EconS 424 – Strategy and Game Theory
Midterm #1 – Answer Key

Exercise #1

a) Starting with player 1, d strictly dominates a. Moving to player 2, we see that y strictly dominates z. Moving then to player 1 again, strategy b strictly dominates d. At this point, we are left with a two-by-two matrix: strategies b and c survive for player 1, and strategies x and y survive for player 2; as depicted in the matrix below. No more strategies can be deleted.

Player 2

		x	y
	b	5, 1	2, 3
Player 1	c	3, 7	4, 6

Because we only have two strategies for each player, we cannot create randomizations between them to try to delete another strategy since, to do that, we need a minimum of three pure strategies (i.e., a randomization between two of them, which dominates the third one).

b) Underlining best response payoffs in the reduced-form matrix that survives IDSDS (see part a), we find the following matrix. Therefore, there is no psNE since no cell has both players' payoffs underlined (no strategy profile is a mutual best response for both players).

Player 2

		x	y
	b	$\underline{5}$, 1	2, $\underline{3}$
Player 1	c	3, $\underline{7}$	$\underline{4}$, 6

c) For player 1, his expected payoffs are

$$EU(b)=5q+2(1-q)=3q+2$$

$$EU(c)=3q+4(1-q)=4-q$$

Implying that player 1 is indifferent between b and c when

$$3q+2=4-q$$

which, solving for q, yields $q=1/2$.

Similarly, for player 2, his expected payoffs are

$$EU(x)=1p+7(1-p)=7-6p$$

$$EU(y)=3q+6(1-q)=6-3p$$

Implying that player 2 is indifferent between x and y when

$$7-6p=6-3p$$

which, solving for p, yields p=1/3. Summarizing, the msNE is {(1/3b,2/3c),(1/2x,1/2y)}.

d) The figure of the best response functions should look like that in the “Tennis game” (see set of slides #6 on the course website, and go to slide #32 for a similar figure). The best response functions cross at an interior point, where p=1/3 and q=1/2, but they do not touch at any border point. If they did, there would be a psNE.

Exercise #2 – Sequential-move game

a) Player 1 has four strategies: Ax, Ay, Bx, and By.
b) The following matrix identifies best response payoffs.

		Player 2	
		L	R
Player 1	Ax	1, $\underline{2}$	$\underline{0}$, 0
	Ay	3, $\underline{4}$	$\underline{0}$, 0
	Bx	$\underline{5}$, $\underline{6}$	$\underline{0}$, 0
	By	$\underline{5}$, $\underline{6}$	$\underline{0}$, 0

c) From the above matrix, there are two psNEs: (Bx, L) and (By, L).
d) There are two subgames: the subgame starting after player 2 chooses L and player 1 responds with A (on the left bottom part of the game tree), and the game as a whole.
e) Solving the subgame after player 2 chooses L and player 1 responds with A, player 1 chooses y since 3>1. The remaining game is a simultaneous-move game, which we represent in the following matrix.

	Player 2	
	L	R
A	3, $\underline{4}$	$\underline{0}$, 0
B	$\underline{5}$, $\underline{6}$	$\underline{0}$, 0

Therefore, the SPNE of the game is (By,L). As a consequence, only one of the two psNEs identified in part c are sequentially rational. The other psNE, (Bx, L) is not sequentially rational since, once player 1 is called to move at the end of the game tree, he prefers y to x.

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Midterm exam #1 – Answer key

Exercise #4 – R&D tournaments

Part (a). Firm 1's optimal expenditure is the value of x_1 for which the first derivative of its profit function equals zero. That is,

$$\frac{\partial \pi_1(x_1, x_2)}{\partial x_1} = 36\left[\frac{x_1 + x_2 - x_1}{(x_1 + x_2)^2}\right] - 1 = 0$$

Rearranging, we find

$$36\left[\frac{x_2}{(x_1 + x_2)^2}\right] - 1 = 0$$

$$36x_2 = (x_1 + x_2)^2$$

and further rearranging

$$6\sqrt{x_2} = x_1 + x_2$$

Solving for x_1, we obtain firm 1's best response function

$$x_1(x_2) = 6\sqrt{x_2} - x_2$$

Figure 1 depicts firm 1's best response function, $x_1(x_2) = 6\sqrt{x_2} - x_2$ as a function of its rival's expenditure, x_2 in the horizontal axis for the admissible set $x_2 \in [0, 25]$.

