EconS 424 – Strategy and Game Theory

Homework #4 – Answer Key

Exercise #13.7 from Harrington – *Collusion among N doctors*

Consider an infinitely repeated game, in which there are $n \geq 3$ doctors, who have created a partnership. In each period, each doctor decides how hard to work. Let e_i^t be the effort chosen by doctor *i* in period *t*, and $e_i^t = 1, 2, \ldots, 10$. Doctor i's discount factor is δ_i.

Total profit for the partnership: $2(e_1^t + e_2^t + e_3^t + \cdots + e_n^t)$

A doctor *i*'s payoff: $\frac{1}{n} \times 2 \times (e_1^t + e_2^t + e_3^t + \cdots + e_n^t) - e_i^t$

a. Assume that the history of the game is common knowledge. Derive a subgame perfect NE in which each player chooses effort $e^* > 1$.

To begin, note that doctor's payoff can be rearranged to:

$$\left(\frac{2}{n}\right)(e_1 + e_2 + \cdots + e_{i-1} + e_{i+1} + \cdots + e_n) - \left(\frac{n-2}{n}\right)e_i$$

Since a doctor's payoff is strictly decreasing in her own effort, she wants to minimize it. $e_i = 1$ is then a strictly dominant strategy for doctor *i* and therefore there is a unique stage game Nash equilibrium in which each doctor chooses the minimal effort level of 1.

- Next, note that each doctor's payoff from choosing a common effort level of *e* is:

$$\left(\frac{1}{n}\right) \times 2 \times (e + e + \cdots + e) - e = \left(\frac{1}{n}\right) \times 2 \times ne - e = e$$

- To determine a doctor's best deviation, we must take a partial derivative with respect to e_i of their payoff function when all other (n-1) players select *e*, yielding

$$\frac{\partial u_i}{\partial e_i} = -\left(\frac{n-2}{n}\right)$$

  which is clearly negative given that $n > 2$. This suggests a corner solution where doctor *i* wants to minimize effort by playing the lowest possible effort, i.e., e_i=1.

- We can now describe a **grim-trigger strategy**. When conditions are met and the strategy is played symmetrically, that will guarantee cooperation at an effort level *e*>1.

Consider the symmetric grim-trigger strategy:

- In period 1: choose $e_i^1 = e^*$
- In period t ≥ 2: choose $e_i^t = e^*$ when $e_j^\tau = e^*$ for all *j*, for all $\tau \leq t-1$; and choose 1 otherwise.

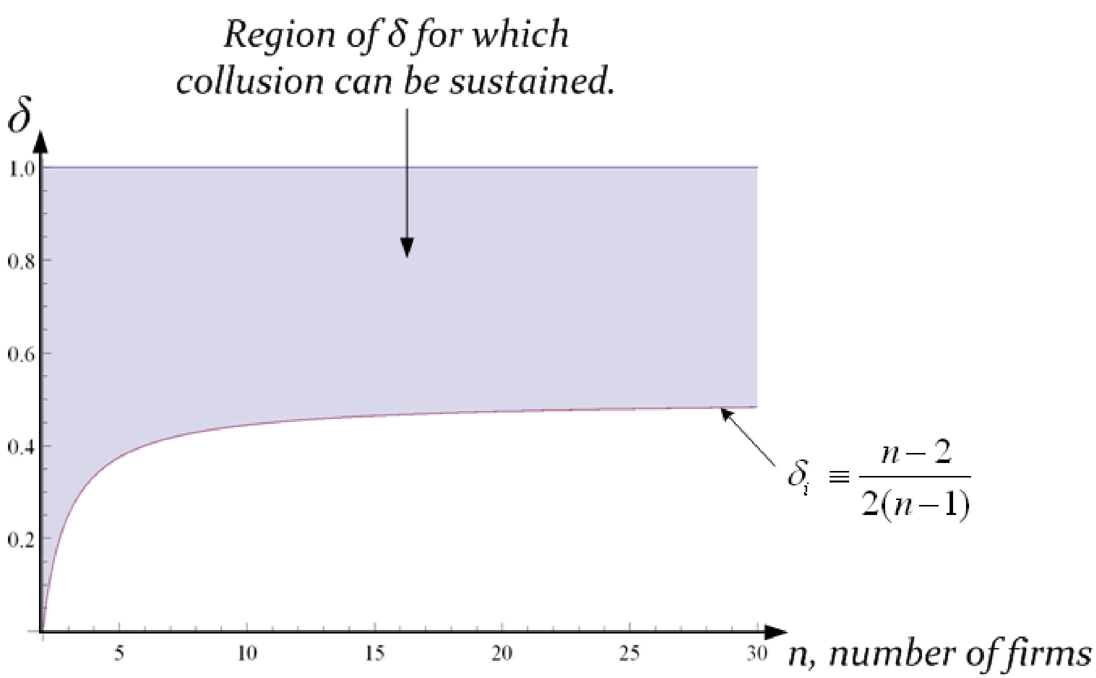
This is a subgame perfect Nash equilibrium if and only if

