For exercise 9C7 from MWG, please see the answer key in Recitation #6.

Substituting for $c$ and simplifying yields

$$p_1 - p_1^2 \frac{4-3\delta}{(2-\delta)^2}.$$

Taking the derivative and solving the first-order condition for $p_1$ yields

$$p_1 = \frac{(2-\delta)^2}{8-6\delta}.$$

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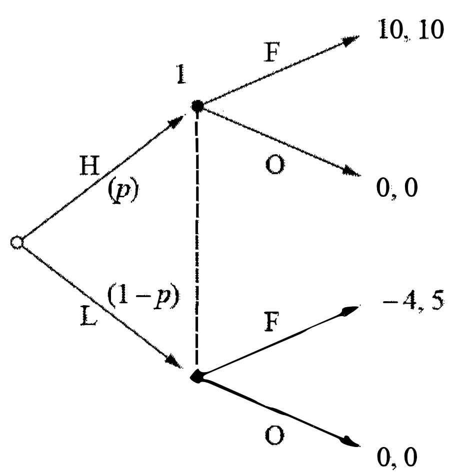
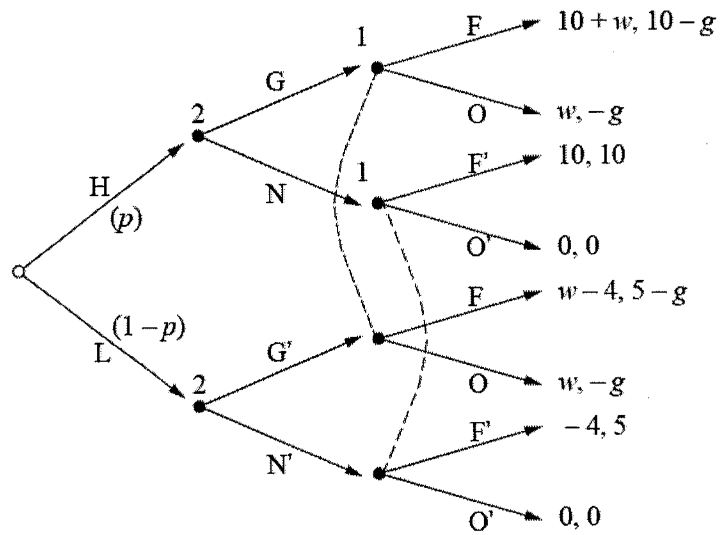
EXERCISE 7.

(a) The extensive form is:




In the Bayesian Nash equilibrium, player 1 forms a firm (F) if $10p - 4(1-p) \geq 0$, which simplifies to $p \geq 2/7$. Player 1 does not form a firm (O) if $p < 2/7$.

(b) The extensive form is:




(c) Clearly, player 1 wants to choose F with the H type and O with the L type. Thus, there is a separating equilibrium if and only if the types of player 2 have the incentive to separate. This is the case if $10 - g \geq 0$ and $0 \geq 5 - g$, which simplifies to $g \in [5, 10]$.

(d) If $p \geq 2/7$, then there is a pooling equilibrium in which NN′ and F′ are played, player 1's belief conditional on no gift is $p$, player 1's belief conditional on a gift is arbitrary, and player 1's choice between F and O is optimal given this belief. If, in addition to $p \geq 2/7$, it is the case that $g \in [5, 10]$, then there is also a pooling equilibrium featuring GG′ and FO′. If $p \leq 2/7$, then there is a pooling equilibrium in which NN′ and OO′ are played (and player 1 puts a probability on H that is less than $2/7$ conditional on receiving a gift).

EXERCISE 8.

(a) A player is indifferent between O and F when he believes that the other player will choose O for sure. Thus, (O, O; O, O) is a Bayesian Nash equilibrium.

(b) If both types of the other player select Y, the H type prefers Y if $10p - 4(1-p) \geq 0$, which simplifies to $p \geq 2/7$. The L type weakly prefers Y, regardless of $p$. Thus, such an equilibrium exists if $p \geq 2/7$.

(c) If the other player behaves as specified, then the H type expects $-g + p(w + 10) + (1 - p)0$ from giving a gift. He expects $pw$ from not giving a gift. Thus, he has the incentive to give a gift if $10p \geq g$. The L type expects $-g+p(9w+5)+(1-p)0$ if he gives a gift, whereas he expects $pw$ if he does not give a gift. The L type prefers not to give if $g \geq 5p$. The equilibrium, therefore, exists if $g \in [5p, 10p]$.

