EconS 503 - Microeconomic Theory II
Homework #4 - Answer key

1. **Cournot competition with both firms uninformed.** Consider the duopoly market where firms face an inverse demand function $p(Q) = 1 - Q$, and $Q = q_1 + q_2 \geq 0$ denotes aggregate output. Let us now assume that both firms are uninformed about each other's costs, that is, every firm i privately observes its cost, c_i, which is $c_H = \frac{1}{2}$ or $c_L = 0$, with probability p and $1-p$, respectively. Firm i, however, does not observe its rival's cost, c_j, which is also $c_H = \frac{1}{2}$ or $c_L = 0$, which occurs with probability p and $1-p$, respectively.

   (a) Find firm i's best response function when its production cost is $c_H = \frac{1}{2}$, and denote it as $q_i^H(q_j^H, q_j^L)$. Is it increasing or decreasing in probability p? Interpret.

   - When firm i has high costs, it chooses q_i^H to solve the following expected profit maximization problem:

$$\begin{aligned}\max_{q_i^H \geq 0} \pi_i^H(q_i^H) &= \underbrace{p\left[(1 - q_i^H - q_j^H)q_i^H\right]}_{\text{profit if } j \text{ is high cost}} + \underbrace{(1-p)(1 - q_i^H - q_j^L)q_i^H}_{\text{profit if } j \text{ is low cost}} - \frac{1}{2}q_i^H \\ &= \left(1 - q_i^H - (1-p)q_j^L - pq_j^H - \frac{1}{2}\right) q_i^H \\ &= \left(\frac{1}{2} - q_i^H - (1-p)q_j^L - pq_j^H\right) q_i^H\end{aligned}$$

     Differentiating with respect to q_i^H, we obtain

$$\frac{1}{2} - 2q_i^H - (1-p)q_j^L - pq_j^H = 0$$

     Solving for q_i^H, we have

$$q_i^H(q_j^L, q_j^H) = \frac{1}{4} - \frac{1}{2}\left[pq_j^H + (1-p)q_j^L\right]$$

     which originates at $\frac{1}{4}$ and decreases in its rival's *expected* output, $pq_j^H + (1-p)q_j^L$, at a rate of $\frac{1}{2}$. Alternatively, firm i decreases its output at a rate of $\frac{p}{2}$ ($\frac{1-p}{2}$) when its rival has high (low) costs.

   - Differentiating $q_i^H(q_j^L, q_j^H)$ with respect to p, we obtain

$$\frac{\partial q_i^H(q_j^L, q_j^H)}{\partial p} = \frac{q_j^L - q_j^H}{2}.$$

     If firm j produces more units when its costs are low, $q_j^L > q_j^H$, this derivative is positive, implying that firm i increases its output as it is more likely that its rival's costs are high. Intuitively, firm j becomes less competitive, in expectation, leading firm to increase its own output.

(b) Find firm i's best response function when its production cost is $c_L = 0$, and denote it as $q_i^L(q_j^H, q_j^L)$. Is it increasing or decreasing in probability p? Interpret.

- When firm i has low costs, it chooses q_i^L to solve the following expected profit maximization problem:

$$\begin{aligned}\max_{q_i^L \geq 0} \pi_i^L(q_i^L) &= \underbrace{p(1 - q_i^L - q_j^H)q_i^L}_{\text{profit if } j \text{ is high cost}} + \underbrace{(1-p)(1 - q_i^L - q_j^L)q_i^L}_{\text{profit if } j \text{ is low cost}} \\ &= \left(1 - q_i^L - (1-p)q_j^L - pq_j^H\right) q_i^L\end{aligned}$$

  Differentiating with respect to q_i^L, we obtain

$$1 - 2q_i^L - (1-p)q_j^L - pq_j^H = 0$$

  Solving for q_i^L, we have

$$q_i^L(q_j^L, q_j^H) = \frac{1}{2} - \frac{1}{2}\left[pq_j^H + (1-p)q_j^L\right]$$

  which originates at $\frac{1}{2}$ and decreases in its rival's *expected* output, $pq_j^H + (1-p)q_j^L$, at a rate of $\frac{1}{2}$. Alternatively, firm i decreases its output at a rate of $\frac{p}{2}$ ($\frac{1-p}{2}$) when its rival has high (low) costs.

