ECONS 424 – STRATEGY AND GAME THEORY

FINAL EXAM – *ANSWER KEY*

Watson-Chapter 23 Exercise 2

a) Country i's payoff function is $2000 + 60x_i + x_i x_j - x_i^2 - 90x_j$

The best response function of player i is given by $BR_i(x_j) = 30 + x_j/2$.
Solving for equilibrium, we find that $x_i = 30 + \frac{1}{2}\left[30 + \frac{x_i}{2}\right]$ which implies that $x_1^* = x_2^* = 60$.
The payoff to each player is equal to $2000 - 30(60) = 200$.

b) *Cooperation.* Under zero tariffs, the payoff to each country is 2000.
Unilateral deviations. Note that, given $x_j = 0$, the optimal deviation by country i can be easily found by using its BRF_i evaluated at $x_j = 0$, as follows:

$$BR_i(0) = 30 + \frac{0}{2} = 30$$

A deviation by player i towards $x_i = 30$ (while player j still selects $x_j = 0$) yields a payoff of $2000 + 60(30) - 30 * 0 - 30^2 - 90 * 0 = 2900$.
Thus, players i's gain from deviating is 900. Therefore, sustaining zero tariffs requires that

$$\frac{2000}{1-\delta} \geq 2900 + \frac{200 * \delta}{1-\delta}$$

Solving for δ, we get $\delta \geq \frac{1}{3}$.

c) *Cooperation.* The payoff to each player of cooperating by setting tariffs equal to k is

$$2000 + 60k + k^2 - k^2 - 90k = 2000 - 30k$$

If player j is cooperating selecting $x_j = k$, player i can find its most profitable deviation by using its BRF_i evaluated at $x_j = k$, as follows

$$BR_i(x_j = k) = 30 + \frac{k}{2}$$

Unilateral deviations. The payoff to a player from unilaterally deviating to $x_i = 30 + \frac{k}{2}$ while country j still selects $x_j = k$ is equal to

$$2000 + 60\left[30 + \frac{k}{2}\right] + \left[30 + \frac{k}{2}\right]k - \left[30 + \frac{k}{2}\right]^2 - 90k = 2000 + \left[30 + \frac{k}{2}\right]^2 - 90k$$

Thus, the gain to player i of unilaterally deviating is

$$\left[30 + \frac{k}{2}\right]^2 - 60k$$

In order to support tariff setting of k, it must be that its payoff from cooperating, $(2000 - 30k) * \frac{1}{1-\delta}$, is larger than its payoff from deviating, $\left[30 + \frac{k}{2}\right]^2 - 60k$, and then be prescribed thereafter, $\frac{\delta}{1-\delta} * 200$, in the NE of the unrepeated game. That is,

$$\left[30 + \frac{k}{2}\right]^2 - 60k + \frac{200\delta}{1-\delta} \leq \frac{[2000 - 30k]}{1-\delta}$$

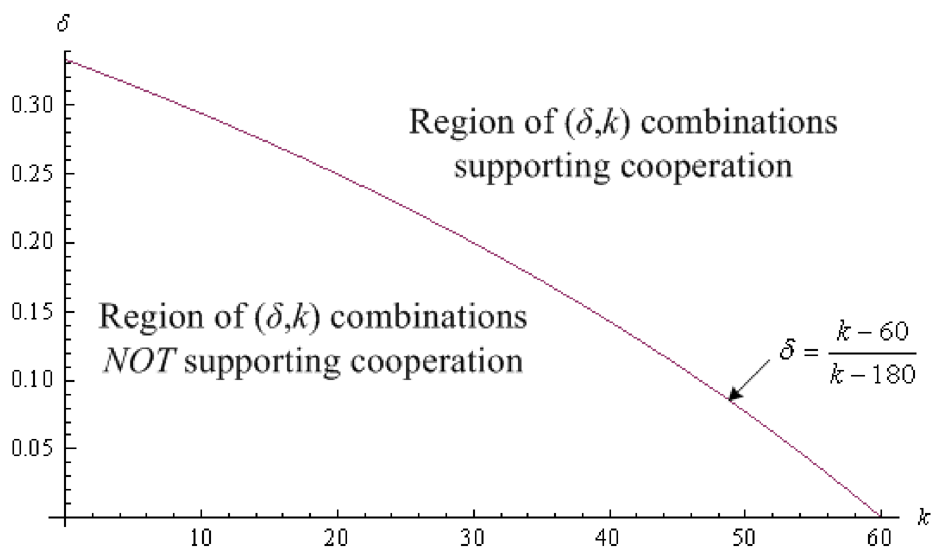
Solving for the discount factor δ yields the condition:

$$\frac{\left[30+\frac{k}{2}\right]^2-60k}{1800-90k+\left[30+\frac{k}{2}\right]^2}\leq\delta$$

or further simplifying,

$$\frac{k-60}{k-180}\leq\delta$$

The following figure depicts the ratio $\frac{k-60}{k-180}$, as a function of k. For simplicity, we restrict the value of k to be lower than the Nash equilibrium tariff level, $x_1^*=x_2^*=60$, since otherwise countries' individual payoffs would not be larger than what they can obtain by independently setting their tariff policy.



Intuition:

- When the tariff-reduction agreement in which countries participate implies very low tariffs, i.e., k approaching zero, countries have strong incentives to defect unilaterally setting higher tariffs, i.e., countries only participate in the agreement if their discount factor δ is sufficiently high.
- However, when the tariff-reduction agreement specifies tariff levels k similar to those in the NE of the game, $x_1^*=x_2^*=60$, countries do not experience a large "temptation" to unilaterally deviate from the treaty and, as a consequence, are willing to stick to the terms of the agreement under most discount factors.

Watson - Chapter 26 Exercise 5

a) For firm 1, whose constant marginal costs of production are 10, we have

$$u_1(p_1,p_2) = p_1 * q_1 - 10q_1 = (p_1 - 10)q_1 = (p_1 - 10)(22 - 2p_1 + p_2)$$
$$= 22p_1 - 2p_1^2 + p_1p_2 - 220 + 20p_1 - 10p_2$$
$$= 42p_1 = p_1p_2 - 2p_1^2 - 10p_2$$

For firm 2, with marginal costs given by c > 0, we have

$$u_2(p_1,p_2) = p_2 * q_2 - c * q_2 = (p_2 - c)q_2 = (p_2 - c)(22 - 2p_2 + p_1)$$
$$= 22p_2 - 2p_2^2 + p_1p_2 - 22c + 2cp_2 - cp_1$$
$$= (22 + 2c)p_2 + p_1p_2 - 2p_2^2 - 22c - cp_1$$

b) Taking FOC with respect to p_1 we have

$$42 + p_2 - 4p_1 = 0 \quad \rightarrow \quad p_1(p_2) = \frac{42 + p_2}{4} \quad \leftarrow \quad (BRF_1)$$

Thus, the BRF of firm 1 originates at $\frac{42}{4}$ and has a vertical slope of $\frac{1}{4}$.
Similarly taking FOC of $u_2(p_1,p_2)$ with respect to p_2,

$$22 + 2c + p_1 - 4p_2 = 0$$

Solving for p_2,

$$p_2(p_1) = \frac{22 + 2c + p_1}{4} \quad \leftarrow \quad BRF_2$$

Which originates at $\frac{22+2c}{4}$ and has a vertical slope of $\frac{1}{4}$. (Note that if firm 2's marginal costs were 10, then $22 + 2c$ would be 42, yielding a BRF_2 symmetric to that of firm 1)

c) Plugging BRF_1 into BRF_2, we obtain

$$p_2 = \frac{22 + 2c + \left[\frac{42 + p_2}{4}\right]}{4}$$

And solving for $p_2^* = \$14$.
Plugging $p_2^* = \$14$ into BRF_1, we have

$$p_1(14) = \frac{42 + 14}{4} = \$14$$

d) Starting with the informed player (Player 2), we have that:

- Firm 2's best-response function when having high costs of c=14, BRF_2^H, is
$$p_2^H(p_1) = \frac{22 + 2 * 14 + p_1}{4} = \frac{50 + p_1}{4}$$
- Firm 2's best-response function when having low costs of c=6, BRF_2^L, is
$$p_2^L(p_1) = \frac{22 + 2 * 6 + p_1}{4} = \frac{34 + p_1}{4}$$

The uninformed firm 1 can now plug this information into its expected profits maximization problem as follows:

$$\max_{p_1} \quad \frac{1}{2}[42p_1 + p_1p_2^H - 2p_1^2 - 10p_2^H] + \frac{1}{2}[42p_1 + p_1p_2^L - 2p_1^2 - 10p_2^L]$$

where the first component represents the case in which firm 2 has high costs, and thus sets a price p_2^H, while the second component denotes the case in which firm 2's costs are low, setting a price p_2^L.

Taking FOC with respect to p_1 we obtain

$$\frac{1}{2}[42 + p_2^H - 4p_1] + \frac{1}{2}[42 + p_2^L - 4p_1]$$

$$21 + \frac{1}{2}p_2^H - 2p_1 + 21 + \frac{1}{2}p_2^L - 2p_1$$

$$42 + \frac{1}{2}p_2^H + \frac{1}{2}p_2^L - 4p_1 = 0$$

And solving with respect to p_1 we obtain firm 1's $BRF_1(p_2^H, p_2^L)$,

$$p_1(p_2^H, p_2^L) = \frac{42 + \frac{1}{2}p_2^H + \frac{1}{2}p_2^L}{4}$$

Plugging p_2^H and p_2^L into $BRF_1(p_2^H, p_1^L)$, we obtain

$$p_1(p_2^H, p_2^L) = \frac{42 + \frac{1}{2}\left[\frac{50 + p_1}{4}\right] + \frac{1}{2}\left[\frac{34 + p_1}{4}\right]}{4}$$

And solving for p_1, we obtain $p_1^* = \$14$.

