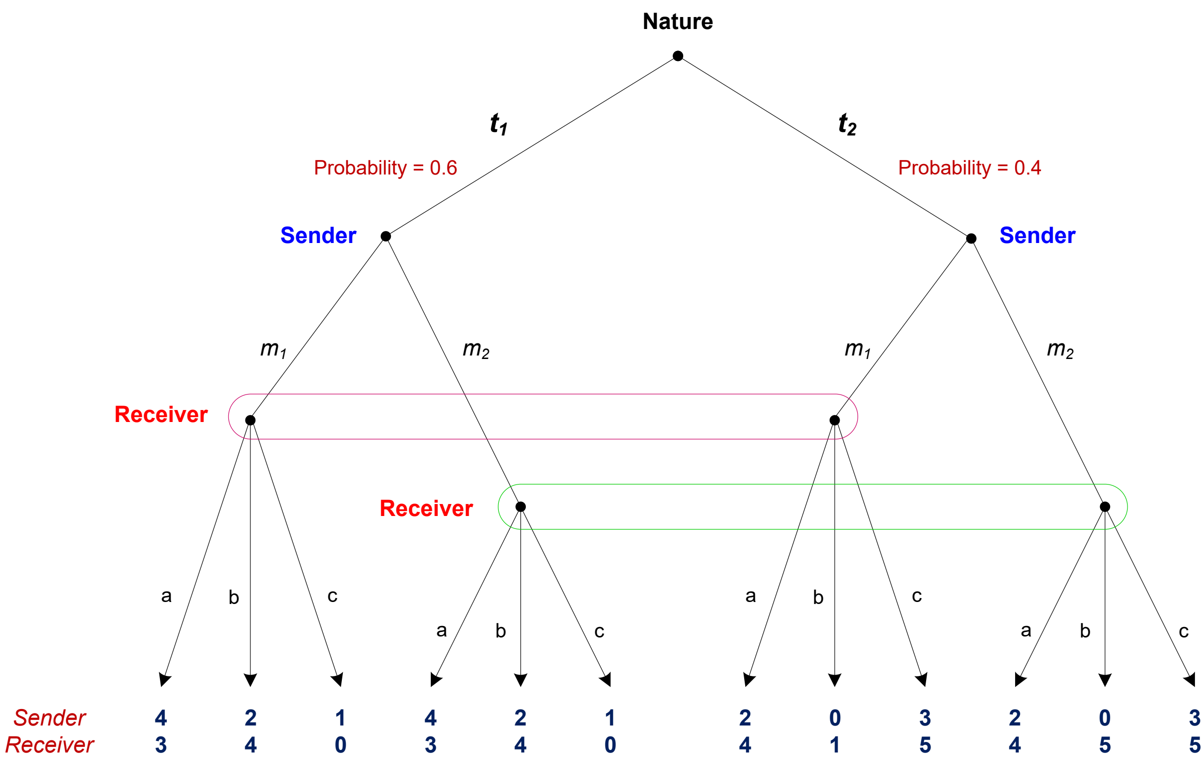
If the action is **y**, then choose action **a.** Then the sender is type t1 with prob. 1/3 and type t2 with prob. 2/3.

Consider the second semi-separating equilibrium where type t1 and t2 chooses X and type t2 chooses Z

If that is part of an equilibrium, then the receiver will infer from x that the sender is either t1 or t2 and thus responds by choosing action a. By choosing x, a type t1 sender will then get a payoff of 7 and a type t2 will get a payoff of 4. But that is not optimal for a type t2 sender, as she can get a payoff of 5 by choosing z (where, recall, the receiver will respond with b).

Exercise 5-Chapter 12-Harrington

Consider the cheap talk game:



a. Find a separating PBNE.

With a separating equilibrium, the sender chooses distinct messages, so let us presume that the sender chooses m_1 when his type is t_1 and chooses m_2 when his type is t_2. (We could instead have supposed that the sender's strategy is to choose m_2 when his type is t_1, and m_1 when his type is t_2.)

Receiver's beliefs

After observing message m_1,

$$\mu(t_1|m_1) = 1$$
$$\mu(t_2|m_1) = 0$$

And after observing message m_2,

$$\mu(t_1|m_2) = 0$$
$$\mu(t_2|m_2) = 1$$

Receiver's optimal response

- After observing m_1, the receiver believes that such a message can only originate from a t_1-type of sender. Hence, his optimal response is b given that it yields a payoff of 4 (higher than what he gets from a, 3, and c, 0.)
- After observing message m_2, the receiver believes that such a message can only originate from a t_2-type of sender. Hence, his optimal response is either b or c, since both yield a payoff of 5, rather than a, which only provides a payoff of 4. For simplicity, we choose c.