## EXERCISE 10.

(a) $1^H$ selects $a_1^H$ to maximize $4a_1^H + 4a_2 - [a_1^H]^2$, which has a first-order condition of $4 - 2a_1^H \equiv 0$ implying $a_1^H = 2$.

Similarly, $1^L$ selects $a_1^L$ to maximize $2a_1^L + 2a_2 - [a_1^L]^2$, which has a first-order condition of $2 - 2a_1^L \equiv 0$ implying $a_1^L = 1$.

Player 2 does not observe $k$ and chooses $a_2$ to maximize $\frac{1}{2}[4a_1^H + 4a_2] + \frac{1}{2}[2a_1^H + 2a_2] - a_2^2$, which has a first-order condition of $2 + 1 - 2a_2 \equiv 0$ implying $a_2 = \frac{3}{2}$.

(b) There is an equilibrium in which both types of player 1 present evidence of their type. This requires that when no evidence is presented, player 2's belief is that $k = 4$. When player 1 shows her type to be H, both player 1 and 2 choose effort of 2, and when player 1 shows her type to be L, both players choose effort of 1. Following the out-of-equilibrium behavior of player 1 not disclosing evidence, player 2 chooses effort of $\frac{3}{2}$, and player 1 chooses effort of 2 when the state is H and 1 when the state is L.

There is also an equilibrium in which player 1 presents evidence in H and does not in L. Upon seeing no evidence presented, player 2 believes that $k = 4$. Player 1 chooses effort of 2 in H and 1 in L. Player 2 chooses effort of 2 when evidence of H is presented and chooses effort of 1 when either no evidence is presented or evidence of L is presented.

In both of these equilibria, player 2 knows the value of $k$ from either direct evidence or from inferring that $k = 4$ due to player 1 not presenting evidence.

(c) After observing $k = 8$, player 1 would like for player 2 to know the value of $k$, but after observing $k = 4$, player 1 would like to not be able to convey the value of $k$.

We can also address this ex ante or prior to the realization of $k$ as follows. When $k = 8$, player 1's payoff is $4[2 + \frac{3}{2}] - 4 = 10$, and when $k = 4$, player 1's payoff is $2[1 + \frac{3}{2}] - 1 = 4$. So player 1's expected payoff is 7. However, when $k$ is known by player 2, player 1's payoffs are the following: when $k = 8$, $u_1 = 4[2 + 2] - 4 = 12$, and when $k = 4$, $u_1 = 2[1 + 1] - 1 = 3$. This yields an expected payoff for player 1 of $7.5 > 7$, so player 1 would prefer that player 2 know the value of $k$.

# Chapter 3

# Hidden Information, Signaling

## 3.1 Question 6

Consider a firm that can invest an amount $I$ in a project generating high observable cash flow $C > 0$ with probability $\theta$ and 0 otherwise: $\theta \in \{\theta_L, \theta_H\}$ with $\theta_H - \theta_L \equiv \Delta > 0$ and $\Pr[\theta = \theta_L] = \beta$. The firm needs to raise $I$ from external investors who do not observe the value of $\theta$. Assume that $\theta_L C - I > 0$. Everybody is risk neutral and there is no discounting.

1. Suppose that the firms can only promise to repay an amount $R$ chosen by the firm (with $0 \leq R \leq C$) when cash flow is $C$ and 0 otherwise. Can a good firm signal its type?

2. Suppose now that the firm also has the possibility of pledging some assets as collateral for the loan: Should a "default" occur (the firm being unable to repay $R$), an asset of value $K$ to the firm is transferred to the creditor whose valuation is $xK$ with $0 < x < 1$. The size of the collateral $K$ is a choice variable. Give a necessary and sufficient condition for the "best" Perfect Bayesian Equilibrium to be separating. How does it depend on $\beta$ and $x$? Explain.

### 3.1.1 No Collateral

Both firms would want to undergo the project since $\theta_L C > I$. A good firm cannot signal its type, since for a separating equilibrium to exist we need $R_H \neq R_L$. However, this cannot be an equilibrium. This can be seen clearly from the incentive compatibility condition for a firm of type $i$

$$\theta_i (C - R_i) \geq \theta_i (C - R_j).$$

Whenever $R_i \neq R_j$ at least one type of firm will want to deviate.

Intuitively speaking, since both firms receive $C$ when the project is successful and 0 when it fails, and we only have one repayment instrument, the bad firm can perfectly mimic the good firm.

