Cheap Talk Games - An Introduction[1]

13.3 Cheap talk with discrete messages but continuous responses

Let us extend the cheap talk model with discrete types, messages, and responses. First, we allow for continuous responses from the receiver (politician), still assuming discrete types and messages. Figure 13.4 depicts the game tree, where the arcs next to the terminal nodes indicate that the last mover chooses his response (policy choice, p, for the politician) from a continuum of available policies, that is, $p > 0$.

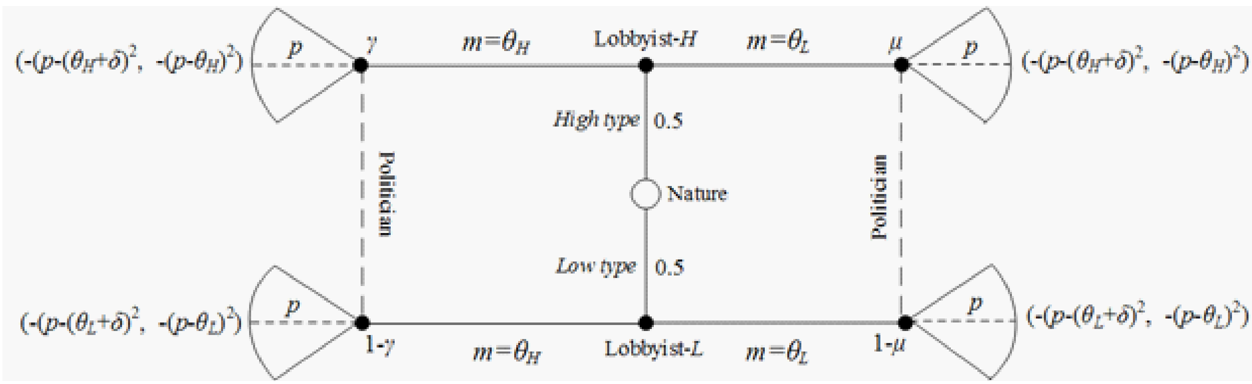

Figure 13.4. Cheap talk with two messages but continuous responses.

As in the previous section, the lobbyist privately observes the state of nature, θ_H or θ_L, both equally likely. The lobbyist, then, chooses a message to send to the politician which, for simplicity, is still binary (either θ_H or θ_L). Upon observing this message, the politician responds with a policy $p > 0$.

Quadratic loss functions. The government (politician) payoff are distributed according to the following quadratic loss functions, as in Crawford and Sobel (1982),

$$U_G(p, \theta) = -(p - \theta)^2$$

which becomes zero when the government responds with a policy that coincides with the true state of the world, that is, $p = \theta$, but is negative otherwise (both when $p < \theta$ and when $p > \theta$). Graphically, $U_G(p, \theta)$ has an inverted-U shape, lying in the negative quadrant for all $p \neq \theta$, but has a height of zero at exactly $p = \theta$; as depicted in figure 13.5. Similarly, the lobbyist's utility is given by a quadratic loss function

$$U_L(p, \theta) = -(p - (\theta + \delta))^2$$

which becomes zero when the policy coincides with the lobbyist's ideal, $p = \theta + \delta$, but is negative otherwise. Intuitively, parameter $\delta > 0$ represents the lobbyist's bias. When $\delta = 0$ the utility functions of both sender and receiver coincide, and we can say that their

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preferences are aligned; but otherwise the lobbyist's ideal policy, $p = \theta + \delta$, exceeds the politician's, $p = \theta$.

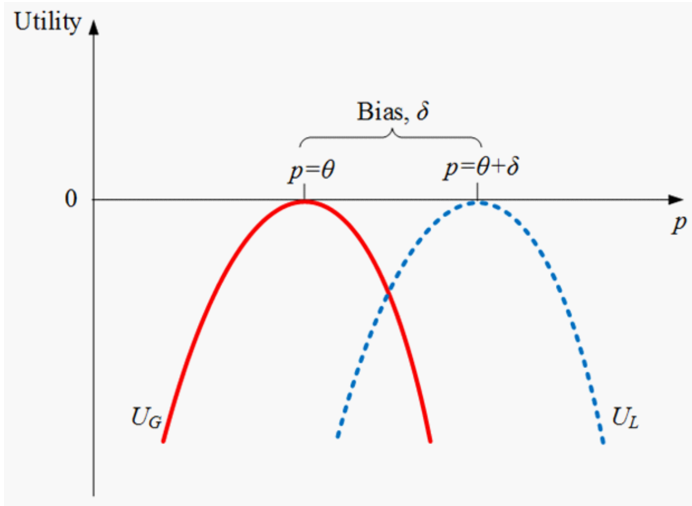

Figure 13.5. Quadratic loss function for each player.