Sender's optimal messages

- If his type is t_1, by sending m_1 he obtains a payoff of 2 (since m_1 is responded with b), but a lower payoff of 1 if he deviates towards message m_2 (since such message is responded with c). Hence, the sender doesn't want to deviate from m_1. [Note that if m_2 were responded with b, then the sender would be indifferent between m_1 and m_2 (both would yield a payoff of 2). Strictly speaking, he wouldn't have incentives to deviate from message m_1].
- If his type is t_2, he obtains a payoff of 3 by sending message m_2 (which is responded with c) and a payoff of 0 if he deviates to message m_1 (which is responded with b). Hence, he doesn't have incentives to deviate from m_2. [Similarly as above, note that if message m_2 were responded with b the sender would obtain the same payoff sending m_2 and m_1, 0. Nonetheless, t_2-sender wouldn't have incentives to deviate from his initially prescribed message of m_2].

Hence, the initially prescribed separating strategy profile can be supported as a PBE.
Also note that there is another separating equilibrium in which m_1 and m_2 are exchanged.

b. Find a pooling PBNE.

With a pooling PBNE, the sender chooses the same message regardless of his type. Let this message be m_1.

Receiver's beliefs

After observing message m_1 (in problem),

$$\mu(t_1|m_1) = \frac{0.6 * 1}{0.6 * 1 + 0.4 * 1} = 0.6$$

$$\mu(t_2|m_1) = \frac{0.4 * 1}{0.6 * 1 + 0.4 * 1} = 0.4$$

After receiving message m_2 (off-the-equilibrium),

$$\mu(t_1|m_1) = \frac{0.6 * 0}{0.6 * 0 + 0.4 * 0} = \frac{0}{0}$$

and hence beliefs must be arbitrarily specified, i.e. $\mu \in [0,1]$.

<u>Receiver's optimal response</u>

- After receiving a message m_1, the receiver's expected utility from responding with actions a, b, and c are

$$\text{Action } a\text{:} \quad 0.6 \times 3 + 0.4 \times 4 = 3.4$$

$$\text{Action } b\text{:} \quad 0.6 \times 4 + 0.4 \times 1 = 2.8$$

$$\text{Action } c\text{:} \quad 0.6 \times 0 + 0.4 \times 5 = 2.0$$

Hence, the receiver's optimal strategy is to choose action a in response to message m_1.

- After receiving message m_2 (off-the-equilibrium), the receiver's EUs from each of his three possible responses are

$$EU_{Receiver}(a|m_2) = \mu * 3 + (1 - \mu) * 4 = 4 - \mu$$

$$EU_{Receiver}(b|m_2) = \mu * 4 + (1 - \mu) * 5 = 5 - \mu$$

$$EU_{Receiver}(c|m_2) = \mu * 0 + (1 - \mu) * 5 = 5 - 5\mu$$

  Where it's clear that the EU from responding with b, $5 - \mu$, is the highest EU payoff the responder can obtain given that $\mu \in [0,1]$.

<u>Sender's optimal message</u>

- If his type is t_1, the sender obtains a payoff of 4 from sending m_1 (since it is responded with a), but a payoff of only 2 when deviating towards m_2 (since it is responded with b). Hence, he doesn't have incentives to deviate from m_1.
- If his type is t_2 , the sender obtains a payoff of 2 by sending m_1 (since it is responded with a), but a payoff of only 0 by deviating towards m_2 (since it is responded with b). Hence, he doesn't have incentives to deviate from m_1.

Therefore, the initially prescribed pooling strategy profile where both types of sender select $m_1$can be sustained as a PBE of the game.

There are other babbling equilibria that differ in terms of the message sent by the sender and the receiver's beliefs in response to a message that the sender never sends (according to his strategy). For any babbling equilibrium, it must be the case that the receiver ends up choosing action a.

c. Suppose the probability that the sender is type t_1 is p and the probability that the sender is type t_2 is $(1 - p)$. Find the values for p such that there is a pooling PBNE in which the receiver chooses action b.