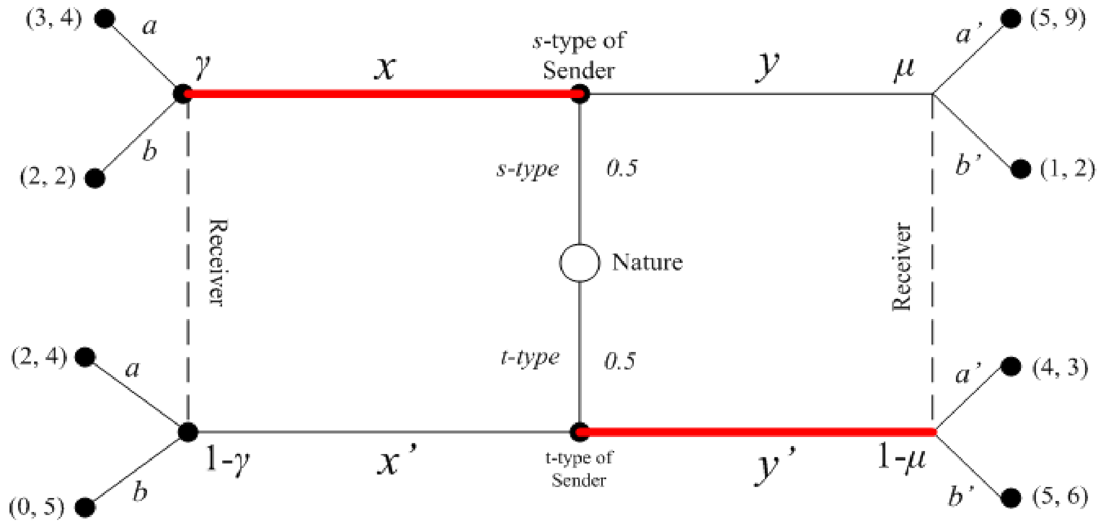
Finally, plugging $p_1^* = \$14$ into firm 2's BRF_2^H and BRF_2^L we have:

$$p_2^H(14) = \frac{50 + 14}{4} = \$16$$

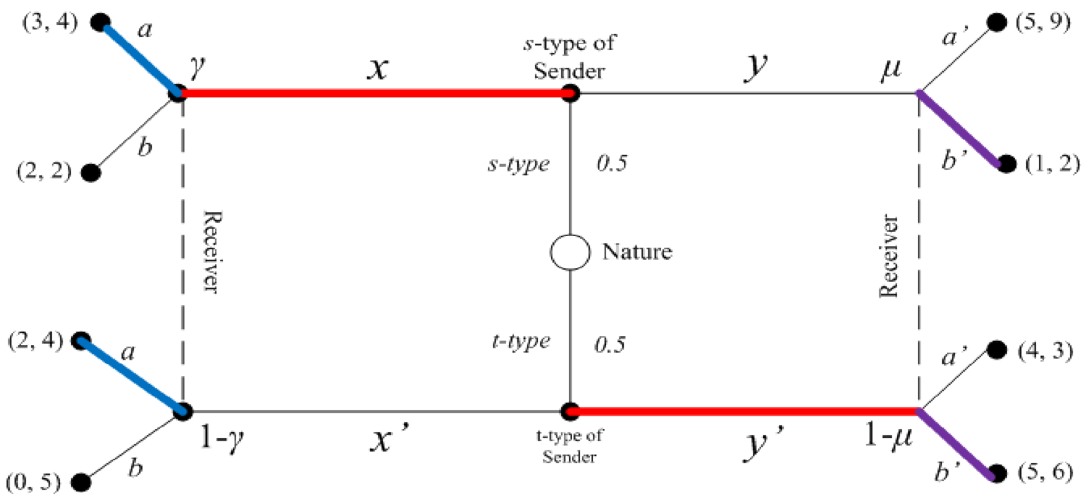
$$p_2^L(14) = \frac{34 + 14}{4} = \$12$$

<u>**Exercise 9 - Chapter 11 - Harrington**</u>

SEPARATING STRATEGY PROFILE (x, y')

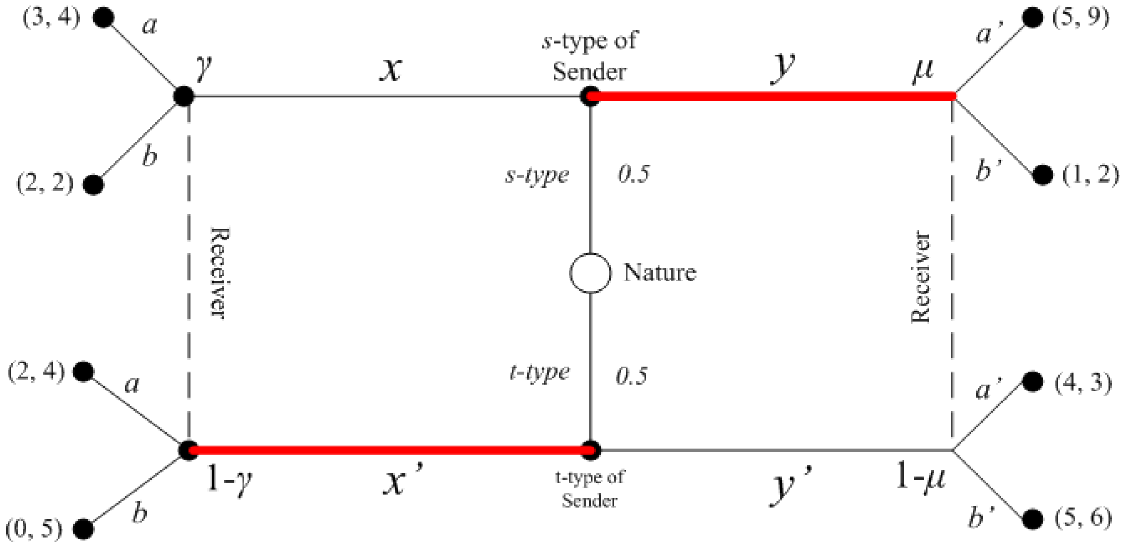


1. **Receiver's beliefs:**
   a. After observing x, the receiver's beliefs are γ=1, since such a message can only originate from the s type according to this separating strategy profile.
   b. After observing y, the receiver's beliefs are μ=0, since such a message can only originate from the t type according to this separating strategy profile.
2. **Receiver's optimal response:**
   a. After observing x, since the receiver assigns full probability to the upper left-hand node, his optimal response is a. Indeed, his payoff from selecting a, 4, exceeds that from selecting b, 2.
   b. After observing y, since the receiver assigns full probability to the lower right-hand node, his optimal response is b. Indeed, his payoff from selecting b, 6, exceeds that from selecting a, 3.
   c. The following figure introduces these optimal responses into our above figure.

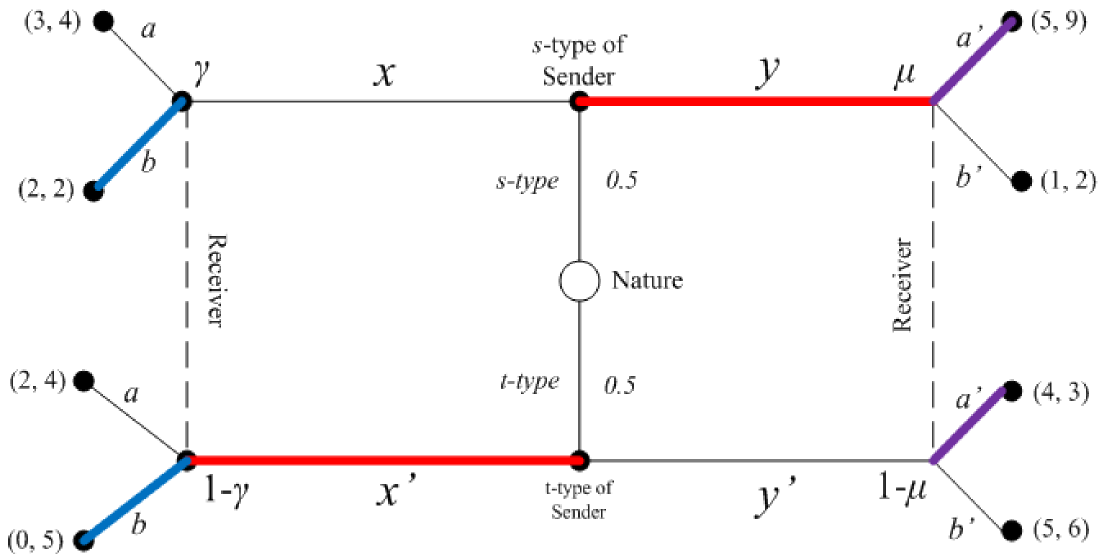


3. **Sender's optimal message:**
    a. When the sender's type is s, if he sticks to the prescribed strategy of choosing x, his payoff is 3 (given that he will be responded with a). By contrast, if he deviates towards y, he will be responded with b, yielding a payoff of only 1. Hence, the s-type of sender has no incentives to deviate from x.
    b. When the sender's type is t, if he sticks to the prescribed strategy of choosing y', his payoff is 5 (given that he will be responded with b). By contrast, if he deviates towards x', he will be responded with a, yielding a payoff of only 2. Hence, the t-type of sender has no incentives to deviate from y'.
    c. Since no type of sender has incentives to deviate from their prescribed strategy profiles, this separating strategy profile can be sustained as a PBE.

