EconS 424 - Strategy and Game Theory
Homework #5 - Answer key

1. **Exercises from Harrington:**

   (a) **Chapter 10:** Exercises 2 and 5.

   - See scanned pages at the end of the handout.

2. **Bargaining under incomplete information, allowing for general valuations.** Consider a bargaining game between a seller and a buyer we described in class, but assume that the buyer's valuation for the object is either high, v^H, with probability $\alpha \in [0,1]$, and low, v^L, with probability $1-q$; where $v^H > v^L > 0$. The seller does not observe the buyer's valuation, but knows that it's distributed according to the above probabilities. The seller is the first mover in this game, and without observing the realization of the buyer's valuation, he sets a price p for the good. Observing this price, but observing his valuation for the object (v^H or v^L), the buyer responds accepting or rejecting the deal.

   (a) *Buyer's response.* Since the buyer is the privately informed player in this game, identify his best response: first, when his valuation is v^H, and, second, when his valuation is v^L.

   - If the buyer's valuation is $v_B = v^H$, the buyer accepts any price p such that

   $$v^H - p \geq 0 \Longrightarrow p \leq v^H$$

   - If the buyer's valuation is, instead, $v_B = v^L$, he accepts any price p such that

   $$v^L - p \geq 0 \Longrightarrow p \leq v^L$$

   - Combining both decision rules, we obtain three regions for price p:
     - *Low prices.* When $p \leq v^L$, the buyer accepts the price regardless of his valuation for the object.
     - *Medium prices.* When $v^L < p \leq v^H$, the buyer accepts the price when his valuation is v^H, but rejects it when his valuation is v^L.
     - *High prices.* When $v^H < p$, the buyer rejects the price regardless of his valuation for the object.

   (b) *Seller's price.* Find the price that the seller sets in the BNE of the game.

   - The seller anticipates the decision rule that we found in part (a), but does not observe the buyer's valuation (v^H or v_L).He sets price $p = v^H$ if and only if

   $$\begin{aligned} EU_S(p = v^H) &\geq EU_S(p = v^L) \\ qv^H + (1-q)0 &\geq qv^L + (1-q)v^L \\ qv^H &\geq v^L \end{aligned}$$

Solving for q, we obtain

$$q \geq \frac{v^L}{v^H}$$

which satisfies $q \in [0,1]$ since $v^H > v^L > 0$ by assumption.

- *Remark.* You may wonder why we only compared the seller's utility in two specific prices: $p = v^H$ and $p = v^L$, among all prices $p \geq 0$ that he can choose from. To understand this point, let's consider a price p' in the following three intervals:
  - If price p' satisfies $v^L < p' < v^H$, it is not as profitable as $p = v^H$. Indeed, low-value buyers do not purchase the good at this price, exhibiting a similar behavior as when the seller sets $p = v^H$. Thefore, the seller does not increase his sales when setting p' but decreases his margin from the unit he sells to high-value buyers (i.e., the margin from price p' is smaller than that with $p = v^H$).
  - If price p' satisfies $p' > v^H$, it is rejected by all types of buyers (regardless of their valuation), which induces zero profits with certainty. The seller is, of course, better off setting a price of $p = v^H$ or $p = v^L$.
  - If price p' satisfies $p' < v^L$, it is accepted by all types of buyers (regardless of their valuation), which implies that the seller could earn a higher profit, and still induce the acceptance by all types of buyers, if he sets a higher price of $p = v^L$.

  We have exhausted all possible prices (strictly above v^H, below v^L, or between these two valuations), showing that the seller is better off setting a price of $p = v^H$, which is profitable when probability q satisfies $q \geq \frac{v^L}{v^H}$, or a price of $p = v^L$ otherwise.
- Summarizing, we found two BNEs in this game:
  - If $q \geq \frac{v^L}{v^H}$, indicating that high-value buyers are relatively likely, the seller sets a price at $p = v^H$ and the buyer accepts any price $p \leq v^H$ if his valuation is high, and $p \leq v^L$ if his valuation is low.
  - If $q < \frac{v^L}{v^H}$, indicating that high-value buyers are relatively unlikely, the seller sets a price at $p = v^L$ and the buyer accepts any price $p \leq v^H$ if his valuation is high, and $p \leq v^L$ if his valuation is low.