For any pooling equilibrium, the sender's strategy has him choose the same message—let it be m_1 – for any type and, in response to observing that message; the receiver's beliefs are her prior beliefs.

<u>Receiver's beliefs</u>

After observing message m_1,

$$\mu(t_1|m_1) = \frac{p * 1}{p * 1 + (1 - p) * 1} = p$$

$$\mu(t_2|m_1) = \frac{(1 - p) * 1}{p * 1 + (1 - p) * 1} = 1 - p$$

After receiving message m_2 (off-the-equilibrium),

$$\mu(t_1|m_1) = \frac{p * 0}{p * 0 + (1 - p) * 0} = \frac{0}{0}$$

And hence beliefs must be arbitrarily specified, i.e. $\mu \in [0,1]$.

<u>Receiver's optimal response</u>

- After receiving a message m_1 (in equilibrium), the receiver's expected utility from responding with actions a, b, and c are

$$\text{Action } a\text{:} \quad p \times 3 + (1 - p) * 4 = 4 - p$$

$$\text{Action } b\text{:} \quad p \times 4 + (1 - p) * 1 = 1 + 3p$$

$$\text{Action } c\text{:} \quad p \times 0 + (1 - p) * 5 = 5 - 5p$$

For it to be optimal to choose action b, it must be the case that

$$1 + 3p \geq 4 - p \quad \rightarrow \quad p \geq \frac{3}{4} \quad and \quad 1 + 3p \geq 5 - 5p \quad \rightarrow \quad p \geq \frac{1}{2}$$

Thus if $p < 3/4$, then the receiver does not choose action b at a pooling equilibrium, as she would prefer action a. If $p \geq \frac{3}{4}$, then it is the receiver's optimal strategy to choose b in response to message m_1.

- After receiving message m_2 (off-the-equilibrium), the receiver's EUs from each of his three possible responses are

$$EU_{Receiver}(a|m_2) = \mu * 3 + (1 - \mu) * 4 = 4 - \mu$$

$$EU_{Receiver}(b|m_2) = \mu * 4 + (1 - \mu) * 5 = 5 - \mu$$

$$EU_{Receiver}(c|m_2) = \mu * 0 + (1-\mu) * 5 = 5 - 5\mu$$

Where it's clear that the EU from responding with b, $5-\mu$, is the highest EU payoff the responder can obtain given that $\mu \in [0,1]$.