13.3.1 Separating PBE

1. *Specifying a strategy profile.* We first specify a "candidate" of strategy profile that we seek to test as a PBE, (θ_H, θ_L), where the θ_H-type lobbyist chooses θ_H, on the top left side of figure 13.6; whereas the θ_L-type lobbyist selects θ_L, on the bottom right side of the figure.

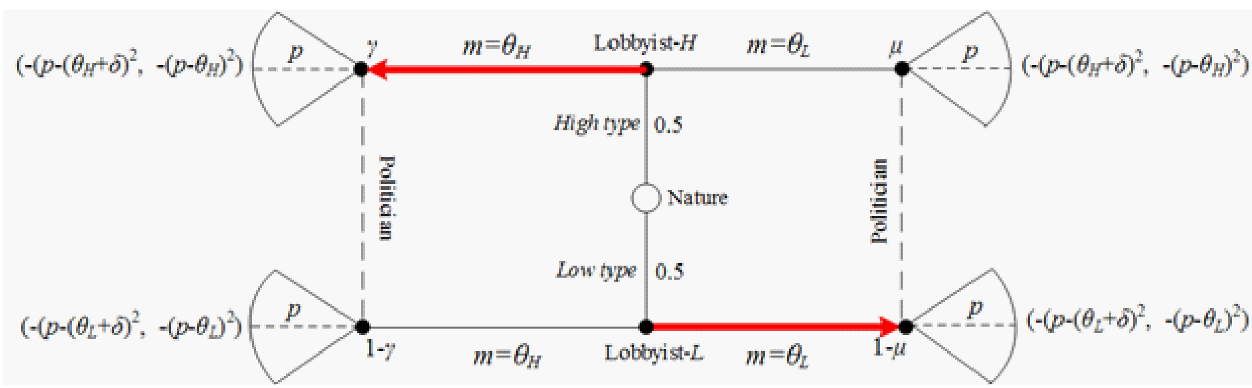

Figure 13.4. Cheap talk with two messages but continuous responses - Separating profile (θ_H, θ_L).

2. *Bayes' rule.* Upon observing θ_H, the politician believes that this message must originate from a high-type lobbyist, entailing that $\mu(\theta_H|\theta_H) = 1$, at the top left side of the figure, which implies that $\mu(\theta_L|\theta_H) = 0$. Similarly, upon observing θ_L, the politician beliefs are $\mu(\theta_L|\theta_L) = 1$, at the bottom right side of the figure, entailing that $\mu(\theta_H|\theta_L) = 0$.

3. *Optimal response.* Given our result from Step 2, the politician responds as follows:

(a) Upon observing θ_H, she responds with policy $p = \theta_H$, as this policy minimizes her quadratic loss, yielding a payoff of zero.[2]

(b) Similarly, upon observing θ_L, she responds with policy $p = \theta_L$, since this policy minimizes her quadratic loss, also yielding a payoff of zero.

4. *Optimal messages.* From our results in Step 3, we now identify the lobbyist's optimal messages.

   (a) *High type.* If he sends a message of θ_H, as prescribed in this strategy profile, he anticipates that the politician will respond with policy $p = \theta_H$, yielding a payoff of $-(\theta_H - (\theta_H + \delta))^2 = \delta^2$. If, instead, this lobbyist deviates towards message θ_L, the politician believes this message, responding with policy $p = \theta_L$, with an associated payoff of $-(\theta_L - (\theta_H + \delta))^2 = -(\theta_L - \theta_H - \delta)^2$ for the lobbyist. Therefore, he does not have incentives to deviate since $\delta^2 > -(\theta_L - \theta_H - \delta)^2$ simplifies to $\theta_H > \theta_L$, which holds by assumption. Intuitively, because of his bias, the lobbyist prefers a policy above the state of nature. Sending message θ_H, he at least induces a policy $p = \theta_H$, but sending message θ_L he would induce a lower policy, $p = \theta_L$, which is further away from his ideal, $p = \theta_H + \delta$; implying that he does not have incentives to send θ_L.

   (b) *Low type.* If he sends a message of θ_L, as prescribed in this strategy profile, the politician responds with $p = \theta_L$, yielding a payoff $-(\theta_L - (\theta_L + \delta))^2 = -\delta^2$ for the lobbyist. If, instead, he deviates to message θ_H (e.g., an overestimation of the true state of the lobbyist industry), the politician responds with $p = \theta_H$, and the lobbyist earns $-(\theta_H - (\theta_L + \delta))^2$. Therefore, the low-type lobbyist does not have incentives to misrepresent the state of nature if

   $$-\delta^2 \geq -(\theta_H - (\theta_L + \delta))^2$$

   simplifying, and solving for δ, yields

   $$\delta \leq \frac{\theta_H - \theta_L}{2}.$$

   Therefore, the low-type lobbyist truthfully report the state of nature if his bias, δ, is sufficiently small or, alternatively, when his preferences and the politician's are sufficiently aligned.

