Exercise #3.12: First-Price Auction with Discrete Valuations^B

3.12 Consider a first-price auction with two bidders, each privately observing his valuation for the object, v_i, which is either high (v_H) or low (v_L), where $1 > v_H > v_L > 0$. The probability of bidder i drawing a high valuation, v_H, is $p \in (0, 1)$. If two bidders submit the same bid, assume that the seller randomly assigns the object between the two bidders.

(a) Show that in any bidding strategy (involving pure or mixed strategies), the low-value bidder submits a bid equal to his true value, $b(v_L) = v_L$.

- When submitting a bid $b(v_L) = v_L$, this bidder either loses the auction when facing a high-value bidder (earning a zero payoff) or wins the object with probability 1/2 when facing another low-value bidder, which also yields a zero payoff since

$$\frac{1}{2}(v_L - b(v_L)) = \frac{1}{2}(v_L - v_L) = 0.$$

- Deviations:
  - If this bidder submits a lower bid, $b(v_L) < v_L$, he loses the auction regardless of the rival he faces, thus earning a zero payoff with certainty. Therefore, the low-value bidder does not have strict incentives to shade his valuation.
  - If this bidder submits a higher bid, $b(v_L) > v_L$, he either loses the auction still earning a zero payoff or wins the auction if his bid satisfies $b(v_L) > b(v_H)$, but earning a negative payoff since $v_L - b(v_L) < 0$. Hence, the low-value bidder does not have incentives to submit a bid above his valuation.
- Overall, this bidder does not have incentives to deviate from submitting a bid $b(v_L) = v_L$. Interestingly, our analysis was unaffected by the bidding function that the high-value bidder uses, $b(v_H)$; that is, whether he submits a bid above/below/equal to his valuation or, as we show below, randomizes his bids.

(b) *Discarding dominated strategies.* Show that the high-value bidder's bid, $b(v_H)$, must satisfy $v_L \leq b(v_H) \leq v_H$.

- Bidding strictly above his valuation is dominated by bidding his valuation, so the high-value bidder has no incentives to submit $b(v_H) > v_H$. A similar argument applies to bids below v_L: if the high-value bidder were to submit a bid $b(v_H) < v_L$, he would lose the auction for sure (regardless of the rival he faces), implying that we can find other bidding functions, such as any bid satisfying $b(v_H) > v_L$, which yields a strictly positive expected payoff. Therefore, the high-value bidder's bid must lie between v_L and v_H, as required.

(c) *No pure-strategy bidding profile.* Show that there is no pure-strategy bidding strategy. [*Hint*: Consider all pure-strategy bidding profiles, and show that at least one bidder has a profitable deviation.]

- To understand the high-value bidder's incentives, note that, under complete information, he would submit a bid $b(v_H) = v_L + \varepsilon$, where $\varepsilon \to 0$, when facing a low-value bidder (winning the auction and retaining the highest surplus). However, if the high-value bidder observes that his rival is another high-value bidder, he would submit a bid $b(v_H) = v_H - \varepsilon$, where $\varepsilon \to 0$, to maximize his chances of winning the auction.
- More generally, we next list the different bidding strategy profiles that can arise according to where the bids lie. For each case, we seek to show that we can always find a profitable deviation, so no bidding strategy profile can constitute a pure-strategy equilibrium of the auction:
  - If $b(v_i) < b(v_j) < v_H$, then bidder i loses the auction, but he can increase his bid above that of bidder j, winning the auction as a result, and making a positive margin if his new bid lies below v_H. Then, a profitable deviation exists.
  - If $v_L \leq b(v_j) < b(v_i) < v_H$, then bidder i wins the auction, but he can further decrease his bid and earn a higher surplus. Then, a profitable deviation exists in this case too.
  - If $b(v_j) \leq b(v_i) = v_H$, and by the same argument as above, bidder i wins the auction, but he would have incentives to lower his bid, still winning the auction and retaining a larger surplus.
- We can then conclude that there is no pure-strategy bidding profile where players have no incentives to deviate, so there is no pure-strategy bidding profile that can be sustained as a Bayesian Nash equilibrium of the auction.