$$\frac{e^*}{1-\delta_i} \geq \left[\left(\frac{n-1}{n}\right)2e^* - \left(\frac{n-2}{n}\right) * 1\right] + \frac{\delta_i}{1-\delta_i} \quad for\ all\ i.$$

That is to say, the equilibrium will only hold so long as the payoff from remaining in the equilibrium is greater than or equal to the one period payoff from deviating plus the payoff from the 'punishment' equilibrium played every period thereafter. Solving for δ_i yields:

$$\delta_i \geq \frac{n-2}{2(n-1)}$$

The following figure depicts this cutoff of δ_i, shading the region of discount factors above δ_i which would support collusion.



It is now possible to see how the equilibrium responds to changes in *n*. Differentiating the about cutoff of δ_i with respect to *n*, we obtain

$$\frac{\partial \delta_i}{\partial n} = \frac{1}{2(n-1)^2}$$

This partial is positive, indicating that as the group size *n* increases, δ_i has to increase to maintain the cooperative equilibrium. So it is more difficult to support cooperation as the group size increases.

b. Assume that the history of the game is not common knowledge, i.e., in each period, only the total effort is observed. Find a subgame perfect NE in which each player chooses effort $e^* > 1$.

Consider the strategy profile in part (a), except that it now conditions on total effort. Let e^t denote total effort for period *t.*

- In period 1: choose $e_i^1 = e^*$
- In period t ≥ 2: choose $e_i^t = e^*$ when $e^\tau = ne^*$ for all *j*, for all $\tau \leq t-1$; and

  choose 1 otherwise.

This is a subgame perfect Nash equilibrium under the exact same conditions as in part (a).

Exercise #13.9 from Harrington: see scanned pages at the end of the handout.

Exercise #2 – *Collusion among N firms*

Consider *n* firms producing homogenous goods and choosing quantities in each period for an infinite number of periods. Demand in the industry is given by $p = 1 - Q$, *Q* being the sum of individual outputs. All firms in the industry are identical: they have the same constant marginal costs $c < 1$, and the same discount factor δ. Consider the following trigger strategy:

- Each firm sets the output q^m that maximizes joint profits at the beginning of the game, and continues to do so unless one or more firms deviate.
- After a deviation, each firm sets the quantity q^{cn}, which is the Nash equilibrium of the one-shot Cournot game.

(a) *Find the condition on the discount factor that allows for collusion to be sustained in this industry.*

First find the quantities that maximize joint profits $\pi = (1-Q)Q - cQ$. It is easily checked that this output level is $Q = \frac{1-c}{2}$, yielding profits of

$$\pi = \left(1 - \frac{1-c}{2}\right)\frac{1-c}{2} - c\,\frac{1-c}{2} = \frac{(1-c)^2}{4}$$

for the cartel.