- Comparing it with firm i's best response function when its costs are high, $q_i^H(q_j^L, q_j^H)$, we can see that, for a given profile of firm j's output, (q_j^L, q_j^H), firm i responds producing a larger output when its own costs are low than when they are high. Graphically, $q_i^L(q_j^L, q_j^H)$ lies above $q_i^H(q_j^L, q_j^H)$.
- Differentiating $q_i^L(q_j^L, q_j^H)$ with respect to p, we obtain

$$\frac{\partial q_i^L(q_j^L, q_j^H)}{\partial p} = \frac{q_j^L - q_j^H}{2}.$$

  This result coincides with that at the end of part (a). If firm j produces more units when its costs are low, $q_j^L > q_j^H$, this derivative is positive, implying that firm i increases its output as it is more likely that its rival's costs are high. Intuitively, firm j becomes less competitive, in expectation, leading firm to increase its own output.

(c) Find equilibrium output levels, q_i^H and q_i^L.

- In a symmetric equilibrium, every firm's output is only a function of its privately observed costs, entailing that

$$\begin{aligned} q_i^L &= q_j^L = q^L, \text{ and} \\ q_i^H &= q_j^H = q^H \end{aligned}$$

  Inserting the above property into the best response functions we found in parts (a) and (b), yields

$$\begin{aligned} q^H &= \frac{1}{4} - \frac{1}{2}[pq^H + (1-p)q^L] \text{ from part (a), and} \\ q^L &= \frac{1}{2} - \frac{1}{2}[pq^H + (1-p)q^L] \text{ from part (b).} \end{aligned}$$

Simultaneously solving for q^L and q^H, we find that the equilibrium output levels are

$$q^{L*} = \frac{4+p}{12} \text{ and } q^{H*} = \frac{1+p}{12}$$

(d) How is equilibrium output q_i^H affected by a marginal increase in probability p? Interpret.

- Differentiating with respect to p, we obtain

$$\frac{\partial q^{H*}}{\partial p} = \frac{1}{12} > 0$$

Therefore, as firm j is more likely to face high costs, firm i responds increasing its output when its own costs are high, q^{H*}.

(e) How is equilibrium output q_i^L affected by a marginal increase in probability p? Interpret.

- Differentiating with respect to p, we obtain

$$\frac{\partial q^{L*}}{\partial p} = \frac{1}{12} > 0$$

Therefore, as firm j is more likely to face high costs, firm i responds increasing its output when its own costs are low, q^{L*}.

2. **Cournot competition under incomplete information-A twist.** Consider a duopoly market where firms face an inverse demand function $p(Q) = 1-Q$, and $Q = q_1+q_2 \geq 0$ denotes aggregate output. Assume that the marginal production cost of firm 1 (2) is high, $c_H = \frac{1}{2} > 0$, with probability p (q, respectively), where $p, q \in [0, 1]$. Similarly, the marginal cost of firm 1 (2) is low, $c_L = 0$, with probability $1-p$ ($1-q$, respectively).

(a) Find firm 1's best response function when its costs marginal costs are low, $q_1^L(q_2^H, q_2^L)$. Find firm 2's best response function when its marginal costs are low, $q_2^L(q_1^H, q_1^L)$.

- *Firm 1.* When firm 1 has low costs, it chooses q_1^L to solve the following expected profit maximization problem:

$$\begin{aligned}\max_{q_1^L \geq 0} \ \pi_1^L\left(q_1^L\right) &= \overbrace{(1-q)\left(1-q_1^L-q_2^L\right)q_1^L}^{\text{Profits if firm2 is low cost}} + \overbrace{q\left(1-q_1^L-q_2^H\right)q_1^L}^{\text{Profits if firm 2 is high cost}} \\ &= \left(1-q_1^L-(1-q)q_2^L-qq_2^H\right)q_1^L\end{aligned}$$

which does not include the production cost of firm 1 because $c_L = 0$. Differentiating with respect to q_1^L and assuming an interior solution, yields

$$\frac{\partial \pi_1^L\left(q_1^L\right)}{\partial q_1^L} = 1-2q_1^L-(1-q)q_2^L-qq_2^H = 0.$$