(c) How are the BNEs you identified in part (b) affected by an increase in v^H? And by an increase in v^L? Interpret.

- An increase in v^H, produces a decrease in ratio $\frac{v^L}{v^H}$, thus making the first BNE more likely to occur or, alternatively, shrinking the range of q's that support the second BNE. Intuitively, high-value buyers become more attractive to the seller, relatively to the low-value buyer, thus inducing him to set a price at $p = v^H$ under larger parameter conditions.
- The opposite argument applies when v^L increases, as that increases ratio $\frac{v^L}{v^H}$, which makes the first (second) BNE less likely (more likely) to occur. In this case, the low-value buyer becomes more attractive, relative to the high-value buyer.

3. **Cournot competition with incomplete information, allowing for a general probability.** Consider the duopoly market under incomplete information discussed in class, where firms face an inverse demand function $p(Q) = 1 - Q$, and $Q \geq 0$ denotes aggregate output, $Q = q_1 + q_2$. Firm 2 privately observes its marginal cost, $c_H = \frac{1}{2}$ or $c_L = 0$, with probability p and $1 - p$, respectively. Firm 1's marginal costs are low, $c_L = 0$, which is common knowledge among both firms.

   (a) *Privately informed firm.* Find firm 2's best response function when its production cost is $c_H = \frac{1}{2}$, and denote it as $q_2^H(q_1)$. Is it increasing or decreasing in probability p? Interpret.

   - When its marginal cost is high, firm 2 solves

$$\max_{q_2 \geq 0} \ (1 - q_1 - q_2)q_2 - \frac{1}{2}q_2$$

   Differentiating with respect to q_2, yields $1 - q_1 - 2q_2 - \frac{1}{2} = 0$. Solving for q_2, we find firm 2's best response function when its marginal cost is high, that is,

$$q_2^H(q_1) = \frac{1}{4} - \frac{1}{2}q_1$$

   (b) *Privately informed firm.* Find firm 2's best response function when its production cost is $c_L = 0$, and denote it as $q_2^L(q_1)$. Is it increasing or decreasing in probability p? Interpret.

   - When its marginal cost is low, firm 2 solves

$$\max_{q_2 \geq 0} \ (1 - q_1 - q_2)q_2 - 0q_2$$

   Differentiating with respect to q_2, yields $1 - q_1 - 2q_2 = 0$. Solving for q_2, we find firm 2's best response function when its marginal cost is low, that is,

$$q_2^L(q_1) = \frac{1}{2} - \frac{1}{2}q_1$$

   Comparing it against that in part (a), we see that $q_2^L(q_1) > q_2^H(q_1)$ for all values of q_1, meaning that firm 2 produces more units when its marginal cost is low than when it is high. Graphically, $q_2^L(q_1)$ originates at $1/2$ while $q_2^H(q_1)$ originates at $1/4$, and they are both parallel to each other (both have a slope of $-1/2$).

   (c) *Uninformed firm.* Find firm 1's best response function, and denote it as $q_1(q_2^H, q_2^L)$. Is it increasing or decreasing in probability p? Interpret.

   - Firm 1 does not know firm 2's marginal cost, so it must maximize its expected profits, as follows

$$\max_{q_1 \geq 0} \ p\underbrace{\left[(1 - q_1 - q_2^H)q_1\right]}_{\text{Firm 1 is high cost}} + (1-p)\underbrace{\left[(1 - q_1 - q_2^L)q_1\right]}_{\text{Firm 1 is low cost}} - 0q_2$$

Differentiating with respect to q_1 yields

$$p\left(1-2q_1-q_2^H\right)+(1-p)(1-2q_1-q_2^L)=0$$

Solving for q_1, we find firm 1's best response function

$$q_1(q_2^H,q_2^L)=\frac{1}{2}-\frac{1}{2}\left[pq_2^H+(1-p)q_2^L\right]$$

which, intuitively, indicates that firm 1 decreases its output by half unit for every one-unit increase in firm 2's expected output, $pq_2^H+(1-p)q_2^L$.