(d) *Mixed-strategy bidding profile.* Show that the following mixed-strategy bidding equilibrium can be sustained, where the low-value bidder submits a bid equal to his value, $b(v_L) = v_L$, while the high-value bidder randomizes with cumulative distribution function

$$F(b) = \frac{1-p}{p}\left(\frac{v_H - v_L}{v_H - b} - 1\right)$$

in the interval $[v_L, E[v]]$, where $E[v] \equiv pv_H + (1-p)v_L$ denotes the expected valuation. For simplicity, assume that, if a tie occurs, the seller assigns the object to the individual with the highest valuation.

- From part (a), we know that the low-value bidder submits a bid equal to his value, $b(v_L) = v_L$, regardless of the strategy that his rival chooses, so we can focus on the high-value bidder.
- *Lower bound.* The lower bound of the randomizing interval is v_L. Otherwise, the bidder could be submitting an unnecessarily high bid. In other words, he could decrease the lower bound, still win the auction when facing a low-value bidder, and extract a larger surplus.
  - As a remark, note that if the high-value bidder submits a bid $b(v_H) = v_L$ (at the lower bound), there is a tie when facing a low-value bidder, but the object is assigned to him because his valuation is higher (one can assume that, after observing bids, the seller can also observe valuations, thus assigning the object to the bidder with the highest valuation in case of a tie).
- *Upper bound.* When the high-value bidder submits a bid $b(v_H) = v_L$, this bidder wins the auction when facing a low-value bidder (which happens with probability $1 - p$) and does not win the auction when facing another high-value bidder (as his rival bidding at exactly $b(v_H) = v_L$ has a probability that converges to zero). This yields an expected payoff of

$$(1 - p)(v_H - b(v_H))$$

which simplifies to

$$(1 - p)(v_H - v_L).$$

When the high-value bidder submits any other bid $b(v_H) = b > v_L$, he wins the auction when facing a low-value bidder (which occurs with probability $1 - p$) but only wins when facing another high-value bidder if his bid is higher than that of his rival (which occurs with probability $pF(b)$, yielding an expected payoff

$$\underbrace{(1 - p)(v_H - b)}_{\text{if facing a low-value bidder}} + \underbrace{pF(b)(v_H - b)}_{\text{if facing another high-value bidder}}$$
$$= (v_H - b)[1 - p(1 - F(b))].$$

If the high-value bidder is randomizing between v_L and points above v_L, it must be that he is indifferent between the above expected payoffs. At the upper bound, $\overline{b}$, we have that $F(\overline{b}) = 1$, implying that the above expected payoff simplifies to $(v_H - b)$, so the indifference in expected payoff entails

$$(1 - p)(v_H - v_L) = v_H - \overline{b}$$

Rearranging, and solving for $\overline{b}$, yields the upper bound

$$\overline{b} = pv_H + (1-p)v_L \equiv E[v]$$

which can be interpreted as the expected valuation and, therefore, lies above v_L but below v_H, as required from part (b).

- *Cumulative distribution function.* From the above indifference condition, evaluated at any bid in the interval $[v_L, E[v]]$, we obtain that

$$(1-p)(v_H - v_L) = (v_H - b)[1 - p(1 - F(b))].$$

Solving for the cumulative distribution function $F(b)$ yields

$$F(b) = \frac{1-p}{p}\left(\frac{v_H - v_L}{v_H - b} - 1\right)$$

with an associated density of

$$f(b) = F'(b) = \frac{1-p}{p}\frac{v_H - v_L}{(v_H - b)^2}.$$

The cumulative distribution function $F(b)$ is well behaved since (1) it originates at zero at its lower bound, v_L, that is,

$$\begin{aligned} F(v_L) &= \frac{1-p}{p}\left(\frac{v_H - v_L}{v_H - v_L} - 1\right) \\ &= \frac{1-p}{p}(1-1) \\ &= 0, \end{aligned}$$

(2) it increases in b since $f(b) > 0$, and (3) it is equal to one at its upper bound, $E[v]$, that is,

$$\begin{aligned} F(E[v]) &= \frac{1-p}{p}\left(\frac{v_H - v_L}{v_H - pv_H - (1-p)\,v_L} - 1\right) \\ &= \frac{1-p}{p}\frac{p}{1-p} \\ &= 1. \end{aligned}$$

- *No profitable deviations.* From our discussion, the high type is indifferent over all bids in interval $[v_L, E[v]]$. We can now check that he does not have incentives to deviate from this randomization:

- If he submits a bid $b(v_i) < v_L$, he loses the auction for sure (regardless of the rival he faces), earning a zero payoff. In contrast, with the above randomization, he earns an expected payoff of $(1-p)(v_H - v_L)$, which is positive since $v_H > v_L$ by assumption.
- If he submits a bid $b(v_i)$ that satisfies $v_H > b(v_i) > E[v]$, he wins the auction but pays more for the object than with the above randomization.