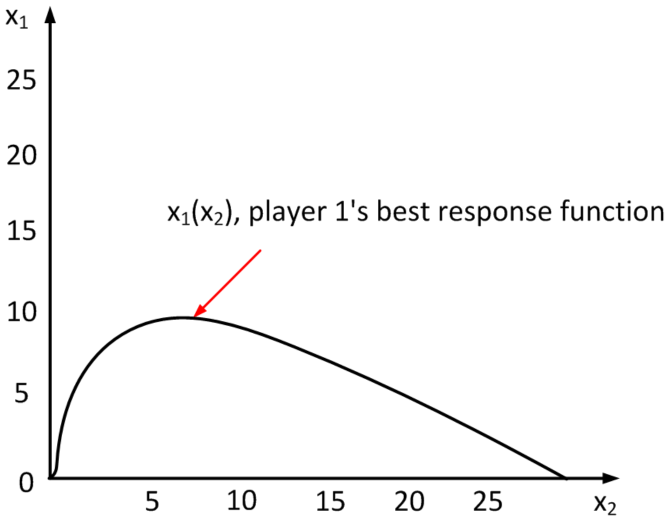

Figure 1. Firm 1’s Best Response Function

Part (b). It is straightforward to show that, for all values of $x_2 \in [0, 25]$, firm 1’s best response also lies in the admissible set $x_1 \in [0, 25]$. In particular, the maximum of BRF$_1$ occurs at x_2=9 since

$$\frac{\partial BR_1(x_2)}{\partial x_2} = \frac{\partial[6\sqrt{x_2} - x_2]}{\partial x_2} = 3(x_2)^{-\frac{1}{2}} - 1$$

Hence, the point at which this best response function reaches its maximum is that in which its derivative is zero, i.e., $3(x_2)^{-\frac{1}{2}} - 1 = 0$, which yields a value of $x_2 = 9$. At this point, firm 1’s best response function informs us that firm 1 optimally spends $6\sqrt{9} - 9 = 9$. Finally, note that the best response function is concave in its rival expenditure, x_2, since

$$\frac{\partial^2 BR_1(x_2)}{\partial x_2} = -\frac{3}{2}(x_2)^{-\frac{3}{2}} < 0.$$

By symmetry, firm 2’s best response function is $x_2(x_1) = 6\sqrt{x_1} - x_1$.

Part (c). In a symmetric Nash equilibrium, x^*, the above best response function simplifies to

$$x^* = 6\sqrt{x^*} - x^*$$

Rearranging, we obtain $2x^* = 6\sqrt{x^*}$, and solving for x^*, we find $x^* = 9$. Hence, the unique symmetric Nash equilibrium has each firm spending 9. As figure 1 depicts, the points at which the best response function of player 1 and 2 cross each other occur at the 45-degree line (so the equilibrium is symmetric). In particular, those points are the origin, i.e., (0,0), but this case is uninteresting since it implies that no firm spends money on R&D, and (9,9).

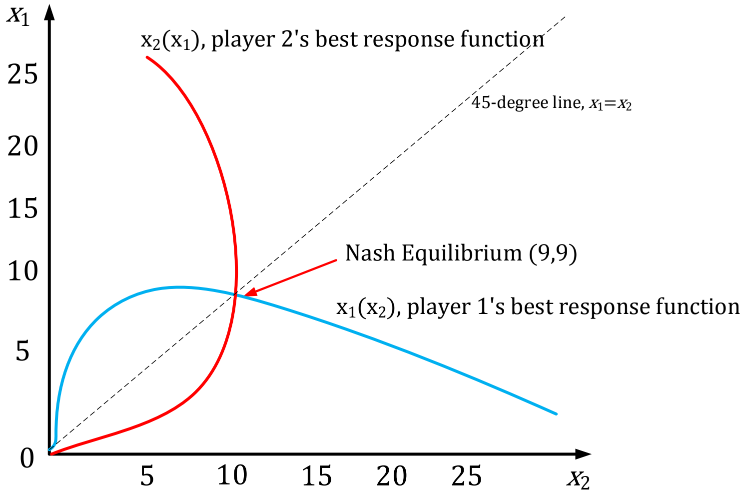

Figure 2. Best Response Functions. Nash-Equilibrium

in period $t - 1$ then player 2 chooses x in period t; 2) player 1 chose different from d and player 2 chose x in period $t - 1$ then player 1 chooses y in all ensuing periods; 3) player 1 chose d and player 2 chose different from x in period $t - 1$ then player 1 chooses v in all ensuing periods; and 4) player 1 chose different from d and player 2 chose different from x in period $t - 1$ then player 1 chooses v in all ensuing periods. Derive the conditions for this strategy pair to be an SPNE.

ANSWER: Take period 1 or any period for which (d,x) has always been played. The profile specifies that (d,x) is to be played. If player 1 deviates this period and follows the strategy afterwards, he can at most get 4 this period by deviating to a. In the ensuing periods, (c,y) will be played and player 1 will get 2 in each period and that means a payoff of $\delta \times 2/(1 - \delta)$. On the other hand, if player 1 does not deviate, he will get 3 in every period, which yields a payoff of $3/(1 - \delta)$. Hence, it is optimal for him to choose d when

$$\frac{3}{1-\delta} \geq 4 + \delta\left(\frac{2}{1-\delta}\right) \Rightarrow 3 \geq 4 - 4\delta + 2\delta \Rightarrow \delta \geq \frac{1}{2}.$$

For player 2, if she deviates this period and follows her strategy thereafter, she can get at most 5 this period by choosing w. In the following periods, (b,v) will be played, which means player 2 will get 2 in each period which yields a payoff of $\delta \times 2/(1 - \delta)$. If he does not deviate he will get $3/(1 - \delta)$. So we need

$$\frac{3}{1-\delta} \geq 5 + \delta\left(\frac{2}{1-\delta}\right) \Rightarrow 3 \geq 5 - 5\delta + 2\delta \Rightarrow \delta \geq \frac{2}{3}.$$

For any period for which the history encompassed a deviation, the strategy profile specifies one of the Nash equilibria of the one-shot game. Hence, no player has a profitable deviation.