Solving for q_1^L, we find the best response function of firm 1 when its costs are low, as follows

$$q_1^L\left(q_2^L, q_2^H\right) = \frac{1}{2} - \frac{(1-q)q_2^L+qq_2^H}{2}$$

which originates at $1/2$, and decreases in its rival's output at a rate of $\frac{1-q}{2}$ ($\frac{q}{2}$) when its rival has low (high) costs. Alternatively, firm 1 decreases its output at a rate of $1/2$ when its rival's expected output, as captured by $(1-q)q_2^L + qq_2^H$, increases by one unit.

- *Firm 2.* When firm 2 has low costs, it chooses q_2^L to solve the following expected profit maximization problem:

$$\max_{q_2^L \geq 0} \ \pi_2^L\left(q_2^L\right) = \overbrace{(1-p)\left(1-q_2^L-q_1^L\right)q_2^L}^{\text{Profits if firm1 is low cost}} + \overbrace{p\left(1-q_2^L-q_1^H\right)q_2^L}^{\text{Profits if firm 1 is high cost}}$$
$$= \left(1-q_2^L-(1-p)q_1^L-pq_1^H\right)q_2^L$$

which does not include the production cost of firm 2 because $c_L = 0$. Differentiating with respect to q_2^L and assuming an interior solution, yields

$$\frac{\partial \pi_2^L\left(q_2^L\right)}{\partial q_2^L} = 1 - 2q_2^L - (1-p)q_1^L - pq_1^H = 0.$$

Solving for q_2^L, we find the best response function of firm 2 when its costs are low, as follows

$$q_2^L\left(q_1^L, q_1^H\right) = \frac{1}{2} - \frac{(1-p)q_1^L + pq_1^H}{2}$$

which originates at $1/2$, and decreases in its rival's output at a rate of $\frac{1-p}{2}$ ($\frac{p}{2}$) when its rival has low (high) costs. Alternatively, firm 2 decreases its output at a rate of $1/2$ when its rival's expected output, as captured by $(1-p)q_1^L + pq_1^H$, increases by one unit.

(b) Find firm 1's best response function when its costs marginal costs are high, $q_1^H(q_2^H, q_2^L)$. Find firm 2's best response function when its marginal costs are high, $q_2^H(q_1^H, q_1^L)$.

- *Firm 1.* When firm 1 has high costs, it chooses q_1^H to solve the following expected profit maximization problem:

$$\max_{q_1^H \geq 0} \ \pi_1^H\left(q_1^H\right) = \overbrace{(1-q)\left(1-q_1^H-q_2^L\right)q_1^H}^{\text{Profits if firm 2 is low cost}} + \overbrace{q\left(1-q_1^H-q_2^H\right)q_1^H}^{\text{Profits if firm 2 is high cost}} - \frac{1}{2}q_1^H$$
$$= \left(1-\frac{1}{2}-q_1^H-(1-q)q_2^L-qq_2^H\right)q_1^H$$

Assuming interior solutions, that is, $q_1^H > 0$, the first order condition satisfies

$$\frac{\partial \pi_1^H\left(q_1^H\right)}{\partial q_1^H} = 1 - \frac{1}{2} - 2q_1^H - (1-q)q_2^L - qq_2^H = 0$$

such that the best response function of firm 1 when its costs are high becomes

$$q_1^H\left(q_2^L, q_2^H\right) = \frac{1}{4} - \frac{(1-q)q_2^L + qq_2^H}{2}$$

which originates at $\frac{1}{4}$, but decreases in its rival's output at a rate of $\frac{1-q}{4}$ ($\frac{q}{4}$) when its rival has low (high) costs.