SEPARATING STRATEGY PROFILE (y, x')

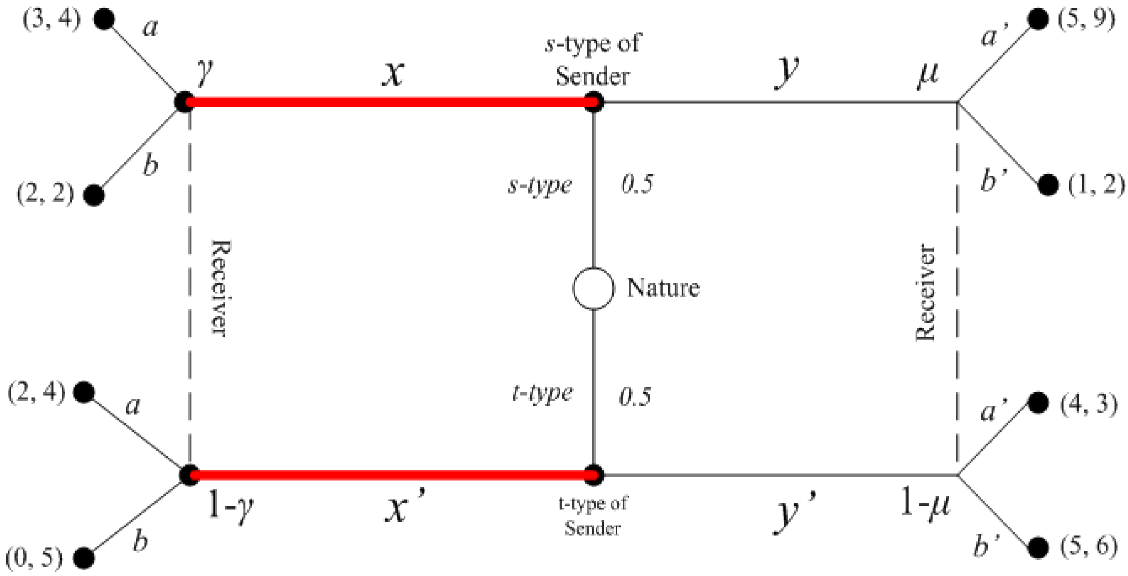


1. **Receiver's beliefs:**
   a. After observing x, the receiver's beliefs are μ=1, since such a message can only originate from the t type according to this separating strategy profile.
   b. After observing y, the receiver's beliefs are γ=0, since such a message can only originate from the s type according to this separating strategy profile.
2. **Receiver's optimal response:**
   a. After observing x, since the receiver assigns full probability to the lower left-hand node, his optimal response is b. Indeed, his payoff from selecting b, 5, exceeds that from choosing a, 4.
   b. After observing y, since the receiver assigns full probability to the upper right-hand node, his optimal response is a. Indeed, his payoff from selecting a, 7, exceeds that from choosing b, 2.
   c. The following figure summarizes these responses.



3. **Sender's optimal message:**
    a. When the sender's type is s, if he sticks to the prescribed strategy of choosing y, his payoff is 5 (given that he will be responded with a). By contrast, if he deviates towards x, he will be responded with b, yielding a payoff of only 2. Hence, the s-type of sender has no incentives to deviate from y.
    b. When the sender's type is t, if he sticks to the prescribed strategy of choosing x', his payoff is 0 (given that he will be responded with b). By contrast, if he deviates towards y,' he will be responded with a, yielding a payoff of only 4. Hence, the t-type of sender *has* incentives to deviate from x' to y'.
    c. Since we found that at least one type of sender has incentives to deviate from their prescribed strategy profile, this separating strategy profile cannot be sustained as a PBE.

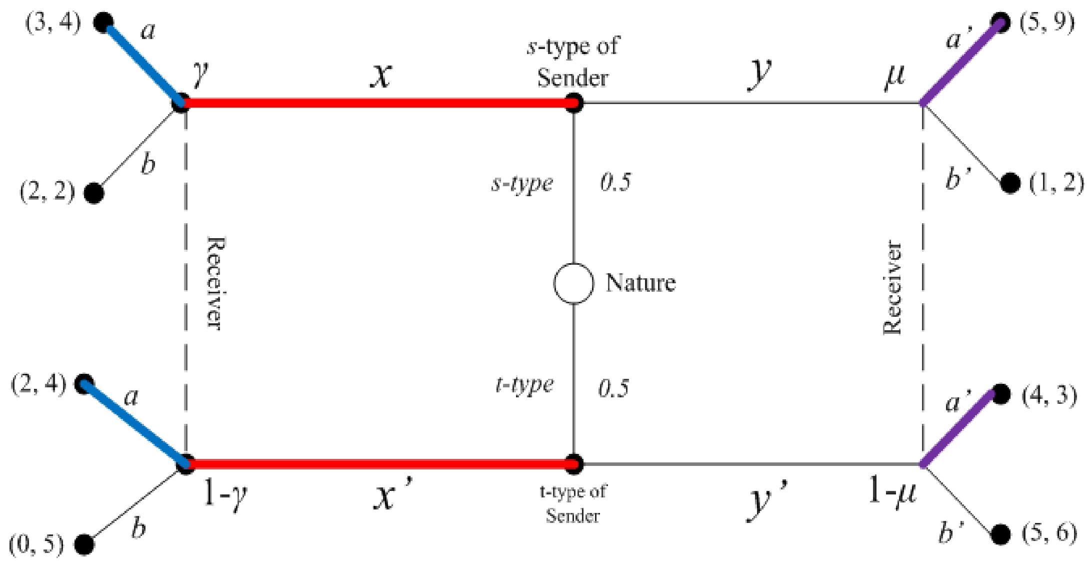
POOLING STRATEGY PROFILE (x,x')



1. **Receiver's beliefs:**
   a. After observing the equilibrium message of x, the receiver's beliefs are γ=0.5 (coinciding with the prior probability distribution over types), since such a message can originate from either type of sender.
   b. After observing the off-the-equilibrium message of y, the receiver's beliefs cannot be updated according to Bayes' rule, and must be arbitrarily specified, i.e., $\mu \in (0,1)$.
2. **Receiver's optimal response:**
   a. After observing the equilibrium message of x, the receiver's optimal response is to respond with a since it yields a larger expected utility. In particular, $EU_R(a|x) = \frac{1}{2}4 + \frac{1}{2}4 = 4$ whereas b only entails a expected payoff of $EU_R(b|x) = \frac{1}{2}2 + \frac{1}{2}5 = 3.5$.
   b. After observing the off-the-equilibrium message of y, the receiver's choice of a or b depends on his specific off-the-equilibrium belief, μ. Indeed, the expected payoff when he responds with a is $EU_R(a|y) = \mu 7 + (1-\mu)3 = 4\mu + 3$ whereas his expected payoff when responding with b is $EU_R(b|y) = \mu 2 + (1-\mu)6 = 6 - 4\mu$. Therefore, the receiver responds with a if and only if $4\mu + 3 > 6 - 4\mu$, or $\mu \geq \frac{3}{8}$.
   c. [As usual, we hereafter divide our analysis into two cases: one in which $\mu \geq \frac{3}{8}$ holds (and therefore the receiver responds with a after observing y), and other case in which $\mu \geq \frac{3}{8}$ is not satisfied (and hence the receiver responds with b after observing y).]

3. **Sender's optimal message:**

   a. *CASE 1:* $\mu \geq \frac{3}{8}$.

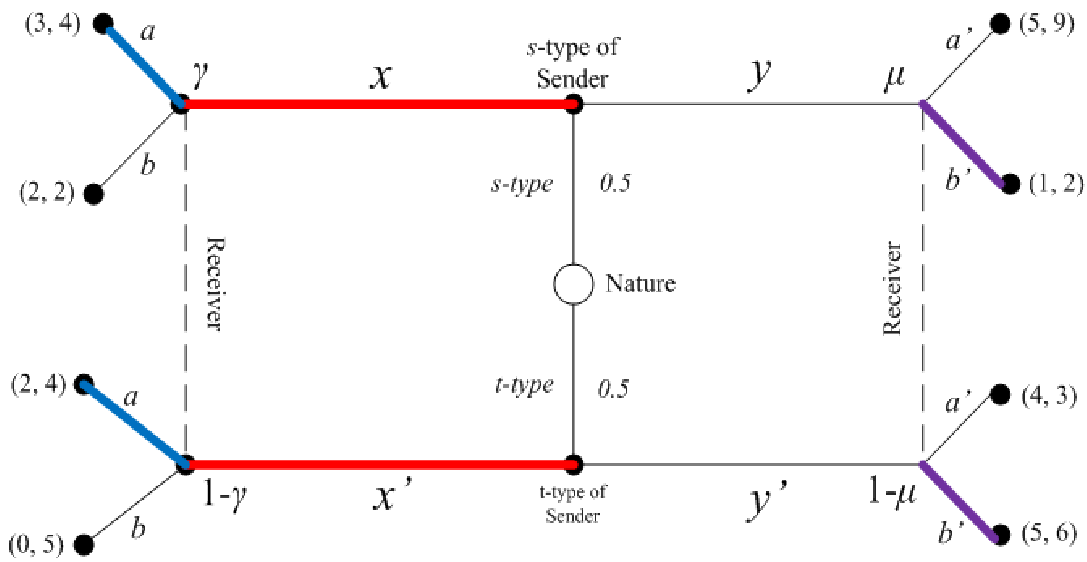


      i. When the sender's type is s, if he sticks to the prescribed strategy of choosing x, his payoff is 3 (given that he will be responded with a). If he instead deviates towards y, he will be responded with a (since $\mu \geq \frac{3}{8}$), yielding a larger payoff of 5. Hence, the s-type of sender *has* incentives to deviate from x.

      ii. Similarly, when the sender's type is t, if he sticks to the prescribed strategy of choosing x', his payoff is 2 (given that he will be responded with a). If he instead deviates towards y,' he will be responded with a (since $\mu \geq \frac{3}{8}$), yielding a larger payoff 4. Hence, the t-type of sender also *has* incentives to deviate from x' to y'.

      iii. Since we found that both types of sender have incentives to deviate from their prescribed strategy profile, this pooling strategy profile cannot be sustained as a PBE when off-the-equilibrium beliefs satisfy $\mu \geq \frac{3}{8}$.

b. *CASE 2:* $\mu < \frac{3}{8}$.