(d) Find equilibrium output levels, q_1, q_2^H, and q_2^L. How is each of them affected by a marginal increase in probability p? Interpret.

- Inserting firm 2's best response functions from parts (a) and (b) into firm 1's best response function, found in part (c), we obtain

$$q_1=\frac{1}{2}-\frac{1}{2}\left[p\underbrace{\left(\frac{1}{4}-\frac{1}{2}q_1\right)}_{q_2^H(q_1)}+(1-p)\underbrace{\left(\frac{1}{2}-\frac{1}{2}q_1\right)}_{q_2^L(q_1)}\right]$$

which is only a function of q_1. Rearranging, yields

$$q_1=\frac{2(1+q)+p}{8}$$

and solving for q_1, we obtain firm 1's equilibrium output,

$$q_1^*=\frac{2+p}{6}.$$

Inserting this result into firm 2's best response functions, yields

$$q_2^H=\frac{1}{4}-\frac{1}{2}\underbrace{\left(\frac{2+p}{6}\right)}_{q_1^*}=\frac{1-p}{12}$$

and

$$q_2^L=\frac{1}{2}-\frac{1}{2}\underbrace{\left(\frac{2+p}{6}\right)}_{q_1^*}=\frac{4-p}{12}$$

- We can summarize the BNE of this game as the following output triplet

$$(q_1,q_2^H,q_2^L)=\left(\frac{2+p}{6},\frac{1-p}{12},\frac{4-p}{12}\right).$$

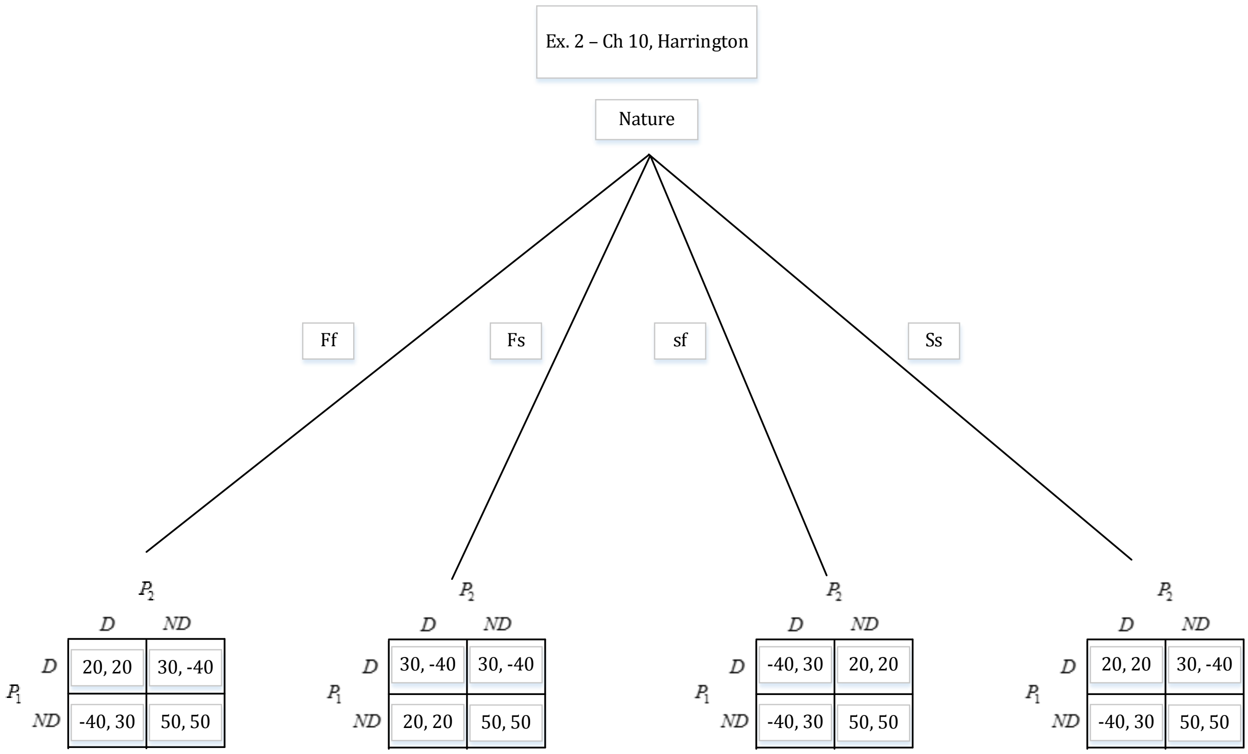
(e) Evaluate the equilibrium output levels, found in part (d), at probability $p=1$. Then, evaluate these output levels at $p=0$. Interpret your results.