5. *Summary.* From Step 4, we found that no sender types have incentives to deviate from (θ_H, θ_L), implying that this separating strategy profile can be supported as a PBE if $\delta \leq \frac{\theta_H - \theta_L}{2}$. In this PBE, the politician, upon observing message θ_i, holds beliefs $\mu(\theta_i|\theta_i) = 1$ and $\mu(\theta_j|\theta_i) = 0$, and responds with policy $p = \theta_i$.

[2] Formally, one can first consider the politician's utility function, $U_G(p, \theta_H) = -(p - \theta_H)^2$, evaluated at θ_H given the policitian's updated beliefs from Step 2, and then differentiate it with respect to policy p, which yields $-2(p - \theta_H) = 0$, which holds if and only if $p = \theta_H$.

13.3.2 Pooling PBEs

1. *Specifying a strategy profile.* We now test if the pooling strategy profile (θ_H, θ_H), where both lobbyists send message θ_H, can be sustained as a PBE. Figure 13.7 depicts this strategy profile.

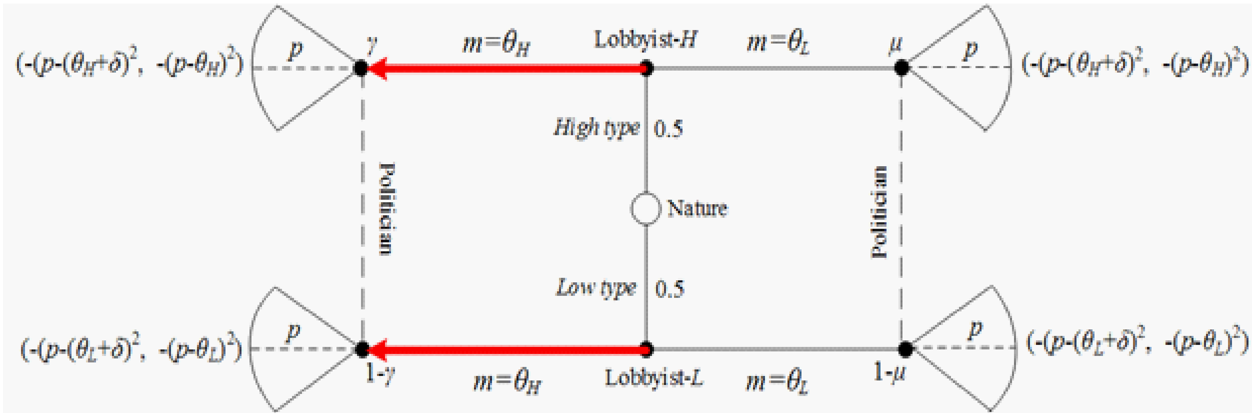

Figure 13.4. Cheap talk with two messages but continuous responses - Pooling profile (θ_H, θ_H).

2. *Bayes' rule.* Upon observing message θ_H, in equilibrium, the politician cannot infer any information from this message, and her posterior beliefs coincide with her priors, that is, $\mu(\theta_H|\theta_H) = 1/2$. Upon observing message θ_L, however, which occurs off-the-equilibrium path, beliefs cannot be updated using Bayes' rule, and we leave them unrestricted, $\mu(\theta_L|\theta_L) = \mu \in [0, 1]$.

3. *Optimal response.* Given our result from Step 2, the politician responds as follows:

   (a) Upon observing θ_H, the politician chooses the policy p that solves

$$\max_{p\geq 0} \underbrace{-\frac{1}{2}(p-\theta_H)^2}_{\text{if } \theta=\theta_H} - \underbrace{\frac{1}{2}(p-\theta_L)^2}_{\text{if } \theta=\theta_L}$$

   since, from Step 2, the politician's beliefs are $\mu(\theta_H|\theta_H) = 1/2$ and $\mu(\theta_L|\theta_H) = 1/2$. Differentiating with respect to p, we obtain $-(p-\theta_H) - (p-\theta_L) = 0$, and solving for p, we find that her optimal response becomes

$$p = \frac{\theta_H + \theta_L}{2},$$

   i.e., the expected state of nature. Intuitively, when the politician believes that both states of nature are equally likely, she implements a policy that coincides with the expected state of nature.