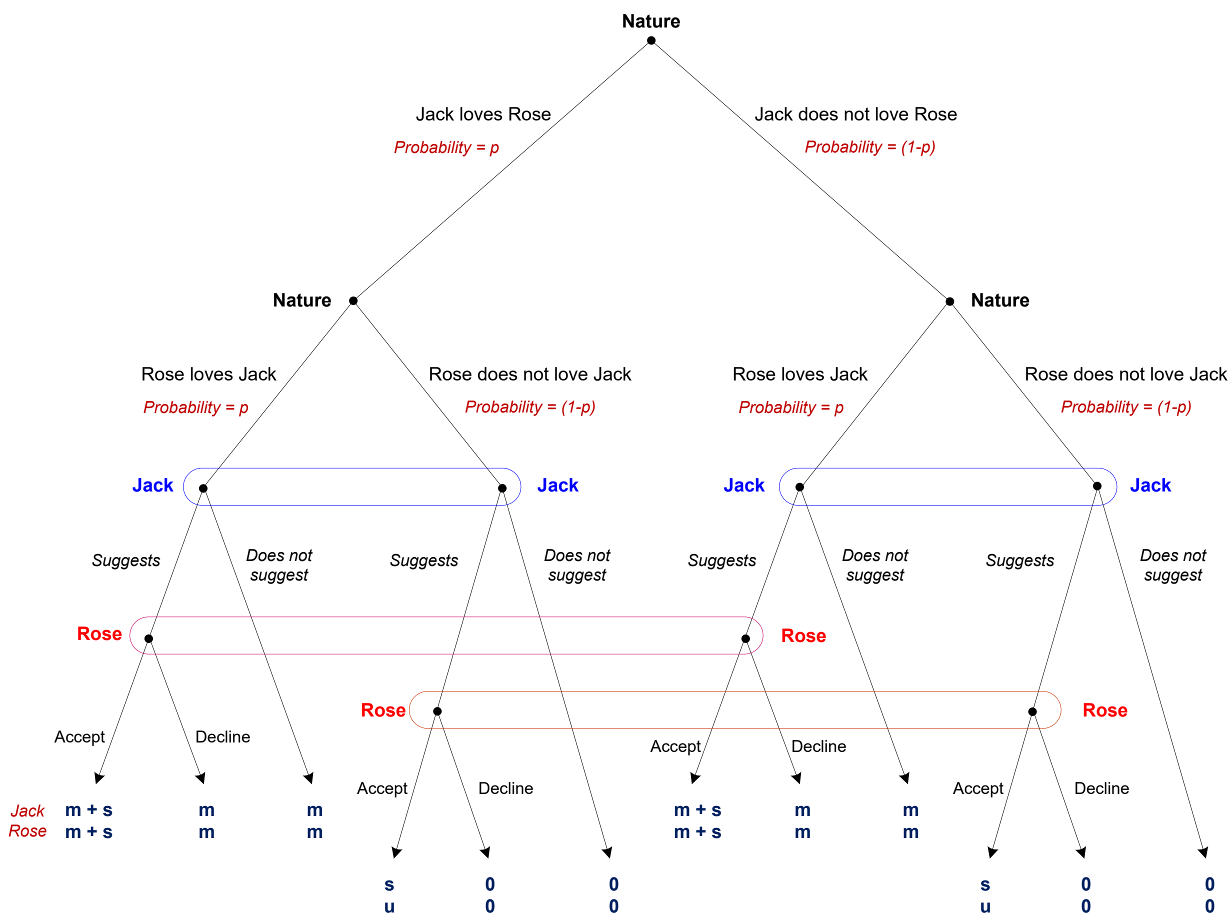
Sender's optimal message

- If his type is t_1, the sender obtains a payoff of 2 from sending m_1 (since it is responded with b), but a payoff of only 2 when deviating towards m_2 (since it is responded with b). Hence, he doesn't have incentives to deviate from m_1.
- If his type is t_2 , the sender obtains a payoff of 0 by sending m_1 (since it is responded with b), but a payoff of only 0 by deviating towards m_2 (since it is responded with b). Hence, he doesn't have incentives to deviate from m_1.

Therefore, the initially prescribed pooling strategy profile where both types of sender select $m_1$can be sustained as a PBE of the game when $p \geq \frac{3}{4}$.

<u>Exercise 7-Chapter 12-Harrington</u>

Consider the Courtship Game with Cheap Talk



Show that there is no PBNE in which premarital sex occurs.

Consider a strategy profile in which Rose accepts Jack's proposal when it is made. To begin, it is clear that she would never accept having sex with someone she doesn't love. Doing so results in a payoff of $u < 0$ (as she knows she isn't going to marry Jack), while not having sex results in a payoff of zero. Thus, if there is an equilibrium with premarital sex, it would only involve Rose having sex with Jack if she loves him. Suppose Rose does act in that manner, accepting if she loves Jack but declining if she does not.

Jack. *What is an optimal response for Jack?* Regardless whether or not he loves Rose, his payoff is higher by having sex. Thus, he'll ask for sex; he has nothing to lose. If Rose doesn't love him, then she'll decline and his payoff is zero. If she does love him, then his payoff is higher by *s*. More specifically, if he loves Rose, then his expected payoff from asking for sex is:

$$EU_{Jack}(sex|loves\ Rose) = p \times (m+s) + (1-p) \times 0 = p(m+s)$$

and from not asking is $EU_{Jack}(not\ sex|loves\ Rose) = p * m + (1-p) * 0 = p * m$. Thus, Jack asks for sex.

Rose. Is Rose's strategy of accepting if she loves Jack optimal given Jack asks regardless whether he loves her? Her payoff from accepting his proposition is:

$$EU_{Rose}(accept|loves\ Jack) = p \times (m+s) + (1-p) \times u$$

Since both Jack types ask, Rose doesn't learn anything about whether he wants to marry her from the fact that he wants to have sex with her. Recall that we are evaluating this in the case when she loves Jack. If she doesn't have sex, then her expected payoff is:

$$EU_{Rose}(reject|loves\ Jack) = p \times m + (1-p) \times 0 = pm$$

Thus, it is indeed optimal for Rose to accept if and only if:

$$p \times (m+s) + (1-p) \times u > p * m$$

which is equivalent to $ps + (1-p)u > 0$.

If this condition does not hold, then Rose would prefer to decline even if she loves Jack. In that case, there is no PBE in which premarital sex occurs.