To sum, it is an SPNE if and only if $\delta \geq \frac{1}{2}$ and $\delta \geq \frac{2}{3}$ or, equivalently, $\delta \geq \frac{2}{3}$.

13. Consider a situation in which there is a group of $n \geq 2$ players and, in each period, a player decides how large an investment to make. For every \$1 a player invests, it is increased by 50% but the total return—which, for example, would be \$150 for a \$100 investment—is equally shared among all members of the group. Thus, if player i contributes z_i then the payoff to player i is $1.5 \times ((z_1 + z_2 + \cdots + z_n)/n) - z_i$, where the total contribution of the n players is $z_1 + z_2 + \cdots + z_n$ which is multiplied by 1.5 and then divided equally among the n players; and from this amount is subtracted the investment of player i, z_i, to derive player i's payoff. In deciding upon his or her investment, assume a player chooses from the set $\{0,10,20, \ldots, 100\}$. This game is played infinitely often where the discount factor is δ and $0 < \delta < 1$. Consider the following symmetric strategy profile. In period 1, invest z. In period $t(\geq 2)$, invest z if, in all past periods, all players invested z. In period $t(\geq 2)$, invest 0 if, in some past period, at least one player did not invest z; z is some value from $\{10,20, \ldots, 100\}$. Derive the conditions for this strategy profile to be an SPNE.

ANSWER: Consider period 1 or period $t(\geq 2)$ when, in all past periods, all players invested z. A player's strategy prescribes that she invests z. Doing so yields an infinite stream of single-period payoffs equal to

$$\frac{1.5 \times n \times z}{n} - z = \frac{z}{2}.$$

Since the best alternative to investing z is investing zero, after which no one will invest, the equilibrium condition is

$$\frac{z/2}{1-\delta} \geq \frac{1.5 \times (n-1) \times z}{n} \Rightarrow \delta \geq \frac{2n-3}{3n-3}.$$

Now consider period $t(\geq 2)$ when, in some past period, at least one player did not invest z. A player's strategy prescribes that she invest 0 which yields an infinite stream of single-period payoffs equal to 0, as all players are not investing. Instead, investing z_i yields a current period payoff of

$$\left(\frac{1.5}{n} - 1\right)z_i < 0$$

when $z_i > 0$; and a future payoff stream of zero (since, according to players' strategies, they will all invest zero). Clearly, it is better to invest zero than some positive amount.

Thus, it is an SPNE if and only if $\delta \geq \frac{2n - 3}{3n - 3}$. Note that the requirement becomes tighter as there are more players.

14. Consider this symmetric two-player game. Assume this stage game is repeated T times and both players have a discount factor δ and $0 < \delta < 1$.

Player 2

Player 1	*a*	*b*	*c*	*d*
a	0,0	2,4	3,0	10,0
b	4,2	3,3	2,1	3,3
c	0,3	1,2	5,5	6,4
d	0,10	3,3	4,6	9,9

a. Assume $T = 1$. Find all SPNE.

ANSWER: There are two Nash equilibria for the stage game: (b,b) and (c,c).

b. Assume $T = 2$ and consider the following symmetric strategy pair: In period 1, choose d; in period 2, choose c if both players chose d in period 1 and choose b otherwise. Determine whether it is an SPNE.

ANSWER: Note that the sub-strategy for period 2 has players act according to a Nash equilibrium for the stage game. Thus, for all period 2 subgames (of which there are 16, each one associated with the 16 possible period 1 outcomes), the sub-strategy profile forms a Nash equilibrium. In period 1, a player's strategy has her choose d which yields a payoff of $9 + \delta 5$, as players will choose c in period 2. If a player chooses any other action in period 1 then both players will choose b in period 2. Hence, the associated payoffs from various actions in period 1 are:

Action a: $10 + \delta 3$
Action b: $3 + \delta 3$
Action c: $6 + \delta 3$
Action d: $9 + \delta 5$

Since the maximal payoff from choosing anything other than d is $10 + \delta 3$ then d is optimal if and only if:

$$9 + \delta 5 \geq 10 + \delta 3 \Leftrightarrow \delta \geq \frac{1}{2}.$$

This strategy profile is an SPNE if and only if $\delta \geq \frac{1}{2}$.

c. Assume T is infinity and consider the following symmetric strategy pair: In period 1, a player chooses d; in period $t(\geq 2)$, a player chooses d if both players chose d in period $t - 1$, and chooses c otherwise. Derive the conditions for it to be an SPNE.