Therefore, the high-value bidder has no incentives to bid below or above the interval $[v_L, E[v]]$. As shown in part (a), the low-value bidder does not have incentives to deviate from submitting a bid equal to his valuation, implying that this strategy profile is a symmetric mixed-strategy equilibrium of the first-price auction with discrete valuations.

(e) How are your equilibrium results affected by a marginal increase in probability p? And by a marginal increase in valuations v_H or v_L? Interpret your results.

- *Higher* p. When the probability of a high valuation, p, increases, the expected value $E[v]$ increases, expanding the support where the high-value bidder randomizes his bid, $[v_L, E[v]]$. In addition, an increase in p shifts $F(b)$ downward since

$$\frac{\partial F(b)}{\partial p} = -\frac{v_H - v_L}{(v_H - b)p^2} < 0,$$

indicating that the high-value bidder assigns less probability weight on low bids when p increases. Technically, for two probabilities p and p', where $p' > p$, $F(b, p')$ first order stochastically dominates $F(b, p)$. Intuitively, this suggests that the high-value bidder becomes more aggressive in his bids as the probability of facing another high-value bidder increases.
- *Higher* v_H. When the high value increases, the expected value $E[v]$ increases, also expanding the support of the randomization and shifting $F(b)$ downward given that

$$\frac{\partial F(b)}{\partial v_H} = -\frac{1-p}{p}\frac{b - v_L}{(v_H - b)^2} < 0.$$

In other words, the bidder randomizes over a larger support and becomes more aggressive in his bids as his valuation (and, thus, the surplus he can retain) increases.
- *Higher* v_L. When the low value increases, the expected value $E[v]$ decreases, shrinking the support of the bid randomization. In addition, a higher v_L produces a downward shift in $F(b)$ since

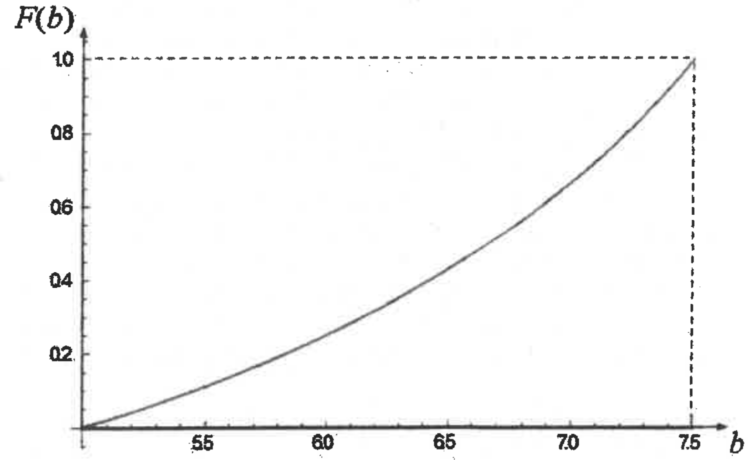

Fig. 3.8 $F(b)$ as a function of b

$$\frac{\partial F(b)}{\partial v_L} = -\frac{1-p}{p}\frac{1}{v_H - b} < 0.$$

Therefore, the high-value bidder bids over a narrower support but becomes more aggressive in his bids as the valuation of the low-value bidder increases.

(f) *Numerical example.* Evaluate your results from part (d) at parameter values $p = 1/2$, $v_H = 10$, and $v_L = 5$.