### 3.1.2 Collateral

#### Separating Equilibrium

The best separating equilibrium is clearly the one with the least amount of $K$ so $K_L = 0$. The loss $(1-x)K$ is higher for a low-type firm since it has a higher probability of being in default.

A separating equilibrium can be supported by the following beliefs:

$$\begin{aligned} \Pr\left(\theta = \theta_L | K > K^*\right) &= 0 \\ \Pr\left(\theta = \theta_L | K \leq K^*\right) &= 1 \end{aligned}$$

and we have

$$\begin{aligned} K_L &= 0 \\ R_L &= \frac{I}{\theta_L}. \end{aligned}$$

This holds since otherwise the low type could offer a higher payment and still be better off–there is no benefit from a positive $K$.

Thus we shall have the following incentive compatibility and individual rationality constraints:

$$\begin{aligned} \theta_L R_L &= I && \text{(IRL)} \\ xK^H(1-\theta_H) + \theta_H R_H &= I && \text{(IRH)} \\ \theta_L(C - R_L) &\geq \theta_L(C - R_H) - (1-\theta_L)K_H && \text{(ICL)} \\ \theta_H(C - R_H) - (1-\theta_H)K_H &\geq \theta_H(C - R_L) && \text{(ICH)} \end{aligned}$$

Rewriting (ICL) we obtain

$$R_L - R_H \leq \frac{1-\theta_L}{\theta_L} K_H.$$

Similarly, re-arranging (ICH) gives

$$R_L - R_H \geq \frac{1-\theta_H}{\theta_H} K_H.$$

Putting these expressions together we obtain

$$\frac{1-\theta_H}{\theta_H} K_H \leq R_L - R_H \leq \frac{1-\theta_L}{\theta_L} K_H.$$

This works even if $x$ is very small. The intuition is that the high type benefits from a lower $R$ more often and suffers from the loss of $K$ less often, since

$\theta_H > \theta_L$. Thus the best separating equilibrium minimizes the use of (socially) wasteful collateral, that is $K_H$ is set as low as possible. Hence, in equilibrium, only (ICL) is binding and (ICH) is slack. Thus, in what follows we can ignore (ICH). Solving the following equalities which we obtained from the constraints using the fact that (ICL) is binding in equilibrium and combining (IRL) and (ICL), we find

$$\begin{aligned} xK^H(1-\theta_H)+\theta_H R_H &= I && \text{(IRH)} \\ K_H(1-\theta_L)+\theta_L R_H &= I. && \text{(IRL, ICL)} \end{aligned}$$

Rewriting these conditions yields

$$\begin{aligned} R_H &= \frac{I}{\theta_H} - \frac{1-\theta_H}{\theta_H} xK_H \\ R_H &= \frac{I}{\theta_L} - \frac{1-\theta_L}{\theta_L} K_H, \end{aligned}$$

and after some algebraic manipulation we obtain

$$\begin{aligned} K_H &= \frac{\Delta I}{\theta_H(1-\theta_L) - x\theta_L(1-\theta_H)} \\ R_H &= \frac{I}{\theta_H} - \frac{1-\theta_H}{\theta_H} \frac{x\Delta I}{\theta_H(1-\theta_L) - x\theta_L(1-\theta_H)}. \end{aligned}$$

From above, we have

$$\begin{aligned} K_L &= 0 \\ R_L &= \frac{I}{\theta_L}, \end{aligned}$$

since this is the best, or the least cost equilibrium. $K$ is costly and hence there is no reason to use it in the low state. Notice that this separating equilibrium always exists.

### Pooling Equilibrium

We can compare this to the best pooling equilibrium, where

$$\begin{aligned} K^P &= 0 \\ R^P &= \frac{I}{\theta_H - \beta\Delta}. \end{aligned}$$

However, this pooling equilibrium may not exist. It will exist if and only if

$$\theta_H\left(C - R^P\right) \geq \theta_H(C-R) - (1-\theta_H)K$$

where

$$\begin{aligned} R &= \frac{I-(1-\theta_L)xK}{\theta_L} \\ I &= \theta_L R + (1-\theta_L)xK, \end{aligned}$$

which follows from the assumption that for any deviation from $K^P = 0$ the investors will believe that the firm is of low type. So the pooling equilibrium is sustainable if there are no deviations given these beliefs. This is the worst belief in the sense that if we cannot find a pooling equilibrium supported by these beliefs then no pooling equilibrium exists (there will always be a profitable deviation from it). Combining the equations, we obtain

$$\theta_H \left( C - \frac{I}{\theta_H - \beta\Delta} \right) \geq \theta_H \left( C - \frac{I - (1-\theta_L)\, xK}{\theta_L} \right) - (1-\theta_H)\, K,$$

which is equivalent to

$$x \leq (\theta_H I (\frac{1}{\theta_L} - \frac{1}{\beta\theta_L + (1-\beta)\theta_H}) + (1-\theta_H)K) \frac{\theta_L}{\theta_H(1-\theta_L)K}.$$