- When firm 2's costs are high with certainty, $p = 1$, the triplet we found in part (d) simplifies to $\left(\frac{1}{2}, 0, \frac{1}{4}\right)$. Intuitively, firm 1 now knows that firm 2 is a high-cost rival with certainty, implying that firm 1 produces the monopoly output, 1/2, while firm 2 does not produce anything. This occurs because the cost advantage of firm 1 is too large to allow a high-cost firm 2 to produce any positive amount. Note that this happens because firm 1 is perfectly informed about firm 2's high costs in this setting. When firm 1 wasn't perfectly informed ($0 < p < 1$), firm 2 could produce a positive output even when its costs were high.
- When firm 2's costs are low with certainty, $p = 0$, the triplet we found in part (d) simplifies to $\left(\frac{1}{3}, 0, \frac{1}{3}\right)$. In this case, firms 1 and 2 know that their costs are low, and symmetric, thus producing the same output in equilibrium, 1/3, as in a standard Cournot game of complete information and symmetric firms.

ECONS 424 – STRATEGY AND GAME THEORY

HOMEWORK #6 – ANSWER KEY

HARRINGTON CHAPTER 10 –EXERCISE 2



Where F/S denotes Fast/Slow for player 1 (Bat)

f/s denotes fast/slow for player 2 (Curly Bill)

Where D represents Draw

ND represents Not Draw

fast, then each has a payoff of 20. If at least one chooses draw, then there is a gunfight.

a. Is it consistent with Bayes–Nash equilibrium for there to be a gunfight for sure? (That is, both gunfighters draw, regardless of their type.)

ANSWER: Yes, as it is an equilibrium for each gunfighter to use a strategy that has him draw regardless of his type. The equilibrium conditions for Bat are

$$\text{Fast Type (Draw): } .6 \times 20 + .4 \times 30 \geq .6 \times (-40) + .4 \times 20 \Rightarrow 24 \geq -16$$

$$\text{Slow Type (Draw): } .6 \times (-40) + .4 \times 20 \geq .6 \times (-40) + .4 \times (-40) \Rightarrow -32 \geq -40.$$

The equilibrium conditions for Curly Bill are

$$\text{Fast Type (Draw): } .65 \times 20 + .35 \times 30 \geq .65 \times (-40) + .35 \times 20 \Rightarrow 23.5 \geq -19$$

$$\text{Slow Type (Draw): } .65 \times (-40) + .35 \times 20 \geq .65 \times (-40) + .35 \times (-40) \Rightarrow -19 \geq -40.$$

b. Is it consistent with Bayes–Nash equilibrium for there to be no gunfight for sure? (That is, both gunfighters wait, regardless of their type.)