- The low-value bidder submits a bid equal to his valuation, $b(v_L) = 5$, and the high-value bidder submits a bid in the interval $[5, 7.5]$ since $E[v] = \frac{1}{2}10 + \frac{1}{2}5 = 7.5$ and uses the following cumulative distribution function

$$F(b) = \frac{1-\frac{1}{2}}{\frac{1}{2}}\left(\frac{10-5}{10-b} - 1\right)$$
$$= \frac{b-5}{10-b}.$$

Figure 3.8 depicts $F(b)$ where the horizontal axis considers that bids lie in the interval $[5, 7.5]$. As expected, the cumulative distribution function puts no probability weight at its lower bound, $F(5) = 0$, increases in b, and puts full probability weight at its upper bound, $F(7.5) = 1$.

$$-e^{-\frac{v_{CE}}{2}} = \frac{e^{-\frac{1}{2}} - 1}{\frac{1}{2}}.$$

Rearranging and simplifying, we find

$$v_{CE} = -2\log\left[-2\left(e^{-\frac{1}{2}} - 1\right)\right],$$

so that

$$p_{NR} = v_{CE} = 0.479,$$

which falls below the price under refund policy, p_R, for all values of $c \in [0, 1]$, since risk-averse consumers discount the price of the product, p_{NR}, to its certainty equivalent value, v_{CE}, when refund cannot be made.

Defining $x \equiv \log\left[-2\left(e^{-\frac{1}{2}} - 1\right)\right]$, the seller's expected profits become

$$\mathbb{E}[\pi_{NR}] = (v_{CE} - c) \times [1 - F(v_{CE})] = -(2x + c)(1 + 2x),$$

implying that the seller earns a higher profit offering return policy if and only if

$$\begin{aligned} \frac{(1-c)^2}{4} &> -(2x + c)(1 + 2x) \\ c^2 + 2c + 1 &> -4[4x^2 + 2x(1 + c)] \\ (1 + c)^2 + 8(1 + c)x + 16x^2 &> 0 \\ (1 + c + 4x)^2 &> 0, \end{aligned}$$

which is always positive. Thus, the seller is more profitable offering a return policy for its product. Intuitively, by allowing consumers to return the product after they have experienced it, the seller is able to charge a higher price to those consumers whose valuation is above the price of the product.

- The seller may, however, not find it profitable to offer a return policy in other contexts (e.g., when valuations are not uniformly distributed or when buyers have different risk-aversion preferences). For a more general presentation, see Proposition 1 in Che (1998).

Exercise #9.8: A Model of Sales, Based on Varian (1980)^C

9.8 Firms offer sales at different times. In this exercise, we show that offering sales (or, more generally, randomizing over prices) is a strategy that helps firms maximize their expected profits. This exercise belongs to the literature on "price dispersion" where firms face a share of consumers who are uninformed about prices, and offer different prices, either at different locations (spatial price dispersion) or at different points in time (temporal price dispersion, as we analyze in this exercise). Price discrimination models, in contrast, assume that consumers can perfectly observe prices.

Consider an industry with N firms and free entry, so firms enter until the profits from doing so are zero. Consumers have a reservation price r for a homogeneous good and purchase at most one unit. A share α^I of consumers is informed about prices, buying from the cheapest firm, and a share $1-\alpha^I$ are uninformed, who purchase from any firm. Therefore, there are $\alpha^U = \frac{1-\alpha^I}{N}$ uninformed consumers per firm. Firms face a symmetric cost function $C(q) = F + cq$, where $F > 0$ denotes fixed costs and c represents its marginal cost. Every firm can only charge one price for its product.

As a reference, note that $C(\alpha^I + \alpha^U) = F + c(\alpha^I + \alpha^U)$ denotes the cost from serving the maximum amount of customers (both informed and uninformed consumers). Therefore, the ratio

$$p_L \equiv \frac{F + c(\alpha^I + \alpha^U)}{\alpha^I + \alpha^U}$$

represents the average cost in this setting.

We next show that, in the above context, every firm has incentives to randomize its pricing over a certain interval. The following questions should help you find the specific cumulative distribution function $F(p)$ that every firm uses in the mixed-strategy Nash equilibrium of the game.

(a) Show that $F(p) = 0$ for all $p < p_L$, and that $F(p) = 1$ for all $p > r$.

- This question essentially asks us to "trim" the support of price randomization in $F(p)$ and characterize its lower and upper bounds.
- *Lower bound.* When charging prices below p_L, a firm must be making losses, since its price lies below its cost in the most favorable scenario (when all types of consumers purchase the good). Therefore, the firm does not assign a probability weight on prices below p_L.
- *Upper bound.* If a firm charges a price above the reservation price r, no customer buys from it, regardless of whether he is informed or uninformed. The firm then has no incentives to assign a probability weight on prices above r. Combining our above results, the price p in $F(p)$ must lie in the interval $[p_L, r]$.

(b) Show that the cumulative distribution function $F(p)$ is nondegenerated, that is, there is no pure strategy Nash equilibrium.