- *Firm 2.* When firm 2 has high costs, it chooses q_2^H to solve the following expected profit maximization problem:

$$\max_{q_2^H \geq 0} \pi_2^H\left(q_2^H\right) = \overbrace{(1-p)\left(1-q_2^H-q_1^L\right)q_2^H}^{\text{Profits if firm 1 is low cost}} + \overbrace{p\left(1-q_2^H-q_1^H\right)q_2^H}^{\text{Profits if firm 1 is high cost}} - \frac{1}{2}q_2^H$$
$$= \left(1-\frac{1}{2}-q_2^H-(1-p)q_1^L-pq_1^H\right)q_2^H$$

Assuming interior solutions, that is, $q_2^H > 0$, the first order condition satisfies

$$\frac{\partial \pi_2^H\left(q_2^H\right)}{\partial q_2^H} = 1-\frac{1}{2}-2q_2^H-(1-p)q_1^L-pq_1^H = 0$$

such that the best response function of firm 2 when its costs are high becomes

$$q_2^H\left(q_1^L, q_1^H\right) = \frac{1}{4} - \frac{(1-p)q_1^L+pq_1^H}{2}$$

which originates at $\frac{1}{4}$, but decreases in its rival's output at a rate of $\frac{1-q}{4}$ $\left(\frac{q}{4}\right)$ when its rival has low (high) costs.

- Comparing it with firm i's best response function when its costs are low, $q_i^L\left(q_j^L, q_j^H\right)$, we can see that, for a given profile of firm j's output, $\left(q_j^L, q_j^H\right)$, firm i responds producing a larger output when its own costs are low than when they are high. Graphically, $q_i^L\left(q_j^L, q_j^H\right)$ originates at $\frac{1}{2}$ while $q_i^H\left(q_j^L, q_j^H\right)$ originates at $\frac{1}{4}$, and both best response functions have the same slope, thus being parallel to each other.

(c) Use your results from parts (a) and (b) to find the BNE of the game. [*Hint*: You cannot invoke symmetry since best response functions are not symmetric in this case.]

- Substituting $q_2^H\left(q_1^L, q_1^H\right)$ and $q_2^L\left(q_1^L, q_1^H\right)$ into $q_1^H\left(q_2^L, q_2^H\right)$ and $q_1^L\left(q_2^L, q_2^H\right)$, we obtain

$$\begin{aligned} q_1^H &= \frac{1}{4} - \frac{(1-q)}{2}\overbrace{\left[\frac{1}{2}-\frac{(1-p)q_1^L+pq_1^H}{2}\right]}^{q_2^L} - \frac{q}{2}\overbrace{\left[\frac{1}{4}-\frac{(1-p)q_1^L+pq_1^H}{2}\right]}^{q_2^H} \\ &= \frac{q}{8} + \frac{p(q_1^H-q_1^L)}{4} + \frac{q_1^L}{4} \end{aligned}$$

$$\begin{aligned} q_1^L &= \frac{1}{2} - \frac{(1-q)}{2}\overbrace{\left[\frac{1}{2}-\frac{(1-p)q_1^L+pq_1^H}{2}\right]}^{q_2^L} - \frac{q}{2}\overbrace{\left[\frac{1}{4}-\frac{(1-p)q_1^L+pq_1^H}{2}\right]}^{q_2^H} \\ &= \frac{2+q}{8} + \frac{p(q_1^H-q_1^L)}{4} + \frac{q_1^L}{4} \end{aligned}$$

Solving for the system of equations simultaneously, we obtain

$$q_1^{H*} = \frac{1-p+2q}{12}$$

$$q_1^{L*} = \frac{4-p+2q}{12}$$

Substituting q_1^{H*} and q_1^{L*} into firm 2's best response functions, we have

$$\begin{aligned} q_2^{H*} &= \frac{1-c_H}{2} - \frac{(1-q)}{2}\overbrace{\left[\frac{4-p+2q}{12}\right]}^{q_1^L} - \frac{q}{2}\overbrace{\left[\frac{1-p+2q}{12}\right]}^{q_1^H} \\ &= \frac{1-q+2p}{12} \end{aligned}$$

$$\begin{aligned} q_2^{L*} &= \frac{1}{2} - \frac{(1-q)}{2}\overbrace{\left[\frac{4-p+2q}{12}\right]}^{q_1^L} - \frac{q}{2}\overbrace{\left[\frac{1-p+2q}{12}\right]}^{q_1^H} \\ &= \frac{4-q+2p}{12} \end{aligned}$$

Therefore, the equilibrium output levels are

$$\begin{aligned} q_1^{L*} &= \frac{4-p+2q}{12} \\ q_1^{H*} &= \frac{1-p+2q}{12} \\ q_2^{L*} &= \frac{4-q+2p}{12} \\ q_2^{H*} &= \frac{1-q+2p}{12} \end{aligned}$$

(d) How are the equilibrium output levels $\left(q_1^H, q_1^L, q_2^H, q_2^L\right)$ affected by a marginal increase in p? And by a marginal increase in q? Interpret.

