ECONS 424 – STRATEGY AND GAME THEORY

HOMEWORK #3 – ANSWER KEY

Exercise 6 - Chapter 16 Watson

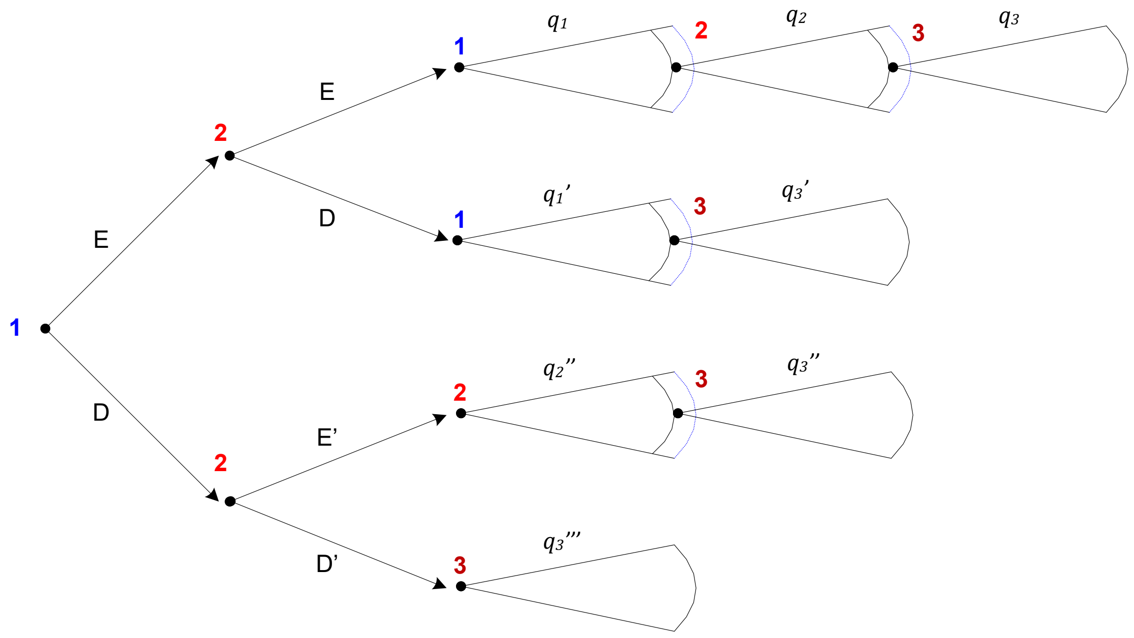
a) If enters against Firm 2, $q^*_1 = q^*_2 = 3$ units. If enters against Firm 3, $q'_1 = q'_3 = 4$ units. In the first stage, Firm 1 enters Firm 3's industry since profits are higher.

b) Yes. If Firm 3's strategy is to choose $q'_3 = 14$ units after Firm 1 enters, then Firm 1's best response is to enter against Firm 2.

Exercise 8 - Chapter 16 Watson

Part a.

Without payoffs, the extensive form is as follows [Note that we are using dashed lines to denote that firm 2 chooses q_2 without observing firm 1's output q_1. Similarly, firm 3 chooses q_3 without observing firm 1 and firm 2's output, q_1 and q_2, respectively.]:



Solving by backward induction, we must first find the output level of every possible entry/no entry scenario. By doing so, we will be able to find the profits resulting from every possible entry/no entry scenario, and then we will be ready to compare firms' profits from entering and not entering:

1. We first solve firms' output in the subgame that starts after firm 1 and 2 enter. [In the figure, this subgame is the upper part, where firms are selecting q_1, q2 and q_3] This is just a Cournot game of quantity competition with three firms competing with each other by simultaneously selecting output. Hence, $q_1 = q_2 = q_3 = 3$.