ANSWER: Consider period 1 or period $t \geq 2$ when (d,d) was chosen in the previous period. The strategy prescribes d which yields a payoff of $\frac{9}{1-\delta}$. The best alternative action is a which yields a payoff of $10 + \frac{\delta 5}{1-\delta}$. Optimality requires:

$$\frac{9}{1-\delta} \geq 10 + \frac{\delta 5}{1-\delta} \Leftrightarrow \delta \geq \frac{1}{5}.$$

Now consider period $t \geq 2$ when (d,d) was not chosen in the previous period. The prescribed action is c and that yields a payoff of $\frac{5}{1-\delta}$ which is at least as high as the payoff from any other action which is the current period payoff plus $\frac{\delta 5}{1-\delta}$. This strategy profile is then an SPNE if and only if $\delta \geq \frac{1}{5}$.

15. Consider this symmetric two-player infinitely repeated version of the stage game where the discount factor is δ and $0 < \delta < 1$.

		Player 2			
		a	***b***	***c***	***d***
Player 1	***a***	6,6	3,10	1,8	2,4
	b	10,3	5,5	2,6	3,2
	c	8,1	6,2	3,3	1,0
	d	4,2	2,3	0,1	2,2

a. Consider the following symmetric strategy pair. In period 1, a player chooses action b. In period $t(\geq 2)$, a player chooses b if both players chose b in period $t-1$, and chooses c otherwise. Derive the conditions for it to be an SPNE.

ANSWER: Consider period 1 or a future period for which, in the preceding period, both players chose b. The payoff to a player from choosing b is $5/(1-\delta)$. The best alternative action is to choose c which has a payoff of $6 + 3\delta/(1-\delta)$. SPNE then requires:

$$\frac{5}{1-\delta} \geq 6 + \delta\left(\frac{3}{1-\delta}\right) \text{ or } \delta \geq \frac{1}{3}.$$

Now suppose in the preceding period either or both players did not choose b. As the prescribed action is c (now and forever), this action is optimal because it is repetition of a stage game Nash equilibrium.

b. Consider the following symmetric strategy pair. In period 1, a player chooses action a. In period $t(\geq 2)$, a player chooses a if both players chose a in all past periods, and chooses d otherwise. Derive the conditions for it to be an SPNE.

ANSWER: This strategy profile is not an SPNE. Consider a history in which at least one player chose differently from a. According to a player's strategy, she is to choose d which results in a payoff of $2/(1-\delta)$. However, the player could do better by instead choosing b and, even if she continued with her strategy thereafter (which means choosing d), her payoff would be higher at $3 + 2\delta/(1-\delta)$. Of course, she could do even better by choosing b in every period and getting a payoff of $3/(1-\delta)$ but, in either case, the prescribed action is not optimal. Thus, this strategy profile is not an SPNE. As an aside, note that for period 1 or some period in which both players chose a in the past, the prescribed action of a is optimal when there is anticipated to be a punishment involving both players choosing d forever:

$$\frac{6}{1-\delta} \geq 10 + \delta\left(\frac{2}{1-\delta}\right) \text{ or } \delta \geq \frac{1}{2}.$$

However, this is not an SPNE because the punishment is not credible; that is, it is not optimal for a player to go through the punishment.

c. Find a SPNE in which the resulting outcome path has: i) player 1 choosing *b* and player 2 choosing *a* in odd periods ($t = 1,3,5,\ldots$); and ii) player 1 choosing *a* and player 2 choosing *b* in even periods ($t = 2,4,6,\ldots$). Be sure to prove that the strategy profile you've derived is an SPNE.

ANSWER: Consider the following strategy profile. Player 1 chooses *b* in period 1. In any future period, when all past odd periods had (*b*,*a*) and all past even periods had (*a*,*b*) then, if the current period is odd, player 1 chooses *b* and, if the current period is even, she chooses *a*. For any other history, player 1 chooses *c*.

Player 2 chooses *a* in period 1. In any future period, when all past odd periods had (*b*,*a*) and all past even periods had (*a*,*b*) then, if the current period is odd, player 2 chooses *a* and, if the current period is even, he chooses *b*. For any other history, player 2 chooses *c*. To derive conditions for it to be an SPNE, start with player 1 and suppose it is period 1 or is an odd period in which in all past periods they have chosen (*b*,*a*) in odd periods and (*a*,*b*) in even periods. Player 1's strategy has him choose *b* and receive a payoff of

$$10 + \delta \times 3 + \delta^2 \times 10 + \delta^3 \times 3 + \cdots = 10 + 3\delta + \delta^2(10 + 3\delta) + \cdots = \frac{10 + 3\delta}{1 - \delta^2}$$

The best alternative action is to choose *c* which yields a payoff of

$$8 + \delta \times 3 + \delta^2 \times 3 + \cdots = 8 + \delta\left(\frac{3}{1 - \delta}\right).$$

It's clear that the latter payoff is lower as it provides a lower payoff in all odd periods and the same payoff in all even periods. Therefore, the prescribed behavior is optimal. Now suppose it is an even period in which in all past periods they have chosen (*b*,*a*) in odd periods and (*a*,*b*) in even periods. Player 1's strategy has him choose *a* and receive a payoff of

$$3 + \delta 10 + \delta^2 \times 3 + \delta^3 \times 10 + \cdots = \frac{3 + 10\delta}{1 - \delta^2}.$$