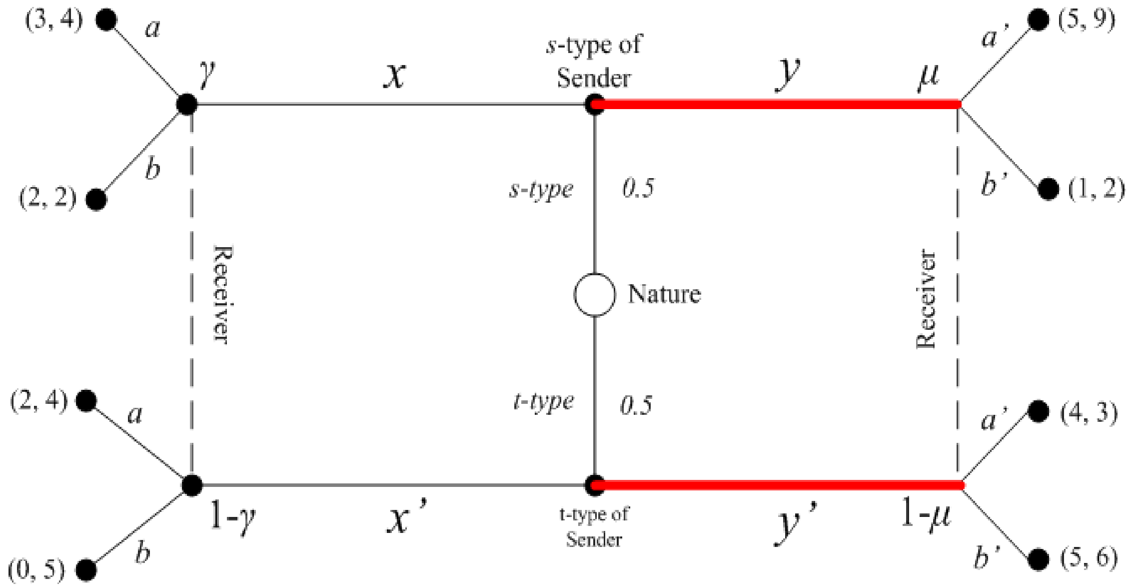
i. When the sender's type is s, if he sticks to the prescribed strategy of choosing x, his payoff is 3 (given that he will be responded with a). If he instead deviates towards y, he will be responded with b (since $\mu < \frac{3}{8}$), yielding a lower payoff of 1. Hence, the s-type of sender *does not have* incentives to deviate from x.

ii. When the sender's type is t, if he sticks to the prescribed strategy of choosing x', his payoff is 2 (given that he will be responded with a). If he instead deviates towards y,' he will be responded with b (since $\mu < \frac{3}{8}$), yielding a larger payoff 5. Hence, the t-type of sender *has* incentives to deviate from x' to y'.

iii. Since we found that at least one type of sender (the t-type) has incentives to deviate from his prescribed strategy profile, this pooling strategy profile cannot be sustained as a PBE when off-the-equilibrium beliefs satisfy $\mu < \frac{3}{8}$.

c. As a result, the pooling strategy profile (x,x') cannot be sustained under any off-the-equilibrium beliefs.

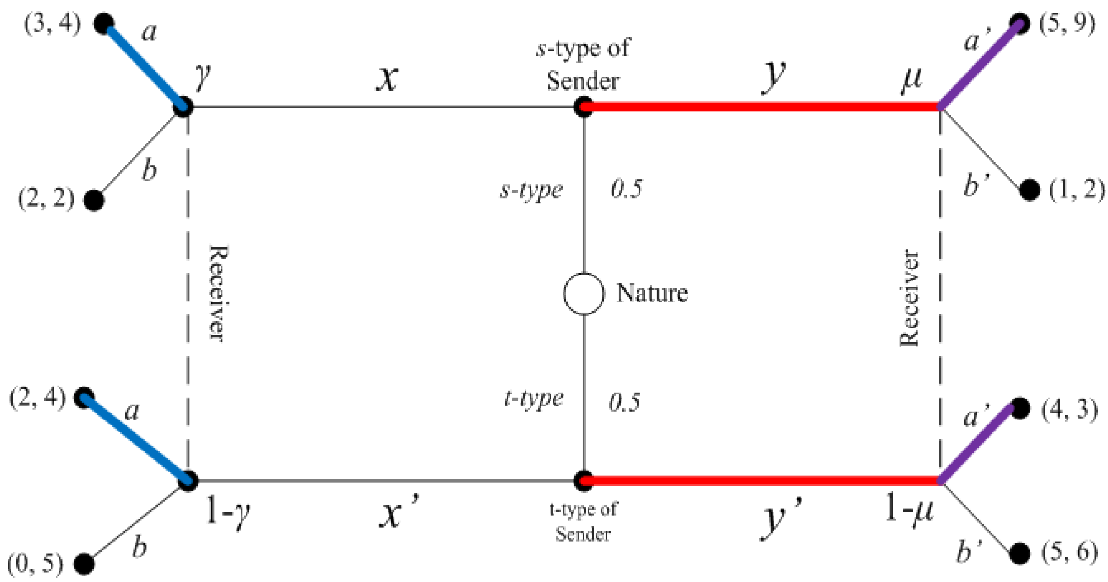
POOLING STRATEGY PROFILE (y,y')



1. **Receiver's beliefs:**
   a. After observing the equilibrium message of y, the receiver's beliefs are μ=0.5 (coinciding with the prior probability distribution over types), since such a message can originate from either type of sender.
   b. After observing the off-the-equilibrium message of x, the receiver's beliefs cannot be updated according to Bayes' rule, and must be arbitrarily specified, i.e., $\gamma \in (0,1)$.
2. **Receiver's optimal response:**
   a. After observing the equilibrium message of y, the receiver's optimal response is to respond with a since it yields a larger expected utility. In particular, $EU_R(a|y) = \frac{1}{2}7 + \frac{1}{2}3 = 5$ whereas b only entails a expected payoff of $EU_R(b|y) = \frac{1}{2}2 + \frac{1}{2}6 = 4$.
   b. After observing the off-the-equilibrium message of y, the receiver's choice of a or b depends on his specific off-the-equilibrium belief, γ. Indeed, the expected payoff when he responds with a is $EU_R(a|x) = \gamma 4 + (1-\gamma)4 = 4$ whereas his expected payoff when responding with b is $EU_R(b|x) = \gamma 2 + (1-\gamma)5 = 5 - 3\gamma$. Therefore, the receiver responds with a if and only if $4 > 5 - 3\gamma$, or $\gamma \geq \frac{1}{3}$.
   c. [As usual, we hereafter divide our analysis into two cases: one in which $\gamma \geq \frac{1}{3}$ holds (and therefore the receiver responds with a after observing x), and other case in which $\gamma \geq \frac{1}{3}$ is not satisfied (and hence the receiver responds with b after observing x).]

3. **Sender's optimal message:**

   a. *CASE 1:* $\gamma \geq \frac{1}{3}$.

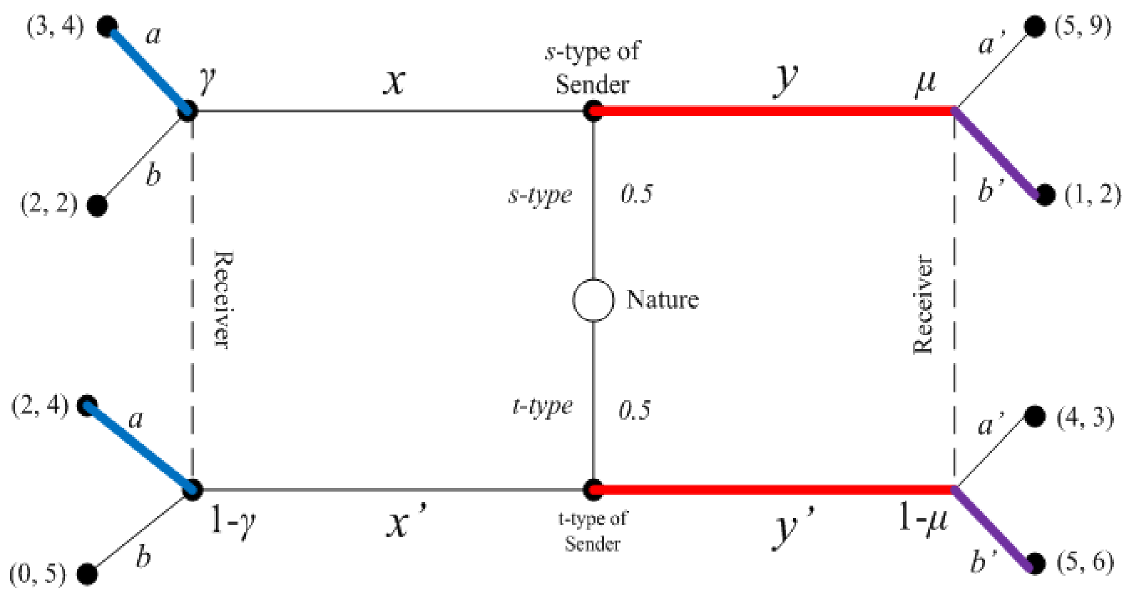


      i. When the sender's type is s, if he sticks to the prescribed strategy of choosing y, his payoff is 5 (given that he will be responded with a). If he instead deviates towards x, he will be responded with a (since $\gamma \geq \frac{1}{3}$), yielding a lower payoff of 3. Hence, the s-type of sender *does not have* incentives to deviate from y.

      ii. Similarly, when the sender's type is t, if he sticks to the prescribed strategy of choosing y', his payoff is 4 (given that he will be responded with a). If he instead deviates towards x', he will be responded with a (since $\gamma \geq \frac{1}{3}$), yielding a lower payoff of 2. Hence, the t-type of sender also *does not have* incentives to deviate from y' to x' either.

      iii. Since we found that no type of sender has incentives to deviate from his prescribed strategy profile, this pooling strategy profile (y,y') *can* be sustained as a PBE when off-the-equilibrium beliefs satisfy $\gamma \geq \frac{1}{3}$.

b. CASE 2: $\gamma < \frac{1}{3}$.