Therefore, at the symmetric equilibrium individual quantities are $q^m = \frac{1}{n}\frac{1-c}{2}$ and individual profits under the collusive strategy are $\pi^m = \frac{1}{n}\frac{(1-c)^2}{4}$.

As for the deviation profits, the optimal deviation by a firm is given by

$$q^d(q^m) = \text{argmax}_q \left[1-(n-1)q^m - q\right]q - cq .$$

where note that all other n-1 firms are still producing their cartel output $q^m = \frac{1}{n}\frac{1-c}{2}$.

It can be checked that the value of q that maximizes the above expression is $q^d(q^m) = (n+1)\frac{(1-c)}{4n}$, and that the profits that a firm obtains by deviating from the collusive output are, hence,

$$\pi^d = \left[1-(n-1)\left(\frac{1}{n}\frac{1-c}{2}\right) - (n+1)\frac{1-c}{4n}\right](n+1)\frac{1-c}{4n} - c(n+1)\frac{1-c}{4n},$$

which simplifies to

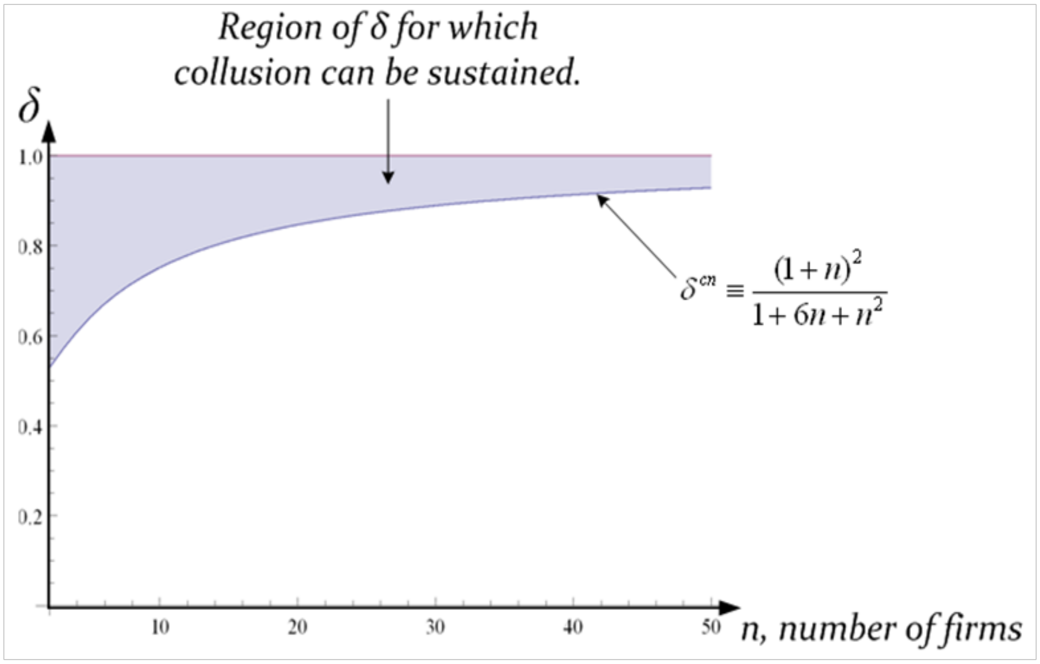
$$\pi^d = \frac{(1-c)^2(n+1)^2}{16n^2}$$

Therefore, collusion can be sustained in equilibrium if

$$\frac{1}{1-\delta}\pi^m \geq \pi^d + \frac{\delta}{1-\delta}\pi^{cn},$$

which after solving for the discount factor, δ, yields $\delta \geq \frac{(1+n)^2}{1+6n+n^2}$. For compactness, we denote this ratio as $\frac{(1+n)^2}{1+6n+n^2} \equiv \delta^{cn}$.