The best alternative action is to choose *c* which yields a payoff of

$$6 + \delta \times 3 + \delta^2 \times 3 + \cdots = 6 + \delta\left(\frac{3}{1 - \delta}\right).$$

Optimality requires:

$$\frac{3 + 10\delta}{1 - \delta^2} \geq 6 + \delta\left(\frac{3}{1 - \delta}\right) \Rightarrow 3\delta^2 + 7\delta - 3 \geq 0,$$

which means, approximately,

$$\delta \geq 0.37$$

Finally, consider a history whereby either in some past odd period (*b*,*a*) was not played and/or in some past even period (*a*,*b*) was not played. The prescribed behavior of playing *c* yields a payoff of $3/(1 - \delta)$, while the best alternative action is *b* and that yields a lower payoff of $2 + (3\delta/(1 - \delta))$. We've shown that, for all periods and histories, player 1's strategy is optimal. By symmetry, this analysis applies as well to player 2. Therefore, this strategy pair is an SPNE when $\delta \geq 0.37$.

14

Cooperation and Reputation:

Applications of Repeated Interaction with Infinitely Lived Players

1. Consider the infinitely repeated game in which the stage game is shown here. Each player's payoff is the present value of her payoff stream, where the discount factor is δ.

Player 2

Player 1	**w**	**x**	**y**	**z**
a	2,2	3,1	2,0	4,−4
b	1,3	4,4	3,1	2,3
c	0,2	1,3	7,7	3,9
d	−4,4	3,2	9,3	0,0

a. Define a grim-trigger strategy that results in player 1's choosing *c* and player 2's choosing *y*, and state conditions for that strategy's resulting in an SPNE.

ANSWER: This game has two Nash equilibria, (a, w) and (b, x). Thus, consider a strategy for player 1 in which, in period 1, he chooses c; in a future period, chooses c if the outcome was (c, y) in all past periods; and, in a future period, chooses b if the outcome was not (c, y) in some past period. For player 2, in period 1, she chooses y; in a future period, chooses y if the outcome was (c, y) in all past periods; and, in a future period, chooses x if the outcome was not (c, y) in some past period. The equilibrium condition is

$$\frac{7}{1-\delta} \geq 9 + \delta\left(\frac{4}{1-\delta}\right) \Rightarrow \delta \geq \frac{2}{5}.$$

b. Consider the following strategy profile: In period 1, player 1 chooses *c*. In any other period, player 1 chooses *c* if, in the previous period, the outcome was either (*c*, *y*) or (*d*, *z*); otherwise he chooses *d*. In period 1, player 2 chooses *y*. In any other period, player 2 chooses *y* if, in the previous period, the outcome was either (*c*, *y*) or (*d*, *z*); otherwise she chooses *z*. Derive conditions for this profile to result in an SPNE.

ANSWER: Suppose it is either period 1 or it is some future period in which the outcome was either (c, y) or (d, z) in the previous period. The prescribed action of c for player 1 is optimal if

$$7 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 9 + \delta \times 0 + \delta^2 \times 7 + \cdots \Rightarrow 7 + 7\delta \geq 9 \Rightarrow \delta \geq \frac{2}{7}.$$

Now consider a history in which in the previous period the outcome was neither (c, y) nor (d, z). The prescribed action of d for player 1 is optimal if

$$0 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 4 + \delta \times 0 + \delta^2 \times 7 + \cdots \Rightarrow 7\delta \geq 4 \Rightarrow \delta \geq \frac{4}{7}.$$

The conditions are the same for player 2. It is then a subgame perfect Nash equilibrium if

$$\delta \geq \frac{2}{7} \text{ and } \delta \geq \frac{4}{7}; \text{ or } \delta \geq \frac{4}{7}.$$

c. Consider the following strategy profile: In period 1, player 1 chooses c. In any other period, player 1 chooses c if, in the previous period, the outcome was either (c, y), (d, w), or (a, z); he chooses d if, in the previous period, player 2 chose y and player 1 did not choose c; he chooses a if, in the previous period, player 1 chose c and player 2 did not choose y; otherwise he chooses b. In period 1, player 2 chooses y. In any other period, player 2 chooses y if, in the previous period, the outcome was either (c, y), (d, w), or (a, z); she chooses w if, in the previous period, player 2 chose y and player 1 did not choose c; she chooses z if, in the previous period, player 1 chose c and player 2 did not choose y; otherwise she chooses x. Derive conditions for this profile to yield an SPNE.