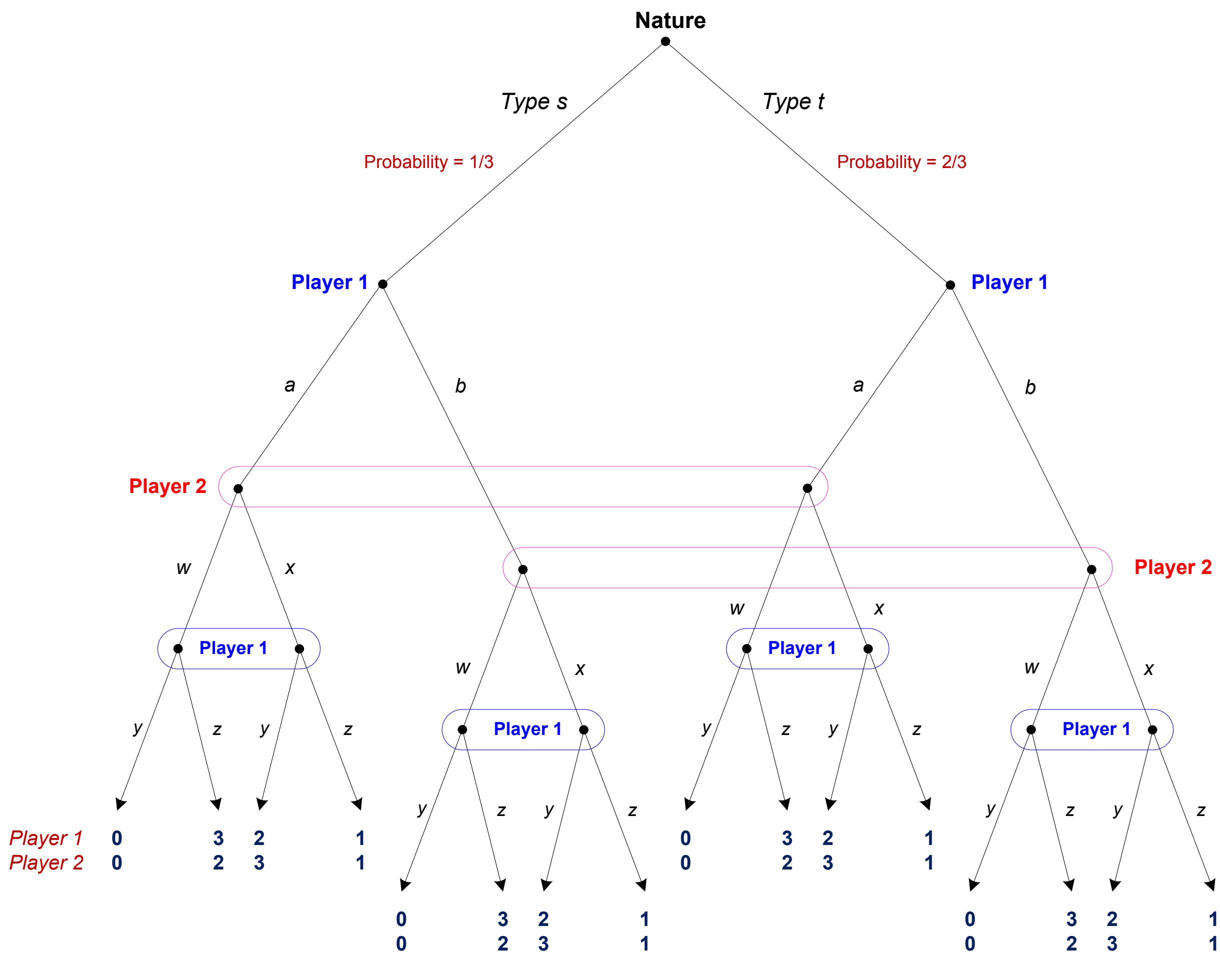
i. When the sender's type is s, if he sticks to the prescribed strategy of choosing y, his payoff is 5 (given that he will be responded with a). If he instead deviates towards x, he will be responded with b (since $\gamma < \frac{1}{3}$), yielding a lower payoff of 2. Hence, the s-type of sender *does not have* incentives to deviate from y.

ii. Similarly, when the sender's type is t, if he sticks to the prescribed strategy of choosing y', his payoff is 4 (given that he will be responded with a). If he instead deviates towards x', he will be responded with b (since $\gamma < \frac{1}{3}$), yielding a lower payoff of 0. Hence, the t-type of sender also *does not have* incentives to deviate from y' to x' either.

iii. Since we found that no type of sender has incentives to deviate from his prescribed strategy profile, this pooling strategy profile (y,y') *can* also be sustained as a PBE when off-the-equilibrium beliefs satisfy $\gamma < \frac{1}{3}$.

Exercise 8-Chapter 12-Harrington

Consider the following game:



Find a separating PBE.

For the case where sender type s chooses a and type t chooses b, there are three separating PBEs. There are three separating equilibria when sender type s chooses b and type t chooses a.

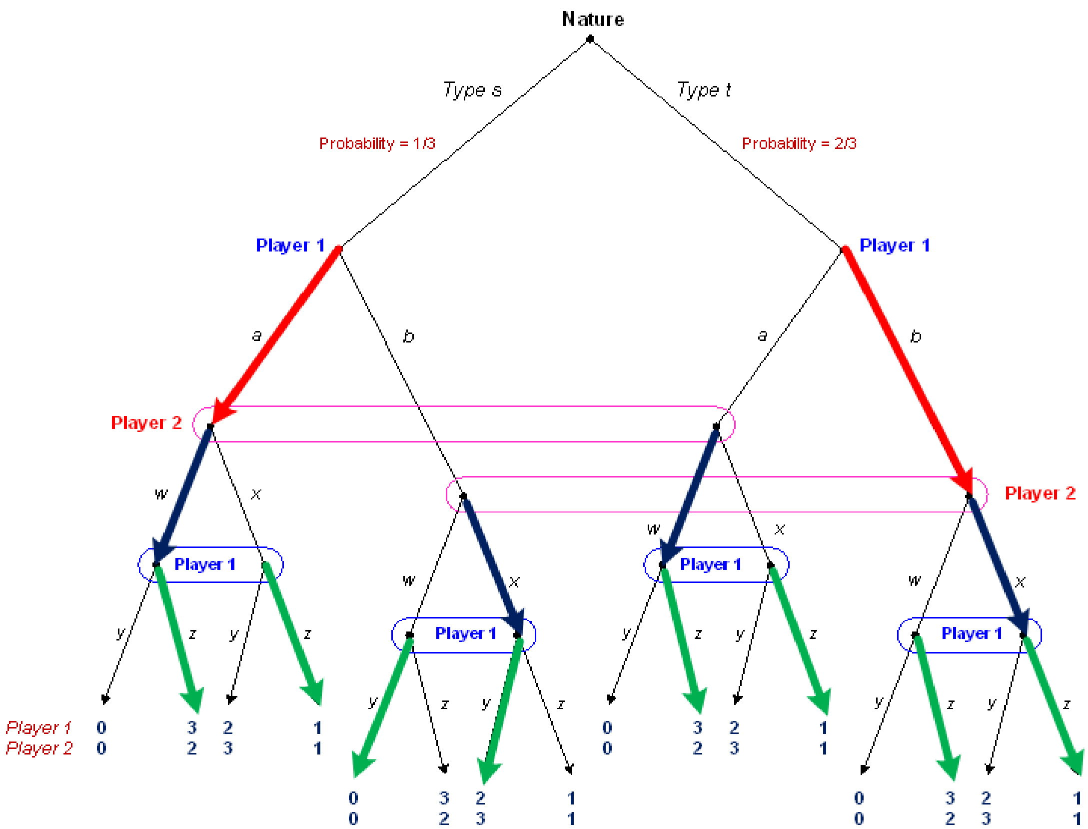
The first separating PBE is:

- Sender's strategy:
  - If my type is s, then choose a.
  - If my type is t, then choose b.
  - Choose z at the information set corresponding with the path $s \rightarrow a$
  - Choose y at the information set corresponding with the path $s \rightarrow b$
  - Choosey z at the information set corresponding with the path $t \rightarrow a$

- Choose z at the information set corresponding with the path $t \rightarrow b$

- Receiver's strategy:
  - If the message is a, then choose w.
  - If the message is b, then choose x.
- Receiver's beliefs:
  - If the message is a, then assign probability 1 to the sender's being type s
  - If the message is b, then assign probability 1 to the sender's being type t

To facilitate our next discussion, the following figure illustrates this strategy profile:



The receiver's (player 2's) beliefs are consistent with the sender's strategy. Given these beliefs and the sender's strategy, the receiver's optimal strategy is to choose w when she observes a since it produces a payoff of 2, which is greater than 1, which is the payoff of she chooses x. When she observes b, it is optimal to play x since it gives her a payoff of 4, which is greater than 0, the payoff if she plays w.

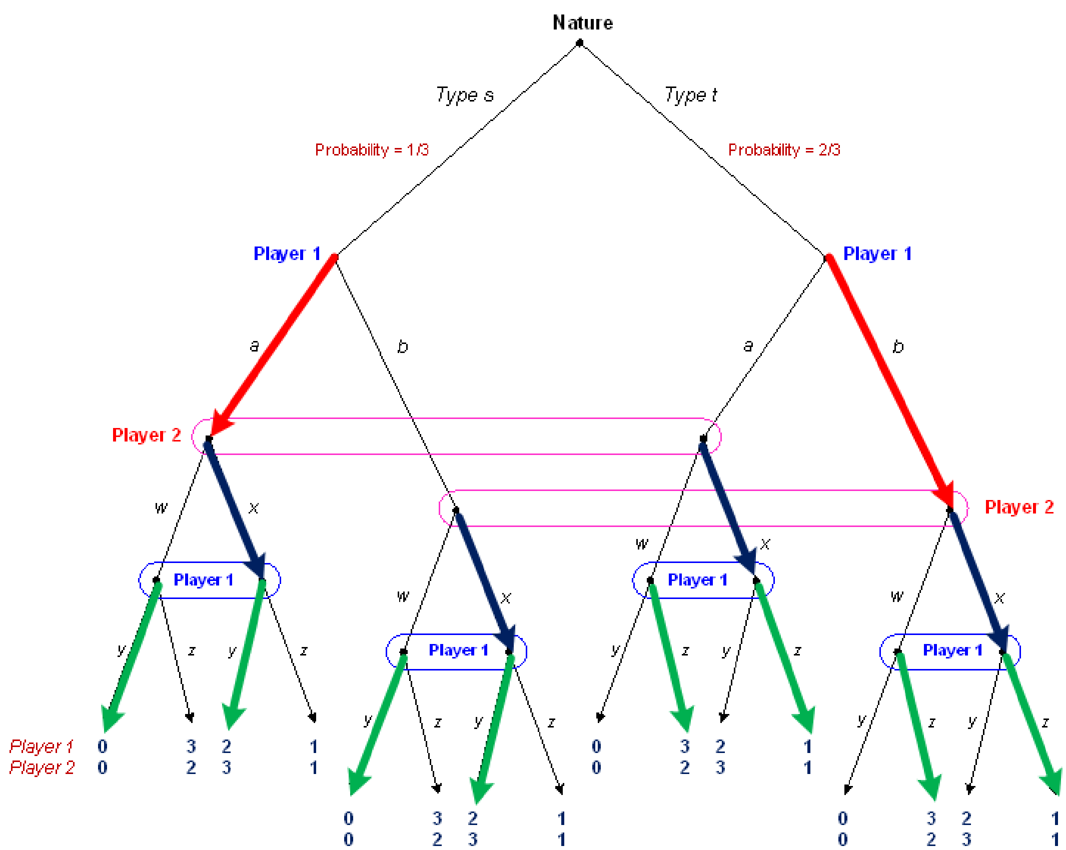
We need to check that the sender's strategy is optimal. If the path is $s \to a$, then the receiver will choose w, which means the sender's payoff is 0 from y and 3 from z, so z is optimal. If the path is $s \to b$, then the receiver will choose x, which means the sender's payoff is 0 from y and 4 from z, so z is optimal. If the path is $t \to a$, then the receiver will choose w, which means the sender's payoff is 1 from y and 0 from z, so y is optimal. If the path is $t \to b$, then the receiver will choose x, which means the sender's payoff is 0 from y and 4 from z, so z is optimal.