    a. PROFITS: In this case, note that the profits of every firm in this Cournot oligopoly game with three firms are:

    $$(12 - Q)q_i = (12 - q_1 - q_2 - q_3)q_i = (12 - 3 - 3 - 3) * 3 = 3 * 3 = 9.$$

    b. Note that we must finally subtract 10 (entry costs) in the profits of firm 1 and firm (You don't have to do so for firm 3, since it was already the incumbent in the market). Hence, the payoff vector would be (9-10, 9-10, 9) = (-1, -1, 9)

2. Now we solve the subgame induced after firm 1 enters (E) but firm 2 does not (D). Here we have a Cournot oligopoly game played by firms 1 and 3 (duopoly), where they simultaneously select q_1' and q_3'. Hence, $q_1' = q_3' = 4$.

    a. PROFITS: In this case, note that the profits of every active firm in this Cournot oligopoly game with two firms are:

    $$(12 - Q)q_i' = (12 - q_1' - q_3')q_i' = (12 - 4 - 4) * 4 = 4 * 4 = 16.$$

    b. Note that we must finally subtract 10 (entry costs) from the profits of firm 1 (entrant), which implies that the payoff vector becomes (16-10, 0, 16) = (6, 0, 16).

3. Now we solve the subgame that starts after firm 1 decides not to enter (D), but firm 2 decides to enter (E'). Now we have a Cournot oligopoly game played by firms 2 and 3 (duopoly), where they simultaneously select q_2'' and q_3''. Hence, $q_2'' = q_3'' = 4$.

    a. PROFITS: In this case, note that the profits of every active firm in this Cournot oligopoly game with two firms are:

    $$(12 - Q)q_i'' = (12 - q_2'' - q_3'')q_i'' = (12 - 4 - 4) * 4 = 4 * 4 = 16.$$

    b. Note that we must finally subtract 10 (entry costs) from the profits of firm 2 (entrant), which implies that the payoff vector becomes (0, 16-10, 16) = (0, 6, 16).

4. Now we solve the subgame induced after firm 1 decides not to enter (D) and firm 2 decides not to enter either (D'). Here firm 3 keeps its monopolistic position, and chooses monopoly output, $q_3''' = 6$.

a. PROFITS: In this case, note that the profits of the only monopoly in the market (firm 3), are:

$$(12 - Q)q_3 = (12 - 6)6 = 36$$

b. Note that we don't have to subtract any entry costs from firm 3's profits, given that it was already the incumbent in the market. Hence, the payoff vector in this case is (0, 0, 36).

Plugging all the payoff vectors in the appropriate nodes (see figure at the end of the answer key), and solving by backward induction, we see that:

1. Firm 2 (last mover in this game):
    - After observing that firm 1 entered the market, firm 2 decides to not enter, since its profit from not entering (0) are higher than from entering a "too crowded" market (profits of -1).
    - After observing that firm 1 didn't enter the market, firm 2 chooses to enter, since its profits from doing so (6, now firm 2 would become the only competitor of firm 3) are higher than from not entering (0).
2. Firm 1 (first mover in this game):
    - Firm 1 decides to enter, given that its profits from entering (and inducing firm 2 to stay out afterwards) are 6, while those from not entering (and inducing firm 2 to enter the market afterwards) are only 0. Hence, firm 1 enters.

Hence, at the subgame perfect equilibrium:

1. firm 1 selects Enter,
2. firm 2 chooses not to enter after observing that firm 1 entered, but chooses to enter after observing that firm 1 didn't enter.
3. Equilibrium output levels at every subgame of this game are:
   $q_1 = q_2 = q_3 = 3$ $\quad q_1' = q_3' = 4$ $\quad q_2'' = q_3'' = 4$ $\quad q_3''' = 6$

Part b.

In the subgame perfect equilibrium only firm 1 enters, inducing firm 2 to stay out of the market.