ANSWER: Consider period 1 or a future period in which the previous period's outcome was either (c, y), (d, w), or (a, z). Player 1's prescribed action of c is optimal if

$$7 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 9 + \delta \times (-4) + \delta^2 \times 7 + \cdots$$

$$\Rightarrow 7 + 7\delta \geq 9 - 4\delta \Rightarrow \delta \geq \frac{2}{11}.$$

Next, consider a history in which, in the previous period, player 1 did not choose c and player 2 chose y. Player 1's prescribed action of d is optimal if

$$-4 + \delta \times 7 + \delta^2 \times 7 + \cdots \geq 2 + \delta \times 4 + \delta^2 \times 4 + \cdots$$

$$\Rightarrow -4 + 7\frac{\delta}{1-\delta} \geq 2 + 4\frac{\delta}{1-\delta} \Rightarrow \delta \geq \frac{2}{3}.$$

Next, consider a history in which, in the previous period, player 1 chose c and player 2 did not choose y. Player 1's prescribed action of a is optimal since any other action lowers the current period payoff and reduces the future payoff stream from

$$\delta \times 7 + \delta^2 \times 7 + \cdots$$

to

$$\delta \times 4 + \delta^2 \times 4 + \cdots.$$

Finally, for any other history, player 1 is to choose b. Note that b maximizes his current payoff given player 2 is to choose x. Furthermore, the future payoff is the same since, come next period, the previous period's history will involve player 2 choosing x and thus the outcome will be (b, x). This applies as well to all ensuing periods. The analysis is analogous for player 2. This strategy profile is then a subgame perfect Nash equilibrium if

$$\delta \geq \frac{2}{11} \text{ and } \delta \geq \frac{2}{3} \Rightarrow \delta \geq \frac{2}{3}.$$

2. As early as the 1820s, the custom of "pairing off" had developed in Congress. If a member of Congress was to miss a formal vote, he would arrange beforehand with a member on the opposing side of the issue for the two not to vote. In modeling this situation, consider two members of Congress—Representatives Smith and Jones—who, on a regular basis, would like to miss a House vote in order to take care of other business. Representative Smith would prefer to be away every three periods, starting with period 1 (hence periods 1, 4, 7, . . .), and receives a value of 3 from being away. Representative Jones would prefer to be away every three periods, starting with period 2 (hence, periods 2, 5, 8, . . .), and also receives value of 3. Call these periods the representatives' "traveling periods." In each such period, there is a House vote, and Smith receives a value of 5 from being in attendance and voting and a value of −5 if Jones is in attendance and votes. Analogously, Jones earns 5 from being in attendance and voting and −5 if Smith is in attendance and votes. Thus, if both are in attendance and vote, then Smith and Jones each have a payoff of

10. In each period, two players simultaneously choose between two actions: *up* and *down*. If both choose *up*, then each receives a payoff of 10 with probability .8 and a payoff of 3 with probability .2. If both choose *down*, then they both receive a payoff of 5 for sure. If one chooses *down* and the other chooses *up*, then the former receives a payoff of 10 for sure and the latter receives a payoff of 10 with probability .6 and a payoff of 3 with probability .4. When choosing an action, each player knows his past choices and both players' past payoffs, but neither knows what the other player actually chose in the past. This stage game is infinitely repeated, where each player's discount factor is δ. Consider the following symmetric strategy pair: In period 1, both players choose *up*. In any other period, choose *up* if both players received the same payoff (either 3, 5, or 10) last period; otherwise choose *down*. Derive conditions whereby this strategy pair is an SPNE.

ANSWER: To begin, derive a player's payoff when it is period 1 or if in the previous period they both had the same payoff, given that they use this strategy (which implies that they choose *up* in the current period). Denoting that payoff as V, it is defined as

$$V = .8 \times 10 + .2 \times 3 + \delta V \Leftrightarrow V = \frac{8.6}{1-\delta}.$$

If it is period 1 or if in the previous period they both had the same payoff, then a player finds it optimal to choose *up* rather than *down* when

$$\begin{aligned} 8.6 + \delta \times 8.6 + \delta^2 \times 8.6 + \cdots \geq\ & 10 + .6 \times [\delta \times 8.6 + \delta^2 \times 8.6 + \cdots] \\ & + .4 \times [\delta \times 5 + \delta^2 \times 8.6 + \cdots]. \end{aligned}$$

If she chooses *down*, then her current payoff is 10. With probability .6, the other player's payoff is 10, in which case the future payoff is V as both will choose *up* next period. With probability .4, the other player's payoff is 3, in which case the payoff next period is 5—as both choose *down*—but after that they return to choosing *up*. Cancelling common terms,

$$8.6 + 8.6\delta \geq 10 + 5.16\delta + 2\delta \Rightarrow \delta \geq \frac{1.4}{1.44} \simeq .97.$$

Now suppose, in the previous period, the players' payoffs were not the same. A player's strategy has him choose *down*, and that is preferable to *up* when

$$5 + \delta \times 8.6 + \delta^2 \times 8.6 + \cdots \geq .6 \times 10 + .4 \times 3 + [\delta \times 5 + \delta^2 \times 8.6 + \cdots].$$

By choosing *down*, a player gets 5 today, and then both players return to choosing *up*. If she chooses *up* today, then her expected current payoff is .6 × 10 + .4 × 3. As players' payoffs will differ, then both players will choose *down* tomorrow—yielding a payoff of 5—and *up* thereafter. Cancelling common terms,

$$5 + 8.6\delta \geq 7.2 + 5\delta \Rightarrow \delta \geq \frac{2.2}{3.6} \simeq .61.$$