   (b) Upon observing θ_L, which occurs off-the-equilibrium path, the politician solves

$$\max_{p\geq 0} \underbrace{-(1-\mu)(p-\theta_H)^2}_{\text{if } \theta=\theta_H} - \underbrace{\mu(p-\theta_L)^2}_{\text{if } \theta=\theta_L}$$

where $\mu(\theta_L|\theta_L) = \mu$ and $\mu(\theta_H|\theta_L) = 1 - \mu$. Differentiating with respect to p, we obtain

$$-2(1-\mu)(p-\theta_H) - 2\mu(p-\theta_L) = 0,$$

and, after solving for p, we find that the politician's optimal response is

$$p = \frac{(1-\mu)\theta_H + \mu\theta_L}{2}$$

which can also be interpreted as the politician's expected state of nature, given her off-the-equilibrium beliefs μ and $1 - \mu$.

4. *Optimal messages.* From our results in Step 3, we now identify the lobbyist's optimal messages.

   (a) *High type.* From point 4a in the separating PBE (see section 13.3.1), the θ_H-type lobbyists sends message of θ_H, and this result holds for all parameter values (i.e., for all δ).

   (b) *Low type.* From point 4a in the separating PBE (section 13.3.1), we know that the θ_L-type lobbyists sends message θ_H, thus misreporting the true state of nature, if

   $$\delta^2 < -(\theta_H - (\theta_L + \delta))^2$$

   which simplifying, and solving for δ, yields

   $$\delta > \frac{\theta_H - \theta_L}{2}.$$

   Therefore, the low-type lobbyist misreports the state of nature, θ_L, if his bias, δ, is sufficiently large or, in other words, if his preferences and the politician's are sufficiently misaligned.

5. *Summary.* From Step 4, we found that no sender types have incentives to deviate from the pooling strategy profile (θ_H, θ_H), sustaining it as a PBE, if $\delta > \frac{\theta_H - \theta_L}{2}$. In this PBE, the politician, upon observing message θ_H, in equilibrium, holds beliefs $\mu(\theta_H|\theta_H) = 1/2$, responding with $p = \frac{\theta_H + \theta_L}{2}$; and upon observing θ_L, off the equilibrium, his beliefs are unrestricted, $\mu(\theta_L|\theta_L) = \mu$, responding with policy $p = \frac{(1-\mu)\theta_H + \mu\theta_L}{2}$.

As a practice, exercise XXXX asks you to examine under which conditions can the opposite pooling strategy profile, (θ_L, θ_L), be supported as a PBE.

13.4 Cheap talk with continuous messages and responses

Let us now extend the above cheap talk model to allow for continuous messages and responses. While the setting in section 13.3 considered continuous responses by the politician, it restricted the lobbyist's messages to only two (binary), as he choose either message θ_H or θ_L. In addition, we allow for the state of nature, θ, to be continuous, in particular, we assume that $\theta \sim U[0, 1]$. Otherwise, it would look unnatural to still consider only two states of nature (θ_H or θ_L) but a continuum of potential messages, $m \geq 0$.

Our discussion, based on Crawford and Sobel (1982), helps us confirm one of the results we found in the previous section, namely, that separating strategy profiles can emerge in equilibrium if the lobbyist and politician's preferences are relatively aligned (low δ). This more general setting, however, allows us a new result: that the quality of information, understook as the number of different messages that the lobbyist sends, also depends on players' preference alignment. Intuitively, as their preferences become more aligned, the lobbyist has incentives to emit a wider array of distinct messages, like different words in a language, ultimately improving the information that the politician receives.

13.4.1 Separating PBE

1. *Specifying a strategy profile.* We first specify a "candidate" of strategy profile that we seek to test as a PBE, as that depicted in figure 13.8, where the lobbyist sends message m_1 when θ lies in the first interval, that is, $\theta \in [\theta_0, \theta_1)$; sends message m_2 when θ lies in the second interval, that is, $\theta \in [\theta_1, \theta_2)$; and similarly for the next intervals until reaching the last interval N.

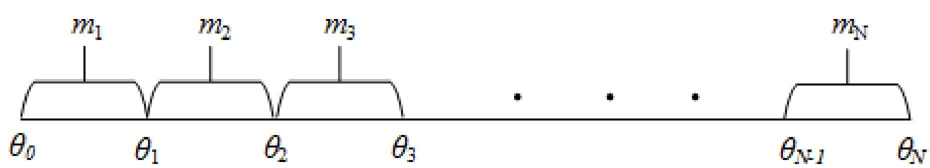

Figure 13.8. Partially informative strategy profile.

2. *Bayes' rule.* Upon observing message m_k, the politician believes that the state of nature must satisfy $\theta \in [\theta_{k-1}, \theta_k]$, such that

$$\mu(\text{Interval } k|m_k) = 1 \text{ and } \mu(\text{Interval } j|m_k) = 0 \text{ for all } j \neq k$$

   That is, after receiving message m_k, the politician believes that θ must lie on that interval, and that θ cannot lie on other intervals.