Hence, under punishment strategies that involve a reversion to Cournot equilibrium forever after a deviation takes place, tacit collusion arises if and only if firms are sufficiently patient. The following figure depicts cutoff δ^{cn}, as a function of the number of firms, n, shading the region of δ that exceeds such a cutoff.



(b) Indicate how the number of firms in the industry affects the possibility of reaching the tacit collusive outcome.

$$\frac{\partial \delta^{cn}}{\partial n} = \frac{4(n^2-1)}{(1+6n+n^2)^2} > 0.$$

(This could be anticipated from our previous figure, where the critical discount factor increases in *n*.)

***Intuition**:* Other things being equal, as the number of firms in the agreement increases, the more difficult it is to reach and sustain tacit collusion. Since firms are assumed to be symmetric, an increase in the number of firms is equivalent to a lower degree of market concentration. Therefore, lower levels of market concentration are associated – *ceteris paribus* – with less likely collusion.

For not using gas to be optimal, the first expression must be at least as great as the second for all $0 < g' \leq G_h$, which is true if and only if

$$-\frac{eg^*}{1-p} \geq (dG_h - eg^*) + \left(\frac{p}{1-p}\right)(dG_h - eG_c) \Rightarrow g^* \leq G_c - \left(\frac{d}{pe}\right)G_h.$$

For histories in which $g_c^\tau > g^*$ and/or $g_h^\tau > 0$ for some $\tau \leq t-1$, it is straightforward to show that the prescribed behavior is optimal. The analysis is the same as that in part (a). We conclude that this strategy pair is a subgame perfect Nash equilibrium if and only if

$$G_c - \left(\frac{d}{pe}\right)G_h \geq g^* \geq G_c - \left(\frac{pb}{a}\right)G_h.$$

That is, the amount of gas that Churchill uses must be sufficiently great that he prefers to continue that use rather than cheat by going to G_c and, in addition, must be sufficiently small so that Hitler prefers to use no gas to using gas and inducing Churchill to use G_c.

9. Consider this two-player symmetric stage game, and suppose it is played three times. After each time it is played, they get to observe what happened. For this three-period game, assume each player's payoff is the sum of her individual period payoffs.

Player 2

Player 1	a	b	c
a	10,10	4,8	1,12
b	8,4	5,5	0,3
c	12,1	3,0	2,2

a. Find three different SPNE. (Note: There may be more than three SPNE but you need only find three.)

ANSWER: When played once, the game has two Nash equilibria: both play b and both play c. Any sequence of these Nash equilibria for the three-period game is an SPNE. For example, consider the symmetric strategy pair: play b in period 1, play c in period 2 (regardless of what happened in period 1), and play b in period 3 (regardless of what happened in periods 1 and 2). Given the other player uses this strategy, a player's payoff from this strategy is $5 + 2 + 5 = 12$. Any other strategy involves a lower payoff because changing an action in any period reduces that period's payoff and leaves unaffected the payoff from other periods because the other player's future actions are independent of what a player does. Under this logic, there are then at least eight subgame perfect Nash equilibria that represent the eight ways in which the two Nash equilibria for the one-period game can be sequenced over three periods.

b. Find an SPNE which has both players choosing action a in the first two periods.

ANSWER: The subgame perfect Nash equilibria described in part (a) had a player's behavior independent of the game's history. For example, what a player did in period 2 was the same for all outcomes in period 1. However, there are also equilibria that depend on the history. Consider the following symmetric strategy pair. A player chooses a in period 1; chooses a in period 2 if both players chose a in period 1 and plays c otherwise; chooses b in period 3 if both players chose a in periods 1 and 2 and plays c otherwise. For this strategy pair, consider a period 3 subgame. There are 81 possible period 3 subgames that are defined by what transpired over the first two periods. Depending on the history over the first two periods, their strategies have them play either b or c. As it is a Nash equilibrium for both to play b and for both to play c, the sub-strategy pair for the period 3 subgame is a Nash equilibrium, as required by the definition of SPNE.