Finally, let's check whether the sender's messages are optimal. Suppose he is type s. he can choose a and induce the receiver to choose w, which means his payoff is 3, or he can choose b and induce the receiver to play x, which results in a payoff of 2. Hence, the sender's optimal message is a. Now, suppose he is type t. Choosing b induces the receiver to choose x, which gives sender's payoff of 4. Or he can choose a and induce the receiver to play w, which results in a payoff of 1. Hence, the sender's optimal message is b.

The second separating PBE is:

- Sender's strategy:
    - If my type is *s*, then send *a*.
    - If my type is *t*, then send *b*.
    - Choose *y* at the information set corresponding with the path $s \to a$
    - Choose *y* at the information set corresponding with the path $s \to b$
    - Choosey *z* at the information set corresponding with the path $t \to a$
    - Choose *z* at the information set corresponding with the path $t \to b$
- Receiver's strategy:
    - If the message is *a*, then choose *x*.
    - If the message is *b*, then choose *x*.
- Receiver's beliefs:
    - If the message is *a*, then assign probability 1 to the sender's being type *s.*
    - If the message is *b*, then assign probability 1 to the sender's being type *t.*

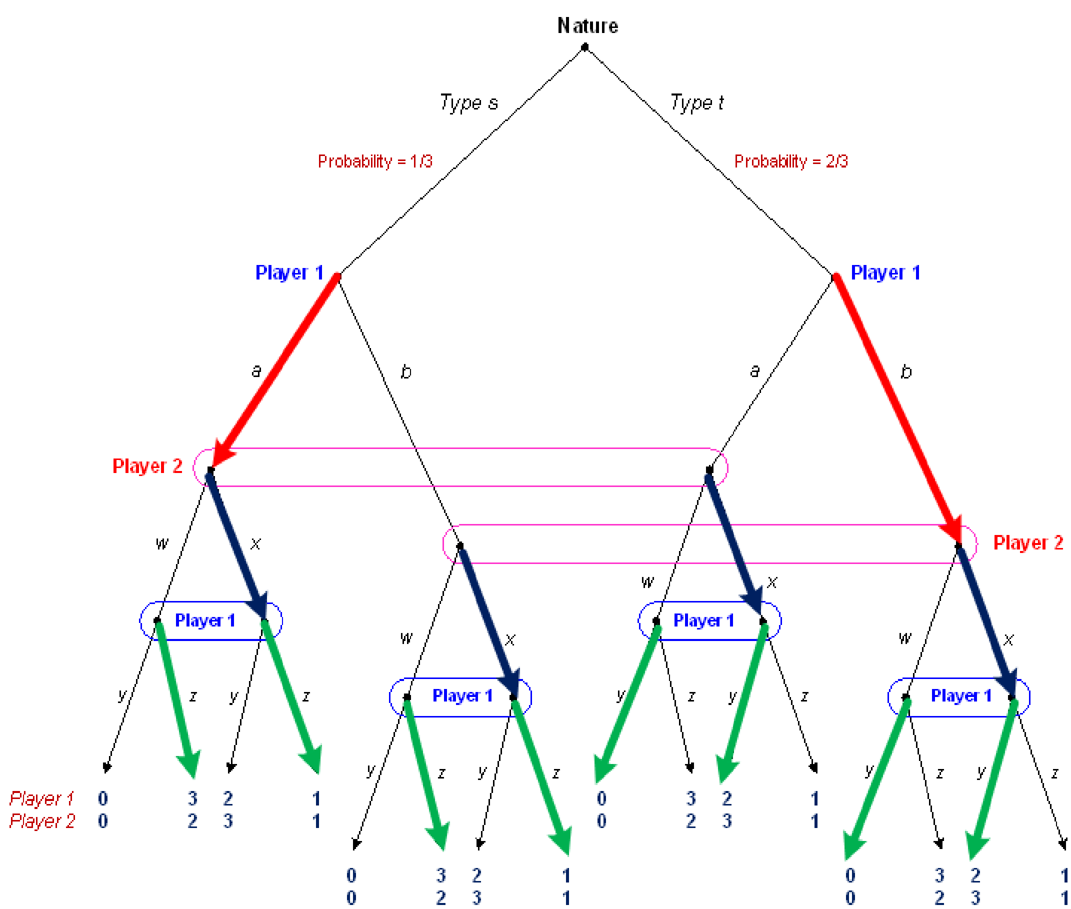
This strategy profile is depicted below:



The third separating PBE is:

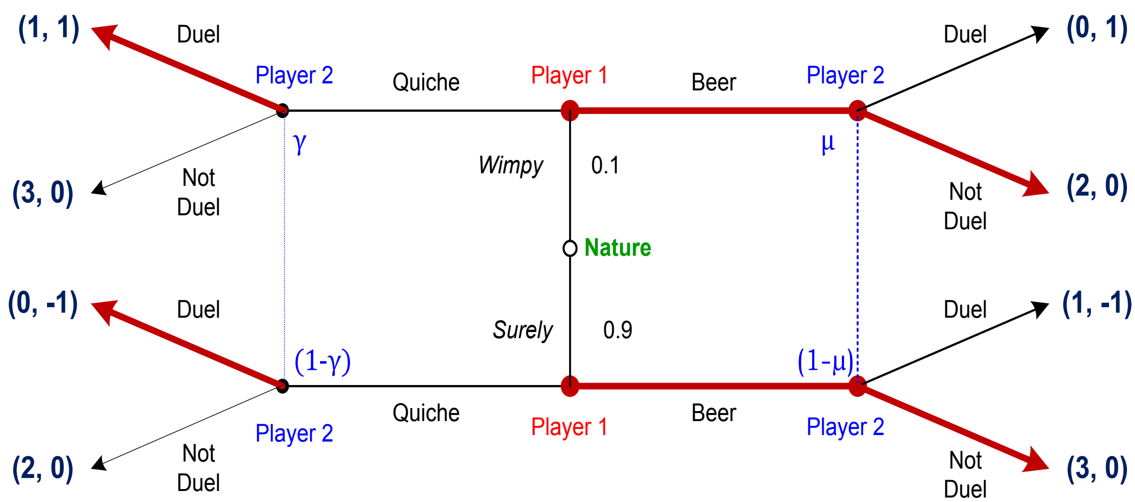
- Sender's strategy:
    - If my type is *s*, then send *a*.
    - If my type is *t*, then send *b*.
    - Choose z at the information set corresponding with the path $s \rightarrow a$
    - Choose z at the information set corresponding with the path $s \rightarrow b$
    - Choosey y at the information set corresponding with the path $t \rightarrow a$
    - Choose y at the information set corresponding with the path $t \rightarrow b$
- Receiver's strategy:
    - If the message is *a*, then choose *w*.
    - If the message is *b*, then choose *w*.
- Receiver's beliefs:
    - If the message is *a*, then assign probability 1 to the sender's being type *s.*
    - If the message is *b*, then assign probability 1 to the sender's being type *t.*

This strategy profile is depicted below:



Exercise 6 – Applying the Cho and Kreps' (1987) Intuitive Criterion in the Far West

a. Check if the pooling equilibrium in which both types of player 1 have Beer for breakfast survives the Cho and Kreps' (1987) Intuitive Criterion.



First step

In the first step we want to eliminate those off-the-equilibrium messages that are equilibrium dominated. For the case of a Wimpy player 1, we need to check if having Quiche can improve his equilibrium utility level (from having Beer). That is, if

$$\underbrace{u_1^*(Beer|Wimpy)}_{Equil.\ Payoff,\ 2} < \underbrace{\operatorname*{Max}_{a_2} u_1(Quiche|Wimpy)}_{\substack{Highest\ payoff\ from\ deviating\\ towards\ Quiche,\ 3}}$$

This condition can be indeed satisfied if player 2 chooses not to duel: the wimpy player 1 would obtain a payoff of 3 instead of 2 in this pooling equilibrium. Hence, the wimpy player 1 has incentives to deviate from this separating PBE. Let us now check the surely type of player 1. We need to check if an (off-the-equilibrium) message of Quiche could ever be convenient for the surely type. That is, if

$$\underbrace{u_1^*(Beer|Surely)}_{Equil.\ Payoff,\ 3} < \underbrace{\operatorname*{Max}_{a_2} u_1(Quiche|Surely)}_{\substack{Highest\ payoff\ from\ deviating\\ towards\ Quiche,\ 2}}$$

But this condition is *not* satisfied: the surely player 1 obtains in equilibrium a payoff of 3 (from having beer for breakfast), and the highest payoff he could obtain from deviating to quiche is only 2 (since he dislikes quiche). Hence, the surely type would never deviate from beer.