3. *Optimal response.* Given our result from Step 2, the politician responds as follows:

   - Upon observing message m_k, she believes to be interval k, thus responding with policy $p = \frac{\theta_{k-1}+\theta_k}{2}$, which minimizes the quadratic loss in that interval. More formally, the politician solves

$$\max_{p \geq 0} \; -E\left[(p-\theta)^2\right]$$
$$\text{such that } \theta \in [\theta_{k-1}, \theta_k]$$

     Since the politician believes that $\theta \in [\theta_{k-1}, \theta_k]$, the expected value of θ is $\frac{\theta_{k-1}+\theta_k}{2}$, which simplifies the above problem to

$$\max_{p \geq 0} \; -\left(p - \frac{\theta_{k-1}+\theta_k}{2}\right)^2$$

Differentiating with respect to policy p, yields

$$-2\left(p-\frac{\theta_{k-1}+\theta_k}{2}\right)(-1)=2\left(p-\frac{\theta_{k-1}+\theta_k}{2}\right)=0$$

which entails an optimal policy

$$p=\frac{\theta_{k-1}+\theta_k}{2}.$$

In other words, after receiving message m_k, the politician (receiver) responds with a policy that coincides with the expected state of the nature in this interval, $p=\frac{\theta_{k-1}+\theta_k}{2}$.

4. *Optimal messages.* From our results in Step 3, we now find the lobbyist's optimal messages.

- When θ_k is the true state of the world, the lobbyist sends a message in the k-th interval, as prescribed by this strategy profile, inducing the politician to respond with policy $p=\frac{\theta_{k-1}+\theta_k}{2}$. We now check that the sender does not have incentives to deviate, separately showing that he does not want to overreport or underreport. For our analysis, it suffices to show that the k-th type sender sends neither m_{k-1} or m_{k+1} (in the intervals immediately below and above interval k).[3]
  - *No incentives to overreport.* First, we check that the k-th sender has no incentive to overreport by sending message m_{k+1}, which occurs if

$$\underbrace{-\left(\frac{\theta_{k-1}+\theta_k}{2}-(\theta_k+\delta)\right)^2}_{\text{Utility from sending message } m_k} \geq \underbrace{-\left(\frac{\theta_{k+1}+\theta_k}{2}-(\theta_k+\delta)\right)^2}_{\text{Utility from sending message } m_{k+1}}$$

    Rearranging the above expression,

$$\theta_{k-1}+\theta_k-2(\theta_k+\delta)\geq-\theta_{k+1}-\theta_k+2(\theta_k+\delta)$$

    Further simplifying,

$$\theta_{k+1}\geq 2\theta_k-\theta_{k-1}+4\delta$$

    As an illustration, let us check the initial condition, for which the sender of type θ_1 has no incentives to overreport by sending message θ_2.

$$-\left(\frac{\theta_0+\theta_1}{2}-(\theta_1+\delta)\right)^2\geq-\left(\frac{\theta_2+\theta_1}{2}-(\theta_1+\delta)\right)^2$$

    After rearranging, we find

$$\theta_0+\theta_1-2(\theta_1+\delta)\geq-\theta_2-\theta_1+2(\theta_1+\delta)$$

    Further simplifying,

$$\theta_2\geq 2\theta_1-\theta_0+4\delta.$$

[3] Formally, if sending message m_{k-1} and m_{k+1} is dominated by message m_k, then sending messages m_{k-j} or m_{k+j}, where $j\geq 2$ thus indicating intervals further away from interval k, would also be dominated.

– *No incentives to underreport.* Second, we check that the $(k+1)$-th sender has no incentive to under-report by sending m_k, which entails

$$-\left(\frac{\theta_{k+1}+\theta_k}{2}-(\theta_{k+1}+\delta)\right)^2 \geq -\left(\frac{\theta_k+\theta_{k+1}}{2}-(\theta_{k+1}+\delta)\right)^2$$

Rearranging the above expression, we obtain

$$\theta_{k+1}+\delta-\frac{\theta_{k+1}+\theta_k}{2} \geq \frac{\theta_k+\theta_{k-1}}{2}-(\theta_{k+1}+\delta)$$

Simplifying,

$$\theta_{k+1} \geq \frac{1}{3}\left(2\theta_k+\theta_{k-1}-4\delta\right)$$

Since $\theta_{k+1} > \theta_k$ by construction, the k-th sender has no incentives to underreport.[4] Intuitively, the lobbyist finds it unprofitable to report a lower type to the politician, for all values of the bias parameter δ. Therefore, in general, the condition for the k-th sender to send the appropriate message is

$$\theta_{k+1} \geq 2\theta_k - \theta_{k-1} + 4\delta.$$

Remark: At this point of our analysis, we have found under which conditions the lobbyist does not have incentives to under- or overreport, meaning that his messages about the interval where θ lies are truthful, and the separating strategy profile described in Step 1 can be supported as a PBE. There are, nonetheless, some details about this strategy profile that we have not characterized yet, in particular: (i) the number of partitions that can be sustained in equilibrium, N; (ii) how is this number of partitions affected by the preference divergence parameter, δ; and (iii) the length of each of these partitions (intervals), as they are not necessarily equally long. We analyze each of them in the next subsections.