Now consider a period 2 subgame, of which there are nine. First consider the subgame associated with both having chosen *a* in period 1. A player's strategy has her choose *a* in period 2. Given she does so and she acts according to her strategy in period 3 and given the other player's strategy, a player's payoff is 10 + 10 + 5 = 25 as both choose *a* in period 2 and then both choose *b* in period 3. If a player chooses any other action in period 2 then the outcome will be both playing *c* in period 3. Hence, if a player chooses *b* in period 2 then her payoff is 10 + 8 + 2 = 20 and if she chooses *c* then her payoff is 10 + 12 + 2 = 24. In either case, she prefers to play *a* as prescribed by her strategy. Now consider the period 2 subgame in which one or both players did not choose *a* in period 1. A player's strategy has her choose *c* which is indeed optimal given that the other player is going to choose *c* in both periods 2 and 3. We conclude that the sub-strategy pair on any period 2 subgame forms a Nash equilibrium.

Finally, consider the period 1 subgame (which is the game itself). A player's strategy has her play *a* which yields a payoff of 10 + 10 + 5 = 25. Choosing *b* yields a payoff of 8 + 2 + 2 = 12 and choosing *c* yields a payoff of 12 + 2 + 2 = 16; both of which are inferior to what is specified by the strategy. In sum, the strategy pair described above forms an SPNE and results in an outcome path of both players choosing *a* in periods 1 and 2 and choosing *b* in period 3. The desirable outcome of both playing *a* is supported in periods 1 and 2, even though it is not a Nash equilibrium for the one-period game, by the threat that if any player deviates from it then they will go to the less desirable of the two Nash equilibria in the final period and, if deviation occurred in period 1, in period 2 as well. Note that, in the final period, players must be at a Nash equilibrium for the stage game; however, which one they're at depends on the history.

10. Consider the infinitely repeated game based on the stage game in the figure below. Each player has a discount factor δ where $0 < \delta < 1$.

Player 1 \ Player 2	*a*	*b*	*c*	*d*
a	2,2	3,0	1,1	11,1
b	1,3	6,4	3,2	1,3
c	0,4	4,1	5,5	4,4
d	2,4	2,5	2,12	8,9

a. Consider a symmetric strategy pair that has a player choose action *d* in period 1. In any other period, a player chooses action *d* as long as, in all past periods, both players chose action *d*; and, for any other history, a player chooses action *b*. Derive the conditions for this strategy pair to be an SPNE.

ANSWER: Take period 1 or any period for which no deviation has occurred (i.e., both players have always chosen *d*). In this period, the profile specifies that (*d*,*d*) continues to be played. If player 1 deviates this period and follows the strategy afterwards, she can at most get 11 this period by deviating to *a*. In the ensuing periods, both players will play *b* and thus player 1 receives 6 in each period; the present value of that payoff stream is $\delta \times 6/(1-\delta)$. On the other hand, if player 1 does not deviate, she will get 8 in every period, which means a payoff of $8/(1-\delta)$. Hence, equilibrium requires

$$\frac{8}{1-\delta} \geq 11 + \delta\left(\frac{6}{1-\delta}\right) \Rightarrow 8 \geq 11 - 11\delta + 6\delta \Rightarrow \delta \geq \frac{3}{5}.$$

For player 2, if he deviates this period and follows the profile afterwards, he can get at most $12 + \delta \times 4/(1-\delta)$. If he does not deviate he will get $9/(1-\delta)$. We then need

$$\frac{9}{1-\delta} \geq 12 + \delta\left(\frac{4}{1-\delta}\right) \Rightarrow 9 \geq 12 - 12\delta + 4\delta \Rightarrow \delta \geq \frac{3}{8}.$$