Player 2's beliefs, after observing the (off-the-equilibrium) message of quiche, can be restricted to $\Theta^{**}(Quiche) = \{Wimpy\}$.

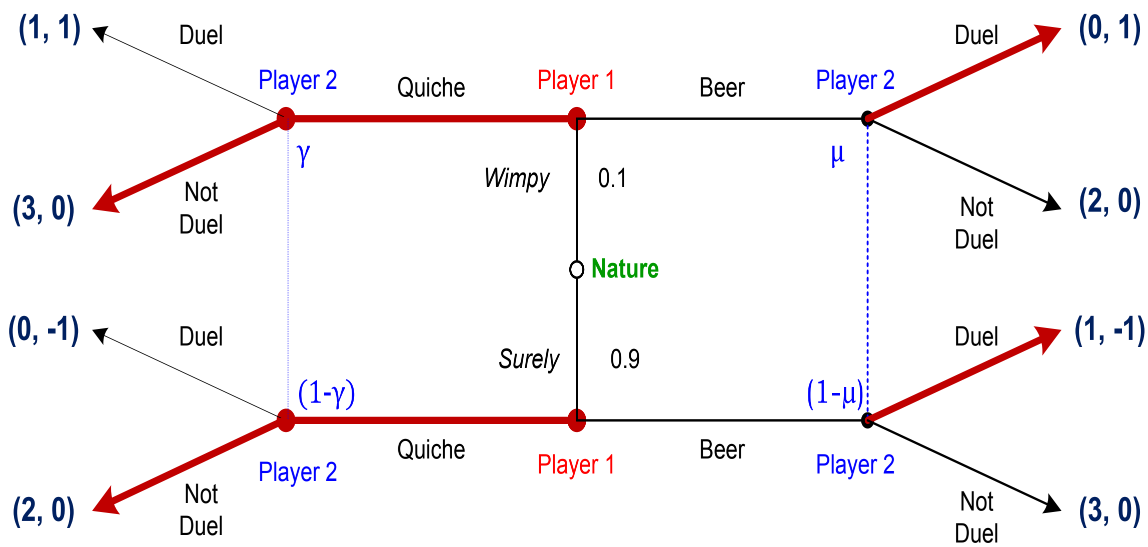
Second step

After restricting the receiver's beliefs to $\Theta^{**}(Quiche) = \{Wimpy\}$, player 2 plays duel every time that he observes the (off-the-equilibrium) message of Quiche, since he is sure that player 1 must be wimpy. The second step of the Intuitive Criterion analyzes if there is any type of sender (wimpy or surely) and any type of message he would send that satisfies:

$$\underset{a \in A^*[\Theta^{**}(m),m]}{Min} u_i\,(m, a, \theta) > u_i^*(\theta)$$

In this context, this condition does not hold for the wimpy type since the minimal payoff he can obtain by deviating (having Quiche for breakfast) *once* the receiver's beliefs have been restricted to $\Theta^{**}(Quiche) = \{Wimpy\}$, is 1 (given that when having quiche for breakfast, player 2 responds by dueling). In contrast, his equilibrium payoff from having beer for breakfast in this pooling equilibrium is 2. A similar argument is applicable to the surely type: in equilibrium he obtains a payoff of 3, and by deviating towards quiche he will be dueled by player 2, obtaining a payoff of 0 (since he does not like quiche, and in addition, he has to fight). As a consequence, no type of player deviates towards quiche, and the pooling equilibrium in which both types of player 1 have beer for breakfast survives the Cho and Kreps' (1987) Intuitive Criterion.

b. Check if the pooling equilibrium in which both types of player 1 have Quiche for breakfast survives the Cho and Kreps' (1987) Intuitive Criterion.



<u>*First step:*</u>

In the first step we want to eliminate those off-the-equilibrium messages that are equilibrium dominated. For the case of a Wimpy player 1, we need to check if having Beer can improve his equilibrium utility level (from having Beer). That is, if

$$\underbrace{u_1^*(Quiche|Wimpy)}_{Equil.\ Payoff,\ 3} < \underbrace{\underset{a_2}{\text{Max}}\, u_1(Beer|Wimpy)}_{\substack{Highest\ payoff\ from\ deviating \\ towards\ Beer,\ 2}}$$

This condition is not satisfied: the Wimpy player 1 obtains a payoff of 3 in this equilibrium, and the highest payoff that he could obtain by deviating towards beer is only by 2. Let us now check if the equivalent condition holds for the surely type of player 1,

$$\underbrace{u_1^*(Quiche|Surely)}_{Equil.\ Payoff,\ 2} < \underbrace{\underset{a_2}{\text{Max}}\, u_1(Beer|Surely)}_{\substack{Highest\ payoff\ from\ deviating \\ towards\ Beer,\ 3}}$$

This condition is satisfied: the surely player 1 obtains in equilibrium a payoff of 2 (from having Quiche for breakfast, which he dislikes), and the highest payoff he could obtain from deviating towards beer is 3. Hence, the surely type is the only type of player 1 who has incentives to deviate towards beer.

Player 2's beliefs, after observing the (off-the-equilibrium) message of beer, can then be restricted to $\Theta^{**}(Beer) = \{Surely\}$.

Second step

After restricting the receiver's beliefs to $\Theta^{**}(Beer) = \{Surely\}$, player 2 responds by not dueling player 1. The second step of the Intuitive Criterion analyzes if there is any type of sender (Wimpy or Surely) and any type of message he would send that satisfies:

$$\underset{a \in A^*[\Theta^{**}(m), m]}{Min} u_i\,(m, a, \theta) > u_i^*(\theta)$$

In this context, this condition holds for the surely type since the minimal payoff he can obtain by deviating (having Beer for breakfast) *after* the receiver's beliefs have been restricted to $\Theta^{**}(Beer) = \{Surely\}$, is 3 (given that player 2 responds by not dueling any player who drinks beer). In contrast, his equilibrium payoff from having quiche for breakfast in this pooling equilibrium is only 2. As a consequence, the surely type deviates to beer, and the pooling equilibrium in which both types of player 1 have Quiche for breakfast *does not survive* the Cho and Kreps' (1987) Intuitive Criterion.

EconS 424 - Strategy and Game Theory
Final Exam - Answer key

5. **Cournot competition with cost externalities.** Consider a Cournot duopoly game where every firm i's inverse demand is $p(Q) = 1 - Q$, and Q denotes aggregate output. Every firm's marginal cost is, instead of c, given by $c - \alpha q_j$, where $\alpha \in [-1, 1]$ measures the degree of cost externalities that firm j's output imposes in firm i. This setting allows for cost externalities to be positive, $\alpha > 0$, if firm j's production decreases its rival's marginal cost; and negative, $\alpha < 0$, if q_j increases firm i's marginal cost.

   (a) Find every firm's best response function. How is it affected by parameter α? Interpret.

   - Firm i's profit maximization problem is:

     $$\max_{q_i \geq 0} \pi_i = (1 - q_i - q_j)q_i - (c - \alpha q_j)q_i$$

     Differentiating with respect to q_i, we obtain

     $$1 - 2q_i - q_j - c + \alpha q_j = 0$$

     Solving for q_i gives firm i's best response function:

     $$q_i(q_j) = \frac{1-c}{2} - \frac{1-\alpha}{2} q_j$$

     which originates at $\frac{1-c}{2}$ and decreases in q_j at a rate of $\frac{1-\alpha}{2}$.
   - The vertical intercept is unaffected by parameter α, but the slope of the best response function increases in α since

     $$\frac{\partial}{\partial \alpha}\left(-\frac{1-\alpha}{2}\right) = \frac{1}{2} > 0$$

     Therefore, as the degree of cost externalities that firm j's output imposes in firm i increases, the best response function becomes steeper, indicating that competition becomes more intense.

   (b) Find every firm' equilibrium output, aggregate output, price and profit. How are they affected by parameter α? Interpret.

   - In a symmetric equilibrium, firms produce the same output level, $q_i = q_j = q$. Inserting this property in the above best response function, yields

     $$q = \frac{1-c}{2} - \frac{1-\alpha}{2} q$$

     which simplifies to $2q = 1 - c - (1 - \alpha)q$. Solving for q, we obtain equilibrium output

     $$q^* = \frac{1-c}{3-\alpha}$$

Given the equilibrium output, the aggregate output is

$$\begin{aligned} Q^* &= q_i^* + q_j^* \\ &= \frac{2(1-c)}{3-\alpha} \end{aligned}$$

the equilibrium price is

$$\begin{aligned} p^* &= 1 - q_i^* - q_j^* \\ &= \frac{1+2c-\alpha}{3-\alpha} \end{aligned}$$

and equilibrium profit is

$$\begin{aligned} \pi^* &= p^* q^* - (c - \alpha q^*) q^* \\ &= \frac{(1-c)^2}{(3-\alpha)^2} \end{aligned}$$

- When we differentiate with respect to α, we obtain that

$$\frac{\partial q^*}{\partial \alpha} = \frac{(1-c)}{(3-\alpha)^2} > 0$$

$$\frac{\partial Q^*}{\partial \alpha} = \frac{2(1-c)}{(3-\alpha)^2} > 0$$

$$\frac{\partial p^*}{\partial \alpha} = -\frac{2(1-c)}{(3-\alpha)^2} < 0$$

$$\frac{\partial \pi^*}{\partial \alpha} = \frac{2(1-c)^2}{(3-\alpha)^3} > 0$$

Therefore, as the cost externalities that firm j's output imposes in firm i increases (becomes more positive as α is higher), both individual and aggregate output increase and individual profits increase as well. In contrast, equilibrium prices decrease. Nonetheless, the price margin, $p^* - (c - \alpha q^*)$, increases since

$$\begin{aligned} p^* - (c - \alpha q^*) &= \frac{1+2c-\alpha}{3-\alpha} - \left(c - \alpha \frac{1-c}{3-\alpha}\right) \\ &= \frac{1+2\alpha-c}{3-\alpha} \end{aligned}$$

and differentiating with respect to α yields

$$\frac{\partial \left[p^* - (c - \alpha q^*)\right]}{\partial \alpha} = \frac{7-c}{(3-\alpha)^2} > 0.$$

- When $\alpha = 0$, one firm's output does not impose cost externality on its rival, thus they are unaffected by each other's outputs. In this context, equilibrium output is $q^* = \frac{1-c}{3}$ as in standard Cournot models.
- When $\alpha = 1$, the cost externality is positive, one firm's production decreases its rival's marginal cost, and equilibrium output becomes $q^* = \frac{1-c}{2}$.
- When $\alpha = -1$, the cost externality is negative, one firm's production increases its rival's marginal cost, and equilibrium output becomes $q^* = \frac{1-c}{4}$.