13.4.2 Equilibrium number of partitions

To answer these questions, recall that, since $\theta \sim U[0,1]$, the first interval starts at $\theta_0 = 0$, and the last interval finishes at $\theta_N = 1$ (see figure 13.8). For easier reference, we denote the length of the first interval as $d \equiv \theta_1 - \theta_0$, where $d \geq 0$. We can now rearrange the incentive compatibility condition describing the lobbyist's no incentives to overreport, $\theta_{k+1} \geq 2\theta_k - \theta_{k-1} + 4\delta$, as follows

$$\theta_{k+1} - \theta_k \geq (\theta_k - \theta_{k-1}) + 4\delta$$

Intuitively, this inequality says that each interval must be at least 4δ longer than its predecessor. If this condition binds, we obtain that the second interval length is

$$\theta_2 - \theta_1 = \underbrace{(\theta_1 - \theta_0)}_{d} + 4\delta = d + 4\delta$$

[4] Check that condition $\theta_{k+1} > \theta_k > \frac{2}{3}\theta_k + \frac{1}{3}\theta_{k-1} = \frac{1}{3}(2\theta_k + \theta_{k-1}) > \frac{1}{3}(2\theta_k + \theta_{k-1} - 4\delta)$ holds for all values of the bias parameter δ, which means that the incentive compatibility constraint for no underreporting becomes slack.

while the length of the third interval is

$$\theta_3 - \theta_2 = \underbrace{(\theta_2 - \theta_1)}_{d+4\delta} + 4\delta = d + (2 \times 4\delta)$$

and similarly for subsequent intervals. By recursion, the length of the k-th interval is, then,

$$\begin{aligned}
\theta_k - \theta_{k-1} &= (\theta_{k-1} - \theta_{k-2}) + 4\delta \\
&= (\theta_{k-2} - \theta_{k-3}) + (2 \times 4\delta) \\
&= (\theta_{k-3} - \theta_{k-4}) + (3 \times 4\delta) \\
&= \ldots \\
&= (\theta_1 - \theta_0) + [(k-1) \times 4\delta] \\
&= d + 4\,(k-1)\,\delta
\end{aligned}$$

As an illustration, the length of the final interval, where $k = N$, is

$$\theta_N - \theta_{N-1} = d + 4\,(N-1)\,\delta$$

We can now express the length of the unit interval, $\theta_N - \theta_0 = 1$, as the sum of N partitions, as follows

$$\begin{aligned}
\theta_N - \theta_0 &= \underbrace{(\theta_N - \theta_{N-1})}_{d+4(N-1)\delta} + \underbrace{(\theta_{N-1} - \theta_{N-2})}_{d+4(N-2)\delta} + \ldots + \underbrace{(\theta_2 - \theta_1)}_{d+4\delta} + \underbrace{(\theta_1 - \theta_0)}_{d} \\
&= d + 4\delta\,[(N-1) + (N-2) + ... + 1]
\end{aligned}$$

where the term in square brackets can be simplfied as follows

$$\begin{aligned}
\underbrace{[(N-1) + (N-2) + ... + 1]}_{N-1 \text{ terms}} &= N(N-1) - [1 + 2 + ... + (N-1)] \\
&= N(N-1) - \frac{N(N-1)}{2} \\
&= \frac{N(N-1)}{2}
\end{aligned}$$

and recall that $1 + 2 + ... + (N-1) = \frac{(N-1)N}{2}$. Therefore, the above expression of $\theta_N - \theta_0$ further simplifies to

$$\begin{aligned}
\theta_N - \theta_0 &= d + 4\delta \frac{N(N-1)}{2} \\
&= Nd + 2\delta N(N-1)
\end{aligned}$$

And since the left-hand side is $\theta_N - \theta_0 = 1$ (θ lies in the unit interval), we can write the above equation as $1 = Nd + 2\delta N\,(N-1)$, and solve for the bias parameter, δ, to obtain

$$\overline{\delta}\,(d) = \frac{1 - Nd}{2N\,(N-1)}.$$

Cutoff $\overline{\delta}\,(d)$ decreases in the length of the first interval, d. Intuitively, as the first interval becomes wider (higher d), a given number of partitions N becomes more difficult to be sustained as a PBE.[5] This result suggest that when the length of the first interval is nil, $d = 0$, we obtain that the maximal value of cutoff $\overline{\delta}(d)$ becomes

$$\overline{\delta}(0) = \frac{1}{2N(N-1)}.$$

Therefore, more partitions (higher N) can only be supported as a PBE if the bias parameter δ becomes smaller. That is, more informative PBEs can be sustained when the preferences of lobbyist and politician are more similar (lower δ). Solving for N, we can also find the maximum number of partitions, $N(\delta)$, as a function of δ, as follows

$$N(N-1) \leq \frac{1}{2\delta}$$

which can be rearranged as

$$N^2 - N - \frac{1}{2\delta} \leq 0$$

Factorizing the above inequality,

$$\left(N - \frac{1+\sqrt{1+\frac{2}{\delta}}}{2}\right)\left(N - \frac{1-\sqrt{1+\frac{2}{\delta}}}{2}\right) \leq 0$$