- If firm i uses a pure strategy, charging price $p_i = p_L$, it makes a loss, thus having incentives to exit the industry. (Recall that, in equilibrium, firms make zero profits.) If, instead, the firm sets a higher price p_i that satisfies $r \geq p_i > p_L$, other firms would have incentives to undercut firm i's price by a small ε. Therefore, firm i does not use a pure strategy.

(c) For simplicity, assume that $F(p)$ is continuous.[1] Find expected profits from the pricing strategy $F(p)$.

- If a firm sets the lowest price, it attracts all consumers, and its profit is

$$\pi_s(p) = p(\alpha^I + \alpha^U) - F - c(\alpha^I + \alpha^U),$$

where the subscript s denotes that the firm is successful at attracting all consumers.
If, instead, the firm is unsuccessful, it only sells its product to uninformed consumers, earning

$$\pi_f(p) = p\alpha^U - F - c\alpha^U,$$

[1] That is, there is no "mass point" in the pricing strategy $F(p)$ that every firm uses. Intuitively, the firm chooses all prices in the $[p_L, r]$ interval with positive probability. More compactly, this means that the density function $f(p) > 0$ for all $p \in [p_L, r]$.

where the subscript f denotes "failure."
The probability that firm i sets a price p higher than its rival $j \neq i$ is

$$F(p) = \text{Prob}\left\{p \geq p_j\right\},$$

so the probability that $p < p_j$ is the converse, $1 - F(p)$. As a result, the probability that p is lower than the prices of all its $N - 1$ rivals is

$$[1 - F(p)]^{N-1},$$

which represents the probability that firm i sells to informed consumers. Finally, the probability that firm i does not sell to informed consumers is

$$1 - [1 - F(p)]^{N-1}.$$

- We are now ready to write firm i's expected profit

$$\int_{p_L}^{r} \left[\underbrace{\pi_s(p)\,[1 - F(p)]^{N-1}}_{\text{Success}} + \underbrace{\pi_f(p)\left[1 - [1 - F(p)]^{N-1}\right]}_{\text{Failure}} \right] f(p)dp.$$

(d) Using the no entry condition, find the cumulative distribution function $F(p)$ with which every firm randomizes.

- Since firms make no profits in equilibrium (otherwise entry or exit would still be profitable), the above expected profit must be equal to zero, which entails

$$\pi_s(p)\,[1 - F(p)]^{N-1} + \pi_f(p)\left[1 - [1 - F(p)]^{N-1}\right] = 0.$$

Rearranging,

$$F(p) = 1 - \left(\frac{\pi_f(p)}{\pi_f(p) - \pi_s(p)}\right)^{\frac{1}{N-1}}.$$

The denominator is negative since $\pi_f(p) < \pi_s(p)$ for any price $p \in [p_L, r]$. Therefore, the numerator must also be negative, $\pi_f(p) < 0$.

(e) Show that the cumulative distribution function $F(p)$ has full support in $p \in [p_L, r]$. That is, $F(p_L + \varepsilon) > 0$ and $F(r - \varepsilon) < 1$ for any $\varepsilon > 0$.

- *Prices slightly above p_L*. If, instead, $F(p_L + \varepsilon) = 0$, firm i is assigning no probability weight to prices slightly higher than the lower bound p_L. Therefore, firm i assigns probability weight to prices strictly above $p_L + \varepsilon$. In that case, another firm j could undercut firm i's price and set for instance a price $p_L + \frac{\varepsilon}{2}$ to make positive profits. Hence, $F(p_L + \varepsilon) > 0$ for any $\varepsilon > 0$.
- *Prices slightly below r.* If, instead, $F(r - \varepsilon) = 1$, firm i assigns no probability to prices slightly below r. At $\widetilde{p} < r$, only uninformed consumers purchase the good and the firm earns $\widetilde{p}\alpha^U - F - c\alpha^U$, yielding zero profits. However, a deviation to price $p = r$ yields

$r\alpha^U - F - c\alpha^U$ which is positive, thus making such deviation profitable. Therefore, $F(r - \varepsilon) < 1$ for any $\varepsilon > 0$.

(f) Taking into account that $\pi_f(r) = 0$, find the equilibrium number of firms in the industry, n^*.