ANSWER: Yes, as it is an equilibrium for each gunfighter to use a strategy that has him wait regardless of his type. Doing so realizes a payoff of 50—as a gunfight is avoided—and all other outcomes yield a lower payoff, so the expected payoff from any drawing must be less. More explicitly, the equilibrium conditions for Bat are

$$\text{Fast Type (Wait): } .6 \times 50 + .4 \times 50 \geq .6 \times 30 + .4 \times 30 \Rightarrow 50 \geq 30$$

$$\text{Slow Type (Wait): } .6 \times 50 + .4 \times 50 \geq .6 \times 20 + .4 \times 30 \Rightarrow 50 \geq 24.$$

The equilibrium conditions for Curly Bill are

$$\text{Fast Type (Draw): } .65 \times 50 + .35 \times 50 \geq .65 \times 30 + .35 \times 30 \Rightarrow 50 \geq 30$$

$$\text{Slow Type (Draw): } .65 \times 50 + .35 \times 50 \geq .65 \times 20 + .35 \times 30 \Rightarrow 50 \geq 23.5.$$

c. Is it consistent with Bayes–Nash equilibrium for a gunfighter to draw only if he is slow?

ANSWER: Yes. Consider a strategy profile in which each draws when slow and waits when fast. The equilibrium conditions for Bat are

Fast Type (Wait): $.6 \times 50 + .4 \times 20 \geq .6 \times 30 + .4 \times 30 \Rightarrow 38 \geq 30$

Slow Type (Draw): $.6 \times 20 + .4 \times 20 \geq .6 \times 50 + .4 \times (-40) \Rightarrow 20 \geq 14.$

The equilibrium conditions for Curly Bill are

Fast Type (Wait): $.65 \times 50 + .35 \times 20 \geq .65 \times 30 + .35 \times 30 \Leftrightarrow 39.5 \geq 30$

Slow Type (Draw): $.65 \times 20 + .35 \times 20 \geq .65 \times 50 + .35 \times (-40) \Leftrightarrow 20 \geq 18.5.$

HARRINGTON CHAPTER 10 – EXERCISE 5

Player 2 has only two strategies $S_2 = \{a, b\}$

Player 1 has four strategies $S_1 = \{xx', xy', yx', yy'\}$ where the first component of every strategy pair denotes what player 1 chooses when his type is H and the second component is what he selects when his type is L.

Here, the Bayesian normal form representation of the game is:

Player 2

Player 1

	a	b
xx’		
xy’		
yx’		
yy’		

Let’s find the EU from strategy profile (xx’, a),

$$EU_1 = p * 3 + (1 - p) * 2 = 2 + p$$

$$EU_2 = p * 1 + (1 - p) * 3 = 3 - 2 * p$$

$$\rightarrow (2 + p, 3 - 2p)$$

Similarly, for strategy profile (xy’, a),

$$EU_1 = p * 3 + (1 - p) * 3 = 3$$

$$EU_2 = p * 1 + (1 - p) * 1 = 1$$

$$\rightarrow (3, 1)$$

Proceeding in this fashion, we find the complete normal form game to be:

Player 2

Player 1

	a	b
xx’	2+p, 3-2p	1, 2+p
xy’	3, 1	4-3p, 3p
yx’	2, 3-2p	1+4p, 2
yy’	3-p, 1	4+p, 2p

A) When p= 0.75 the above matrix becomes:

Player 2

Player 1

	a	b
xx'	2.75, 1.5	1, 2.75
xy'	3, 1	1.75, 2.25
yx'	2, 1.5	4, 2
yy'	2.25, 1	4.75, 1.5

Doing the usual underlining to find best responses for each player, we find that there is a unique BNE: (yy', b).