Since $N \geq 1$ (that is, there must be at least one partition), we can rule out the negative root such that

$$N \leq \frac{1+\sqrt{1+\frac{2}{\delta}}}{2}$$

Furthermore, since N is a positive integer, we have that

$$N \leq \overline{N}(\delta) \equiv \left\lfloor \frac{1+\sqrt{1+\frac{2}{\delta}}}{2} \right\rfloor$$

where the $\lfloor \cdot \rfloor$ sign rounds to the next integer from below, e.g., $\lfloor 3.7 \rfloor = 3$. Cutoff $\overline{N}(\delta)$ represents the maximum number of partitions that can be supported for a given value of δ. As the bias parameter, δ, increases, fraction $\frac{2}{\delta}$ becomes smaller, ultimately decreasing cutoff $\overline{N}(\delta)$. Intuitively, as the lobbyist and the politician become more divergent in their preferences, the lobbyist has more incentives to overreport his type, so his messages becomes less informative.

Figure 13.9 depicts cutoff $\overline{N}(\delta)$ and illustrates that the PBE yields a smaller number of partitions as the bias parameter δ increases, thus supporting less information transmission from the (privately informed) lobbyist to the (uninformed) politician. Alternatively, solving for δ in cutoff $\overline{N}(\delta)$, we find that

$$\delta \leq \frac{1}{2N(N-1)},$$

[5] Note that this finding cannot be interpreted as a standard comparative statics result since we have not found yet the length d that arises in equilibrium. We do that below.

which also indicates that, as we seek a larger number of partitions (higher N) in equilibrium, the preference divergence parameter must be lower.

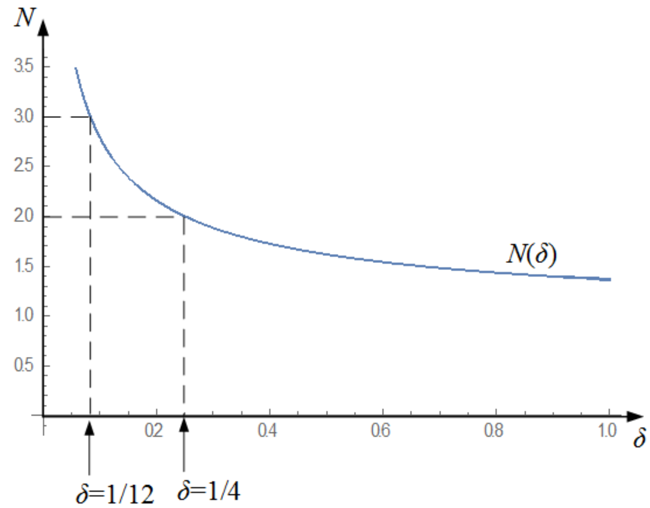

Figure 13.9. Cutoff $\overline{N}(\delta)$ as a function of the bias parameter δ.

Example 13.1. Equilibrium number of partitions. If we seek to support $N = 2$ partitions in equilibrium, we need $\delta \leq \frac{1}{2\times2(2-1)} = \frac{1}{4}$. To sustain $N = 3$ partitions, however, we require $\delta \leq \frac{1}{23(3-1)} = \frac{1}{12}$, which imposes a more restrictive condition on players' preference alignment. $\square$

13.4.3 Interval lengths in equilibrium

Let us now find the equilibrium length of the first interval, d^*, which will provide us with the length of all subsequent intervals. Consider again the lobbyist incentive compatibility condition for not overreporting,

$$\theta_{k+1} - 2\theta_k + \theta_{k-1} = 4\delta$$

This expression can be understood as a second-order linear difference equation.[6] Since the unit interval starts at $\theta_0 = 0$, we can write the above equation for $k = 2$ as follows

$$\theta_2 = 2\theta_1 - \theta_0 + 4\delta = 2\theta_1 + 4\delta$$

and, simlarly, for $k = 3$ this equation becomes

$$\theta_3 = 2\theta_2 - \theta_1 + 4\delta = 2\underbrace{(2\theta_1 + 4\delta)}_{\theta_2} - \theta_1 + 4\delta = 3\theta_1 + 12\delta$$

which helps us express it more generally, for any value of k, as follows

$$\theta_k = k\theta_1 + 2k\,(k-1)\,\delta.$$

[6] Recall that linear second-order difference equation generally take the form $x_{t+2} + ax_{t+1} + bx_t = c_t$, where a and b are constants and c_t is a number for all values of t. In our setting, $x_{t+2} = \theta_{k+1}$, $a = -2$, $x_{t+1} = \theta_k$, $b = 1$, $x_t = \theta_{k-1}$, and $c_t = 4\delta$ for all t. For an introduction to linear difference equations, see for instance, Simon and Blume (1994, chapter 23).