In sum, this symmetric strategy pair is a subgame perfect Nash equilibrium if

$$\delta \geq .97 \text{ and } \delta \geq .61 \Rightarrow \delta \geq .97.$$

11. In a village, there are $n \geq 2$ households. In each period, each household works and earns money. After paying for necessities, the household is left with $x > 0$ dollars. Each household only considers using this residual money to buy appliances. There are an infinite number of different types of appliances that each household would like to buy: a washer, a dryer, a television set, and so forth. For simplicity, each appliance has the same price of $p > 0$ and the same lifetime utility, which is denoted $Z > 0$ (where Z is measured in dollars). Assume $Z > p$ so that the value of an appliance to a household exceeds its price.

At the start of the game, each household has zero savings. Assume that no interest is paid on savings. A household's total payoff is calculated as shown here and is based only on buying appliances. Imagine that dollars are worthless except when used to buy appliances. If a household buys appliances in periods $t_1, t_2, \ldots,$ then its payoff is

$$\delta^{t_1-1}Z + \delta^{t_2-1}Z + \cdots \text{ or } \sum_{i=1}^{\infty} \delta^{t_i-1}Z$$

where δ is the discount factor and $0 < \delta < 1$. One final assumption is that $p = nx$; that is, the price of an appliance happens to be an integer multiple of what a household can save in each period, and this multiple happens to equal the number of households. In each period, a household has $n + 1$ choices: save x, spend its savings to buy an appliance (assuming that its savings is at least p), or give x to another household (of which there are $n - 1$ households). Let us provide two descriptions of behavior. The *self-sufficiency rule* has a household save x dollars each period until savings is nx, at which time it buys an appliance. Thus, this rule results in a household buying an appliance in periods n, $2n$, $3n, \ldots.$ Alternatively, the *rotational credit rule* has all households giving x dollars to household i in period t when $t = i, n + i, 2n + i, \ldots$ and household i buys an appliance. Thus, when used by all households, the rotational credit rule has household 1 buying an appliance in period 1, household 2 buying an appliance in period 2, . . . , household n buying an appliance in period n, household 1 buying an appliance in period $n + 1$, and so forth.

a. Derive the payoff to a household from using the self-sufficiency rule.

ANSWER: With the self-sufficiency rule, a household purchases an appliance every n periods starting with period n. Its payoff is then:

$$\delta^{n-1}Z + \delta^{2n-1}Z + \cdots = \sum\nolimits_{r=1}^{\infty} \delta^{rn-1}Z = \frac{\delta^{n-1}z}{1-\delta^{n}}.$$

The last equality is derived as follows. Define S as

$$S = \delta^{n-1} + \delta^{2n-1} + \delta^{3n-1} + \cdots$$

Now multiply each side by δ^n,

$$\delta^n S = \delta^{2n-1} + \delta^{3n-1} + \delta^{4n-1} + \cdots$$

Subtracting the second line from the first line yields

$$S - \delta^n S = \delta^{n-1} \Rightarrow S = \frac{\delta^{n-1}}{1-\delta^{n}}.$$

b. Consider the following strategy profile. In period 1, each player uses the rotational credit rule. In period $t \geq 2$: 1) if all players have acted according to the rotational credit rule in all past periods, then use the rotational credit rule; and 2) if one or more players did not act according to the rotational credit rule in some past period, then use the self-sufficiency rule. Derive the conditions for this strategy profile to be an SPNE.

ANSWER: The space of histories can be partitioned into $n + 1$ sets. There are n sets of the following form: 1') period 1 and period t where $t = 1, n + 1, 2n + 1, \ldots$ and in all past periods all players acted according to the rotational credit rule; 2') period t where $t = 2, n + 2, 2n + 2, \ldots$ and in all past periods all players acted according to the rotational credit rule; . . . ; and n') period t where $t = n, 2n, 3n, \ldots$ and in all past periods all players acted according to the rotational credit rule. Note that a history in set h' results in household h buying an appliance. Finally, there is the last set of histories: $n + 1$')$t \geq 2$ and, in some past period, one or more players did not act according to the rotational credit rule. For all histories within each of these $n + 1$ sets, the prescribed behavior is the same.

Consider household 1 and a history in set 1'. Its strategy has it buy an appliance which results in a total payoff of

$$Z + \delta^n Z + \delta^{2n} Z + \cdots = \frac{Z}{1 - \delta^n}.$$

Implicit in this expression is that the other households act according to their strategies in the current period which means giving their savings to household 1 and all players acting according to their strategies in future periods. Alternatively, household 1 can choose not to buy an appliance. It then has savings of $(n + 1)x$ in period 2 so, according to the self-sufficiency rule, it buys an appliance. This leaves savings of x going into period 3. Come period $n + 1$, it has savings of nx and thus buys an appliance and subsequently buys one every n periods thereafter. Its total payoff in that case is

$$\delta Z + \delta^n Z + \delta^{2n} Z + \cdots$$

which is strictly lower than that from buying in period 1 (note that the first term is smaller and the other terms are equal). It is easy to see that the remaining option—giving x to another household—is inferior as it yields a total payoff of

$$\delta Z + \delta^{n+1} Z + \delta^{2n+1} Z + \cdots = \frac{\delta Z}{1 - \delta^n}.$$

Thus, household 1's strategy is clearly optimal for 1' histories.