- Differentiating with respect to p, we obtain

$$\begin{aligned} \frac{\partial q_1^{L*}}{\partial p} &= -\frac{1}{12} < 0 \\ \frac{\partial q_1^{H*}}{\partial p} &= -\frac{1}{12} < 0 \\ \frac{\partial q_2^{L*}}{\partial p} &= \frac{1}{6} > 0 \\ \frac{\partial q_2^{H*}}{\partial p} &= \frac{1}{6} > 0 \end{aligned}$$

As firm 1's costs are more likely to be high (p increases), firm 1's output level decreases regardless of its own marginal costs. However, firm 2 will produce more units.

- Differentiating with respect to q, we find that

$$\begin{aligned}\frac{\partial q_1^{L*}}{\partial q} &= \frac{1}{6} > 0 \\ \frac{\partial q_1^{H*}}{\partial q} &= \frac{1}{6} > 0 \\ \frac{\partial q_2^{L*}}{\partial q} &= -\frac{1}{12} < 0 \\ \frac{\partial q_2^{H*}}{\partial q} &= -\frac{1}{12} < 0\end{aligned}$$

As firm 2's costs are more likely to be high (q increases), firm 2's output level decreases regardless of its own marginal costs. However, firm 1 will produce more units.

(e) *Symmetric probabilities.* Evaluate your equilibrium results in the special case where both firms' costs occur with the same probability, $p = q$. What if, in addition, these probabilities are both 1/2?

- When $p = q$, firms are symmetric, the equilibrium output levels become

$$q_1^{L*} = q_2^{L*} = q^{L*} = \frac{4+p}{12}$$

$$q_1^{H*} = q_2^{H*} = q^{H*} = \frac{1+p}{12}$$

- In addition, if $p = q = \frac{1}{2}$, the equilibrium output levels become

$$q_1^{L*} = q_2^{L*} = q^{L*} = \frac{3}{8}$$

$$q_1^{H*} = q_2^{H*} = q^{H*} = \frac{1}{8}$$

(f) *Special cases.* Evaluate your equilibrium results in the special case where both firms' types are certain, as under complete information: (1) $p = q = 1$, (2) $p = 1$ and $q = 0$, (3) $p = 0$ and $q = 1$, and (4) $p = q = 0$. Interpret.

- When $p = q = 1$, both firms have high marginal costs, the equilibrium output levels become

$$q_1^{H*} = q_2^{H*} = q^{H*} = \frac{1}{6}$$

which coincides with the results in the standard Cournot game with complete information when both firms face same marginal cost, $c_H = \frac{1}{2}$.

- When $p = 1$ and $q = 0$, firm 1 has high marginal costs and firm 2 has low marginal costs, the equilibrium output levels are

$$\begin{aligned}q_1^{H*} &= 0 \\ q_2^{L*} &= \frac{1}{2}\end{aligned}$$

As expected, the firm with low marginal cost produces more units than the firm with high marginal cost.

- When $p = 0$ and $q = 1$, firm 2 has high marginal costs and firm 1 has low marginal costs, the equilibrium output levels are

$$q_1^{L*} = \frac{1}{2}$$
$$q_2^{H*} = 0$$

As expected, the firm with low marginal cost produces more units than the firm with high marginal cost.

- When $p = q = 0$, both firms have low marginal costs, the equilibrium output levels become

$$q_1^{L*} = q_2^{L*} = q^{L*} = \frac{1}{3}$$

which is exactly the same results as in standard Cournot game with complete information when both firms face zero marginal costs.