Therefore, evaluating this expression at $k = N$, we obtain $\theta_N = Nd + 2N(N-1)\delta$, which helps us write $\theta_N - \theta_0 = Nd + 2N(N-1)\delta$ since $\theta_0 = 0$. In addition, because $\theta_N - \theta_0 = 1$, we can express $1 = Nd + 2N(N-1)\delta$ and, solving for d, we find that the length of the first interval is

$$d^* = \frac{1}{N} - 2\,(N-1)\,\delta$$

which also satisfies $d^* = \theta_1 - \theta_0 = \theta_1$ given that $\theta_0 = 0$. Interestingly, this length decreases in the number of partitions in that equilibrium, N, since $\frac{\partial d^*}{\partial N} = -2d - \frac{1}{N^2} \leq 0$, which means that the first interval shrinks to "make room" for subsequent partitions to its right side.

Example 13.2. First interval decreasing in N**.** If $\delta = \frac{1}{20}$ and $N = 2$, which is compatible with condition $N \leq \overline{N}(\delta)$ found in the previous section, we obtain that the length of the first interval is

$$d^* = \frac{1}{2} - 2\,(2-1)\,\frac{1}{20} = \frac{2}{5}.$$

However, when N increases to $N = 3$, this length shrinks to $d^* = \frac{1}{2} - 3\,(3-1)\,\frac{1}{20} = \frac{2}{15}$. □

We can now use the above results to find the length of the k^{th} interval, $\theta_k - \theta_{k-1}$. First, substituting $\theta_1 = \frac{1}{N} - 2\,(N-1)\,\delta$ into expression $\theta_k = k\theta_1 + 2k\,(k-1)\,\delta$, we obtain

$$\begin{aligned}\theta_k &= k\underbrace{\left(\frac{1}{N} - 2\,(N-1)\,\delta\right)}_{\theta_1} + 2k\,(k-1)\,\delta \\ &= \frac{k}{N} - 2k\,(N-k)\,\delta\end{aligned}$$

which implies that the length of the k^{th} interval is

$$\begin{aligned}\theta_k - \theta_{k-1} = &\underbrace{\left(\frac{k}{N} - 2k\,(N-k)\,\delta\right)}_{\theta_k} - \underbrace{\left(\frac{k-1}{N} - 2\,(k-1)\,(N-k+1)\,\delta\right)}_{\theta_{k-1}} \\ = &\frac{1}{N} - 2\,(N+1-2k)\,\delta.\end{aligned}$$

As a remark, we can confirm that the length of the first interval coincides with the expression found above, d^*. Indeed, evaluating $\theta_k - \theta_{k-1}$ at the first interval (i.e., $k = 1$), this expression simplifies to

$$\begin{aligned}\theta_1 - \theta_0 &= \frac{1}{N} - 2\,(N+1-2)\,\delta \\ &= \frac{1}{N} - 2\,(N-1)\,\delta = d^*\end{aligned}$$

which coincides with the result of d^* found above.

Example 13.3. Length of each interval in equilibrium. Following with example 13.2, in a context with $\delta = \frac{1}{20}$ and $N = 2$, the length of the first interval is $d^* = \frac{2}{5}$. The length of the second interval is

$$\theta_2 - \theta_1 = \frac{1}{2} - 2\,[2+1-(2\times 2)]\,\frac{1}{20} = \frac{3}{5}$$

which, together, account for the total length of the unit interval, i.e., $\frac{2}{5} + \frac{3}{5} = 1$. Similarly, in a setting with $N = 3$ partitions, it is easy to show that the first interval's length is $d^* = \frac{2}{15}$, as shown in example 13.2, that of the second interval is

$$\theta_2 - \theta_1 = \frac{1}{2} - 3\left[3 + 1 - (2 \times 2)\right]\frac{1}{20} = \frac{1}{3},$$

and that of the third interval is

$$\theta_3 - \theta_2 = \frac{1}{2} - 3\left[3 + 1 - (2 \times 3)\right]\frac{1}{20} = \frac{8}{15},$$

with their sum satisfying $\frac{2}{15} + \frac{1}{3} + \frac{8}{15} = 1$, spanning all unit interval. □