- Condition $\pi_f(r) = 0$ entails

$$r\alpha^U - F - c\alpha^U = 0.$$

Substituting $\alpha^U = \frac{1-\alpha^I}{N}$ into the above expression yields

$$F = (r-c)\underbrace{\frac{1-\alpha^I}{N}}_{\alpha^U}.$$

Solving for N, we obtain

$$N^* = \frac{(r-c)\left(1-\alpha^I\right)}{F}.$$

Therefore, the higher the profit margin $r - c$, the larger share of the uninformed consumers $1 - \alpha^I$, and the lower the entry cost F, the more firms in equilibrium.

(g) Taking into account that $\pi_s(p_L) = 0$, and the equilibrium number of firms N^*, find the lower bound of firms' randomization strategy, p_L.

- Condition $\pi_s(p_L) = 0$ entails

$$p_L(\alpha^I + \alpha^U) - F - c(\alpha^I + \alpha^U) = 0.$$

Substituting $\alpha^U = \frac{1-\alpha^I}{N}$ into the above expression yields

$$F = (p_L - c)\left(\alpha^I + \frac{1-\alpha^I}{N}\right).$$

Further inserting the equilibrium number of firms, N^*, found in part (f), we have

$$F = (p_L - c)\left(\alpha^I + \frac{1-\alpha^I}{\frac{(r-c)(1-\alpha^I)}{F}}\right).$$

Rearranging, we obtain

$$F = (p_L - c)\left(\frac{(r-c)\alpha^I + F}{r-c}\right).$$

Solving for p_L, we find the lower bound of firms' randomization strategy

$$p_L = \frac{c\,(r-c)\,\alpha^I + rF}{(r-c)\,\alpha^I + F}.$$

(h) Evaluate your above results in the special case in which all consumers are uninformed.

- When all consumers are uninformed, $\alpha^I = 0$, the lower bound of firms' randomization strategy, p_L, becomes

$$p_L = \frac{rF}{F} = r,$$

which coincides with the upper bound of firms' randomization strategy. In other words, firms put full probability weight on one price, $p = r$, with every firm extracting all surplus from a share $\frac{1}{N}$ of consumers.

(i) *Numerical example.* Evaluate your results in parts (d), (f), and (g) at parameter values $r = 1$, $F = \frac{2}{9}$, $c = 0$, and $\alpha^I = \frac{1}{3}$.

- In this setting, the equilibrium number of firms becomes

$$N^* = \frac{(1-0)\left(1-\frac{1}{3}\right)}{\frac{2}{9}} = 3.$$

In addition, the lower bound is

$$p_L = \frac{0\,(1-0)\,\frac{1}{3} + \frac{2}{9}\times 1}{(1-0)\,\frac{1}{3} + \frac{2}{9}} = \frac{2}{5}.$$

In this context, the share of uninformed consumers for every firm becomes

$$\alpha^U = \frac{1-\alpha^I}{N*} = \frac{1-\frac{1}{3}}{3} = \frac{2}{9}.$$

- Finally, the cumulative distribution function is

$$F(p) = 1 - \left(\frac{\pi_f(p)}{\pi_f(p) - \pi_s(p)}\right)^{\frac{1}{N-1}},$$

where profits from successfully attracting all customers are

$$\pi_s(p) = (p - c)\left(\alpha^I + \alpha^U\right) - F$$
$$= (p - 0)\left(\frac{1}{3} + \frac{2}{9}\right) - \frac{2}{9}$$
$$= \frac{5p - 2}{9},$$

and profits from only attracting uninformed consumers are

$$\pi_f(p) = (p - c)\,\alpha^U - F$$
$$= (p - 0) \times \frac{2}{9} - \frac{2}{9}$$
$$= -\frac{2(1 - p)}{9}.$$

Therefore, the above function $F(p)$ becomes

$$F(p) = 1 - \left(\frac{2(1 - p)}{5p - 2 - 2(1 - p)}\right)^{\frac{1}{2}}$$
$$= 1 - \sqrt{\frac{2(1 - p)}{7p - 4}},$$

which is distributed between the lower bound $p_L = \frac{2}{9}$ and the upper bound $r = 1$. Differentiating $F(p)$ with respect to p, we find its probability density function

$$f(p) = \frac{3(7p - 4)^{-\frac{3}{2}}}{\sqrt{2(1 - p)}},$$

which is positive so that firms randomize over the full support of the interval $\left[\frac{2}{9}, 1\right]$.