3. **PAs with budget constrained bidders.** Consider a FPA with $N \geq 2$ bidders, but assume that every bidder privately observes his valuation for the object, v_i, and his budget, w_i, both being uniformly and independently drawn from the $[0, 1]$ interval. For simplicity, assume that if a bidder wins the auction and the winning price is above his budget, w_i, he cannot afford to pay this price, and the seller imposes a fine on the buyer, $F > 0$, for having to renege from his bid.

   (a) Show that bidding above his budget, $b_i > w_i$, is a strictly dominated strategy for every bidder i.

   - Consider a bidder who submits a bid above his own budget, $b_i > w_i$, and wins the auction. In this case, he would have to pay his bid, b_i, which he cannot afford, thus not receiving the object and, in addition, facing a penalty from the seller after reneging (for a total utility of $-F$). Therefore, deviations to any bid satisfying $b_i \leq w_i$ weakly increase bidder i's payoff, implying that bidding according to $b_i > w_i$ is a strictly dominated strategy.

   (b) If bidder i's valuation, v_i, satisfies $\frac{N-1}{N} v_i \leq w_i$, show that bidding according to $b_i(v_i) = \frac{N-1}{N} v_i$ (as found in section 9.4) is still a weakly dominant strategy.

   - When bidder i's bid in an unconstrained setting, $b_i(v_i) = \frac{N-1}{N} v_i$, is affordable, i.e., $b_i(v_i) \leq w_i$, the bidder behaves as in section 9.4, where we found that this bidding function was optimal for the unconstrained bidder.

   (c) If bidder i's valuation, v_i, satisfies $\frac{N-1}{N} v_i > w_i$, show that submitting a bid equal to his budget, $b_i = w_i$, is a weakly dominant strategy.

   - If bidder i's bid in an unconstrained setting, $b_i(v_i) = \frac{N-1}{N} v_i$, is unaffordable, i.e., $b_i(v_i) > w_i$, he cannot submit $b_i(v_i)$. Doing so would make him renege on the bid he submitted and, thus, not receiving the object but paying a fine F to the seller. Therefore, deviating to $b_i = w_i$ weakly increases bidder i's payoff (never decreases it).
   - In addition, deviating from bid $b_i = w_i$ cannot strictly increase bidder i's payoff:

– If he wins by submitting $b_i = w_i$, he exhausts his budget, yielding a payoff of zero. If he deviates to $b_i < w_i$, he would lose the auction, which entails a zero payoff.

– If he loses by submitting $b_i = w_i$, his payoff is zero. Deviating to a higher bid increases his chances of winning, but he would not be able to pay his bid, not receiving the object, and facing a fine from the seller, ultimately producing a negative payoff.

In summary, if $\frac{N-1}{N}v_i > w_i$, so $v_i > \frac{N}{N-1}w_i$, bidder i submits a bid equal to his budget, $b_i = w_i$, and he cannot strictly increase his payoff by submitting any other bids, making $b_i = w_i$ a weakly dominant strategy in this context.

(d) Combine your results from parts (b) and (c) to describe the equilibrium bidding function in the first-price auction with budget constraints, $b_i(v_i, w_i)$. Depict it as a function of v_i.

- We found that, every bidder i submits a bid $\frac{N-1}{N}v_i$ when his budget constraint is not binding (as in part b), and a bid equal to his budget, w_i, otherwise (as in part c). More compactly,

$$b_i(v_i, w_i) = \begin{cases} \frac{N-1}{N}v_i & \text{if } v_i \leq \frac{N}{N-1}w_i \\ w_i & \text{otherwise.} \end{cases}$$

When $N = 2$ bidders, the equlibrium bid is $\frac{v_i}{2}$, when $\frac{v_i}{2} \leq w_i$ or $v_i \leq 2w_i$; but becomes $b_i = w_i$ otherwise. Graphically, when $v_i \leq 2w_i$, the equilibrium bid increases in bidder i's valuation, but otherwise, the bidding function become flat. In other words, when $v_i \leq 2w_i$, the bidder behaves as in a standard FPA, but otherwise he submits a bid equal to his budget w_i, thus being unaffected by his valuation.