Next consider household 1 and a history in set 2'. Its strategy has it give x dollars to household 2 and this yields a total payoff of

$$\delta^{n-1} Z + \delta^{2n-1} Z + \cdots = \frac{\delta^{n-1} Z}{1 - \delta^n},$$

as it doesn't get to buy an appliance for another $n - 1$ periods. The other possible actions are to give it to another household (which is clearly nonoptimal) or to retain its savings in which case households move to the self-sufficiency rule. With the latter case, household 1's total payoff is then

$$\delta^{n-1} Z + \delta^{2n-1} Z + \cdots = \frac{\delta^{n-1} Z}{1 - \delta^n}.$$

as it has enough savings by period $n + 1$ to buy an appliance (and will buy an appliance every n periods thereafter). As the payoffs from the two actions are the same, household 1 is content to act according to its strategy.

Next consider household 1 and a history in set m' where $m \in \{3, 4, \ldots, n\}$. The total payoff from its strategy is:

$$\delta^{n-m+1} Z + \delta^{2n-m+1} Z + \cdots = \frac{\delta^{n-m+1} Z}{1 - \delta^n},$$

while its total payoff from not giving its savings to household m is:

$$\delta^{n-1} Z + \delta^{2n-1} Z + \cdots = \frac{\delta^{n-1} Z}{1 - \delta^n},$$

since household 1 won't have enough saved up for another $n - 1$ periods. The strategy is optimal because

$$\frac{\delta^{n-m+1} Z}{1 - \delta^n} > \frac{\delta^{n-1} Z}{1 - \delta^n} \Leftrightarrow \delta^{m-2} < 1,$$

which is true since $m > 2$ and $0 < \delta < 1$.

The final case for household 1 is for $n + 1$' histories. As the strategy prescribes the self-sufficiency rule, this is clearly optimal. Given the other players are not going to share their savings, it is not optimal for household 1 to do so.

We conclude that the strategy is optimal for household 1 and this is independent of the value of δ.

In turning to household 2, note that the situation it faces for histories in $1'$ is exactly the same as that faced by household 1 for histories in n' as, in both cases, each household is to give its savings to another household and its turn to buy an appliance is in the next period. Similarly, the situation household 2 faces for histories in $2'$ is exactly the same as that faced by household 1 for histories in $1'$ and, more generally, the situation household 2 faces for histories in m' is exactly the same as that faced by household 1 for histories in $m - 1'$ for $m \in \{2, 3, \ldots, n\}$. All households face the same situation for $n + 1'$ histories. Thus, if the conditions hold for household 1 then they hold for household 2. A similar logic applies when we consider households 3, 4, . . . , n.

In conclusion, this strategy profile is an SPNE for any discount factor.

12. Consider the infinitely repeated version of this symmetric two-player stage game. Each player's discount factor is δ where $1/2 < \delta < 1$. Find a symmetric SPNE.

Player 2

		a	*b*	*c*
	a	1,1	2,2	0,2
Player 1	***b***	2,2	3,3	2,4
	c	2,0	4,2	1,1

ANSWER: Note that there is no symmetric Nash equilibrium for the stage game. Consider a symmetric strategy that has a player choose action b in period 1. In period t, a player chooses b if, in the previous period, both players chose b or both chose a; otherwise, a player chooses a. In deriving the conditions for this strategy pair to be an SPNE, consider either period 1 or period t where, in the previous period, both chose b or both chose a:

$$\frac{3}{1-\delta} \geq 4 + \delta \times 1 + \delta^2\left(\frac{3}{1-\delta}\right) \Leftrightarrow 3 + \delta \times 3 \geq 4 + \delta \times 1 \Leftrightarrow \delta \geq \frac{1}{2}$$

Suppose instead that in the previous period either both did not choose a or both did not choose b:

$$1 + \delta\left(\frac{3}{1-\delta}\right) \geq 2 + \delta \times 1 + \delta^2\left(\frac{3}{1-\delta}\right) \Leftrightarrow 1 + \delta \times 3 \geq 2 + \delta \times 1 \Leftrightarrow \delta \geq \frac{1}{2}$$

Given the assumption $\frac{1}{2} < \delta < 1$ then these conditions are satisfied and it is a SPNE. Similarly, it is a SPNE to replace action a in that strategy with action c.

13. Consider this infinitely repeated version of the symmetric two-player stage game below. Each player's discount factor is δ where $0 < \delta < 1$.

Player 2

		a	*b*	*c*	*d*
	a	−1,−1	2,2	0,2	3,2
Player 1	***b***	2,2	4,4	4,3	13,0
	c	2,0	3,4	6,6	7,7
	d	2,3	0,13	7,7	9,9