Thus, the smaller $x$ or $\beta$ the more likely is the existence of a pooling equilibria. This is intuitive. A smaller $x$ means that the signal is more costly, and hence a profitable deviation from the least-cost pooling equilibrium that has no costly collateral, is very difficult. Similarly, with a smaller $\beta$ the less likely it is that the firm is a bad type (so a smaller cross-subsidy is needed).

**Comparison**

One way to compare the different equilibria would be to compare ex-ante expected profits of the firm for the separating and pooling equilibrium (compare section 3.1.1). The expected profits for the separating equilibrium are given by

$$\begin{aligned} \pi^S &= (1-\beta)\, \pi_H^S + \beta \pi_L^S \\ &= (1-\beta) \left[ \theta_H (C - R_H) - (1-\theta_H) K_H \right] + \beta \theta_L (C - R_L) \\ &= C \left[ \theta_L + (1-\beta)\, \Delta \right] - I - (1-\beta)\, (1-\theta_H)(1-x) K_H. \end{aligned}$$

where $K_H$ is as defined above.

In contrast, expected profits for the pooling equilibrium are

$$\begin{aligned} \pi^P &= (1-\beta)\, \pi_H^P + \beta \pi_L^P \\ &= (1-\beta)\, \theta_H (C - R^P) + \beta \theta_L (C - R^P) \\ &= \left[ \theta_L + (1-\beta)\, \Delta \right] (C - R^P) \\ &= \left[ \theta_L + (1-\beta)\, \Delta \right] C - I. \end{aligned}$$

Hence, we have

$$\pi^P > \pi^S,$$

whenever the pooling equilibrium exists. The pooling equilibrium leads to higher profits as it avoids the use of (wasteful) collateral. Thus, the best perfect Bayesian equilibrium (when defined in this way) is separating if and only if the pooling equilibrium does not exist. Another way would be look for one equilibria Pareto-dominating the other. Here we would see when both types prefer the pooling equilibrium. This happens when $\beta$ is very small and thus the tiny gain from signalling (avoiding the infinitesimal cross subsidy) is smaller than the costly signal. See section 3.1.1 for details.

a) Standard figure.

b) At period 1, the buyer buys if $v - p_1 \geq 0$, or $v \geq p_1$. Hence, if $p_1 < 10$ the buyer buys regardless of his valuation; if $p_1$ lies between 10 and 30, he buys only when his valuation is v=30; and if $p_1$p1 is above 30 the buyer rejects regardless of his valuation. Hence, the seller expected profit is $\frac{1}{2}p_1$ since both types are equally likely. (In addition, if a first-period price of 30 is rejected the game proceeds to period 2, where the seller infers that the buyer's valuation must be lower than 30)

At period 2, let $\mu = Prob(v = 30|History\ at\ t = 2)$, and the buyer buys if his valuation v satisfies $v \geq p_2$

In addition, when beliefs satisfy

- $\mu > \frac{1}{2}$, the seller's optimal second-period price is $p_2 = \$30$ ,
- when $\mu < \frac{1}{2}$ the seller's optimal second-period price is $p_2 = \$10$, and
- when $\mu = \frac{1}{2}$ the seller is indifferent between setting a second-period price of $p_2 = \$10$ and $p_2 = \$30$.

We can now apply Bayes' rule to update the seller's beliefs about the seller's valuation upon observing that a price $p_1$ was rejected in the first period. In particular,

$$\mu(p_1) = \frac{\frac{1}{2}\alpha}{\frac{1}{2}\alpha + \left(1 - \frac{1}{2}\right)} = \frac{1}{2}$$

where $\alpha$ is the probability that the low-value buyer does not buy the product at a price $p_1$. Rearranging, we obtain

$$\frac{\alpha}{1 + \alpha} = \frac{1}{2}$$

And solving for probability $\alpha$ yields $\alpha = 1$, implying that in the second period the seller believes that the buyer who rejected price $p_1$ in the first-period game must be a high-value buyer with certainty.