<u>Exercise #6</u> – Entry Deterring Investment

(a) Consider the timing: The incumbent chooses whether to invest; then the entrant observes the incumbent's investment decision; then the firms compete in prices. Show that in subgame perfect equilibrium the incumbent does not invest.

ANSWER: 1st period, Incumbent decides to I or Not invest, while in the 2nd period, firms compete in a Bertrand oligopoly.

<u>Second Stage</u>

If the incumbent invests, then costs satisfy $c_1 = c_2 = 0$, and each firm $i \neq j$ maximizes

$$\max_{p_i} \quad (1 - 2p_i + p_j)p_i$$

Taking first-order conditions with respect to p_i, we obtain $1 - 4p_i + p_j = 0 \Rightarrow 4p_i = 1 + p_j \Rightarrow p_i = \frac{1+p_j}{4}$

And notice that second-order conditions for a maximum in this case are -4 < 0.

By symmetry, $p_i = \frac{1+\frac{1+p_i}{4}}{4} = \frac{4+1+p_i}{16} = \frac{5+p_i}{16}$

And solving for p_i, we have $16p_i = 5 + p_i \Rightarrow p_i = p_j = \frac{1}{3}$

We can now find which are the associated profits:

$$\pi_i = \frac{1}{3}\cdot\left(1 - \frac{2}{3} + \frac{1}{3}\right) = \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$$

Hence, the incumbent's *net* profits (after including the investment in cost reducing technology) are,

$$\pi_1 = \frac{2}{a} - 0.205 = 0.017$$

while the entrant's profits are just $\frac{2}{9}$.

If the incumbent does not invest, then costs are $c_1 = \frac{1}{2}$ and $c_2 = 0$. In such case, the incumbent's profit maximization problem is

$$\max_{p_1}(1 - 2p_1 + p_2)(p_1 - 0.5) = p_1 - 2p_1^2 + p_1 p_2 - 0.5 + p_1 - 0.5p_2$$

Taking first-order conditions with respect to p_1 yields

$$1 - 4p_1 + p_2 + 1 = 0$$

And solving for p_i we obtain the incumbent's best-response function

$$2 + p_2 = 4p_1 \Rightarrow p_1 = \frac{2 + p_2}{4} \qquad \text{(A)}$$

[Notice that in this case second-order conditions are also satisfied, since -4 < 0].

The entrant's profit-maximization problem in the case that the incumbent does not invest is

$$\max_{p_2}(1 - 2p_2 + p_1)p_2$$

Taking first-order conditions with respect to p_2 yields $1 - 4p_2 + p_1 = 0$, and solving for p_2, we obtain

$$p_2 = \frac{1 + p_1}{4} \qquad \text{(B)}$$

[Notice that in this case second-order conditions are also satisfied, since -4 < 0].
Solving for (p_1, p_2) in (A) and (B), we obtain

$$p_1 = \frac{2 + \left(\frac{1 + p_1}{4}\right)}{4} = \frac{8 + 1 + p_1}{16} = \frac{9 + p_1}{16}$$

and solving for p_1 yields

$$16p_1 = 9 + p_1 \Rightarrow 15p_1 = 9 \Rightarrow p_1 = \frac{3}{5}$$

Plugging this result into the entrant's best response function (as described in B) yields

$$p_2 = \frac{1 + \frac{3}{5}}{4} = \frac{8}{20} = \frac{2}{5}$$

Hence, profits for the incumbent and entrant when the incumbent does not invest are:

$$\pi_1 = \left(1 - 2 \cdot \frac{3}{5} + \frac{2}{5}\right)\left(\frac{3}{5} - \frac{1}{2}\right) = 0.02$$

$$\pi_2 = \left(1 - 2 \cdot \frac{2}{5} + \frac{3}{5}\right) \cdot \frac{2}{5} = 0.32$$

<u>First Stage</u>

The incumbent decides <u>not</u> to invest in cost-reducing technologies since its profits from not investing, 0.02, are higher than from investing, 0.017.

(b) Show that if the investment decision is *not* observed by the entrant, the incumbent's investing is part of the equilibrium. Comment.

ANSWER: Now the Entrant only chooses a single p_2. We know from part (a) that $p_2 = \frac{1}{3}$ if the Incumbent chose invest, and that $p_2 = \frac{2}{5}$ if the Incumbent chose not to invest.

1) If $p_2 = \frac{1}{3}$, then the Incumbent profits are

   If the incumbent invests:

$$\max_{p_1}\left(1-2p_1+\frac{1}{3}\right)p_1$$

FOCs are

$$1-4p_1+\frac{1}{3}=0\Rightarrow 4p_1=\frac{2}{3}\Rightarrow p_1=\frac{1}{3}$$

and therefore the incumbent's profits are

$$\pi_1=\left(1-\frac{2}{3}+\frac{1}{3}\right)\frac{1}{3}-0.205=0.017$$

If, instead, the incumbent does not invest:

$$\max_{p_1}\left(1-2p_1+\frac{1}{3}\right)\left(p_1-\frac{1}{2}\right)$$

FOCs are now:

$$1-4p_1+\frac{1}{3}+1=0\Rightarrow 4p_1=\frac{7}{3}\Rightarrow p_1=\frac{7}{12}$$

And therefore the incumbent's profits are

$$\pi_1=\left(1-\frac{14}{12}+\frac{1}{3}\right)\cdot\left(\frac{7}{12}-\frac{1}{2}\right)=\frac{1}{72}=0.014$$

Hence, when the entrant's price is $p_2=\frac{1}{3}$, the incumbent prefers to invest, since its profits from so doing, 0.017, exceed those from not investing, 0.014.

2) If $p_2=\frac{2}{5}$, then the incumbent profits are

If the incumbent invests:

$$\max_{p_1}\left(1-2p_1+\frac{2}{5}\right)p_1$$

FOCs are:

$$1-4p_1+\frac{2}{5}=0\Rightarrow 4p_1=\frac{7}{5}\Rightarrow p_1=\frac{7}{20}$$

And thus the incumbent's profits become

$$\pi_1=\left(1-\frac{14}{20}+\frac{2}{5}\right)\cdot\frac{7}{20}-0.205=\frac{49}{200}-0.205=0.04$$

If, instead, the incumbent does not invest:

$$\max_{p_1}\left(1+2p_1+\frac{2}{5}\right)\left(p_1-\frac{1}{2}\right)$$

And FOCs now become:

$$1-4p_1+\frac{2}{5}+1=0 \Rightarrow 4p_1=\frac{12}{5} \Rightarrow p_1=\frac{12}{20}=\frac{3}{5}$$

Implying that the incumbent's profits are

$$\pi_1=\left(1-2\cdot\frac{3}{5}+\frac{2}{5}\right)\left(\frac{3}{5}-\frac{1}{2}\right)=\frac{1}{5}\cdot\frac{1}{10}=\frac{1}{50}=0.02$$

Therefore, when the entrant's price is $p_2=\frac{2}{5}$, the incumbent also prefers to invest, since its profits from so doing, 0.04, exceed those from not investing, 0.02. Hence, the Incumbent always <u>invests</u> in pure strategies.

Entrant. We finally need to check if the entrant prefers to set a price $p_2=\frac{1}{3}$ or $p_2=\frac{2}{5}$. Comparing its profits from each pricing policy, we obtain that the entrant prefers to set a price of $p_2=\frac{1}{3}$.

Therefore, the unique SPNE of the entry game is (Invest, $p_1=p_2=\frac{1}{3}$)

(iii) Explain why the conclusion is question (i) may be affected if the entrant faces a fixed entry cost, *F*.

- Does it matter whether the potential entrant makes its entry decision before or after the incumbent's investment decision?

ANSWER: If the entrant makes its entry decision AFTER the incumbent's investment decision, the investment decisions may act as an entry barrier if and only if

- $\pi_{entrant}$ with investment = 2/9 – fixed costs of entry < 0, and
- $\pi_{entrant}$ without investment = 0.32 – fixed costs of entry > 0

If, instead, the entrant's entry decision is made BEFORE the incumbent's investment decision,

- Incumbent must decide I/NI and p_1
- Entrant must decide E/NE and p_2

<u>Incumbent Best Response:</u>

If Entrant enters (E) and $(E, p_2=1/3)$, then Incumbent $(I, p_1=1/3)$

If Entrant enters (E) and $(E, p_2=2/5)$, then Incumbent $(I, p_1=7/20)$

But from part (b) we know that $p_1=\frac{7}{20}$ and $p_2=\frac{2}{5}$ will never happen in equilibrium.

<u>Entrant Best Response:</u>

If Incumbent $(I, p_1=1/3)$, then Entrant:

$(E, p_2 = 1/3)$ if 2/9-FCE > 0
Not enter if 2/9-FCE < 0

If Incumbent $(I, p_1 = 7/20)$, then Entrant:

$(E, p_2 = 2/5)$ if 0.22 – FCE > 0
Not enter if 0.22 – FCE <0

But we know from part (b) that $p_1 = \frac{7}{20}$ and $p_2 = \frac{2}{5}$ cannot be part of a subgame perfect equilibrium.

Therefore, the unique equilibrium with entry is:

- Entrant: (E, p_2=1/2) if 2/9 –F > 0
- Incumbent: (I, p_1=1/2)