B) Player 1:

When player 2 selects a, he prefers: 3>2+p for all p

3>2

3>3-p Hence, he selects xx'

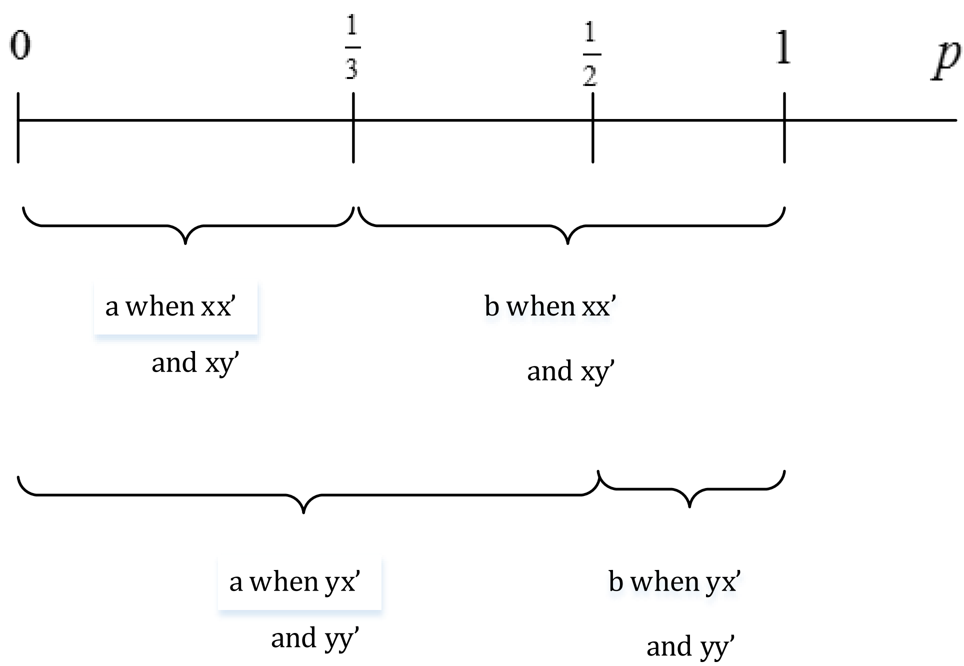
Player 2:

When player 1 selects xx', he prefers a if 3-2p>2+p→ $1 > 3p \rightarrow \frac{1}{3} > p$

When player 1 selects xy', he prefers a if $1 > 3p \rightarrow \frac{1}{3} > p$ (otherwise he prefers b)

When player 1 selects yx', he prefers a if $3 - 2p > 2 \rightarrow 1 > 2p \rightarrow \frac{1}{2} > p$

When player 1 selects yy', he prefers a if $1 > 2p \rightarrow \frac{1}{2} > p$ (otherwise he prefers b)



We can then divide our analysis into three different matrices

- One matrix for $p < \frac{1}{3}$
- Another matrix for $p \in \left[\frac{1}{3}, \frac{1}{2}\right]$
- Another matrix for $p > \frac{1}{2}$

First Case: $p < \frac{1}{3}$

		Player 2	
		a	b
Player 1	xx'	2+p, 3-2p	1, 2+p
	xy'	3, 1	4-3p, 3p
	yx’	2, 3-2p	1+4p, 2
	yy'	3-p, 1	4+p, 2p

Unique BNE: (xy’, a)

Second Case: $p \in \left[\frac{1}{3}, \frac{1}{2}\right]$

Player 2

Player 1	a	b
xx'	2+p, 3-2p	1, $\underline{2+p}$
xy'	$\underline{3}$, 1	4-3p, $\underline{3p}$
yx’	2, $\underline{3-2p}$	1+4p, 2
yy'	3-p, $\underline{1}$	$\underline{4+p}$, 2p

Third Case: $p > \frac{1}{2}$

Player 2

Player 1		a	b
	xx'	2+p, 3-2p	1, 2+p
	xy'	3, 1	4-3p, 3p
	yx'	2, 3-2p	1+4p, 2
	yy'	3-p, 1	4+p, 2p

Unique BNE: (yy', b)

(This BNE is consistent with part (a) of this exercise, where $= 0.75 > \frac{1}{2}$).