4. **Equilibrium bidding function in a lottery auction.** Consider a lottery auction with $N \geq 2$ bidders, all of them assigning the same value to the object, v. Every bidder i's utility from submitting a bid b_i is

$$EU_i\left[b_i|v\right] = \frac{b_i}{b_i + B_{-i}}v - b_i$$

where the ratio represents bidder i's probability of winning, which compares his bid relative to the aggregate bids submitted by all players, where $B_{-i} \equiv \sum_{j \neq i} b_j$. The second term indicates that bidder i must pay his bid both when winning and losing, as in APAs. For simplicity, assume that bidders use a symmetric bidding strategy, $b(v)$.

(a) Find bidder i's equilibrium bidding function.

- Differentiating $EU_i\left[b_i|v\right]$ with respect to b_i, we find

$$\frac{B_{-i}}{(b_i + B_{-i})^2}v - 1 = 0$$

In a symmetric bidding strategy, $b^* = b_1^* = b_2^* = \cdots = b_N^*$, so that $B_{-i}^* = (N-1)b^*$. Therefore, the above first-order condition becomes

$$\frac{(N-1)b^*}{\left[b^* + (N-1)b^*\right]^2}v = 1$$

Since $b^* + (N-1)b^* = Nb^*$, we can rearrange this expression as follows

$$\frac{N-1}{N^2 b^*}v = 1$$

Solving for b^*, yields an equilibrium bidding function of

$$b^*(v) = \frac{N-1}{N^2}v.$$

(b) *Comparative statics.* How does bider i's equilibrium bid change with v and N?

- Differentiating $b^*(v)$ with respect to v, we find that

$$\frac{\partial b^*(v)}{\partial v} = \frac{N-1}{N^2} \geq 0,$$

implying that bidder i increases his bid when his valuation of the object (and that of all other bidders, since valuations are common in our setting) increases.

- Differentiating $b^*(v)$ with respect to N, yields

$$\frac{\partial b^*(v)}{\partial N} = -\frac{N-2}{N^3}v \leq 0$$

Since $N \geq 2$ by assumption, the more bidders competing in the auction, the less that every bidder i bids in equilibrium. For instance, evaluating quilibrium bidding function $b^*(v)$ at $N = 2$, yields $b^*(v) = \frac{1}{4}v$; evaluating it at $N = 3$, we obtain $b^*(v) = \frac{2}{9}v$; and evaluating it at $N = 10$, we find $b^*(v) = \frac{9}{100}v$. Graphically, the bidding function rotates clockwise as the number of bidders increases.

(c) *Bidding coordination.* Find equilibrium bids if bidders could coordinate their bidding decisions. Compare your results with those of part (b).

- If bidders choose the bidding profile $(b_1, ..., b_N)$ to maximizes their joint expected utility, they would solve the following problem

$$\begin{aligned}\max_{b_1,\dots,b_N \geq 0} \quad & \sum_{i=1}^{N} \left(\frac{b_i}{b_i + B_{-i}}v - b_i \right) \\ = \quad & \frac{\sum_{i=1}^N b_i}{\sum_{i=1}^N b_i + \sum_{i=1}^N B_{-i}}v - \sum_{i=1}^N b_i\end{aligned}$$

Since $B = \sum_{i=1}^N b_i$ and $\sum_{i=1}^N B_{-i} = B_{-i}$ (it is unaffected by the sum), we can compactly express the above problem as follows

$$\begin{aligned}\max_{b_1,\dots,b_N \geq 0} \; & \frac{B}{b_i + B_{-i}}v - B \\ & = v - (b_i + B_{-i}).\end{aligned}$$

Differentiating with respect to b_i, we obtain -1, indicating that we face a corner solution. In other words, bidders maximize their joint expected payoff when bidding $b_i^* = b_j^* = 0$ for every bidder $i \neq j$.

- *Comparison.* When every bidder simultaneously and independently chooses his bid, he submits a bid of $b^*(v) = \frac{N-1}{N^2}v$, ignoring the negative externality that his bid imposes on the other player, namely, it reduces bidder j's probability of winning the auction. In contrast, when all bidders coordinate their bids, they internalize this externality, reducing their bid to zero.