Exercise #3 - Excessive entry in an industry

- Since the equilibrium has to be found by backward induction, first solve the last stage of the game, where firms choose quantities given the number of firms, *n*, that have entered the market in the previous stage. In particular, the n^{th} firm entering produces an output level of $\frac{1-c}{n+1}=\frac{1}{n+1}$, thus obtaining profits of $\pi^c(n)=\frac{1}{(n+1)^2}-F$. Since *n* is a real number, the equilibrium number of firms in the industry will be given by solving for *n* in $\boldsymbol{\pi^c(n)=0}$, given that the n^{th} entrant must be indifferent between entering and staying out. Solving for *n* we obtain

$$n^c=\frac{1}{\sqrt{F}}-1.$$

- The social planner will choose *n* to maximize total welfare (the sum of consumer and producer surplus). Let us first find consumer surplus, *CS*. Notice that the CS is given by the area of the triangle between the vertical intercept of the demand curve, 1, and the equilibrium price

$$p=1-n\frac{1}{n+1}=\frac{1}{n+1}.$$

Hence, CS is given by

$$CS=\frac{1}{2}\left(1-\frac{1}{n+1}\right)n\frac{1}{n+1}=\frac{n^2}{2(n+1)^2}.$$

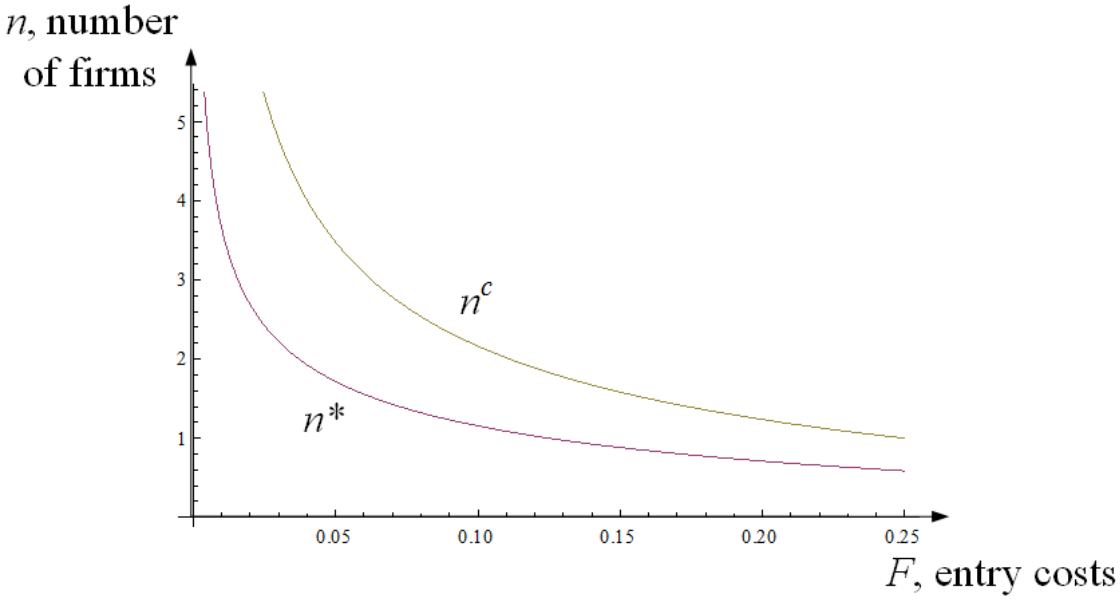
Producer surplus is simply given by the aggregate profits all firms make in the industry, i.e., $n\pi^c(n)=n\frac{1}{(n+1)^2}-nF$. Therefore, the sum of consumer and producer surplus yields a total welfare of

$$W=CS+PS=\frac{n(n+2)}{2(n+1)^2}-nF.$$

Taking first-order conditions with respect to *n*, and solving for *n*, we obtain

$$n^* = \sqrt[3]{\frac{1}{F}} - 1 .$$

- By comparing the optimal number of firms n^* we just found with the number of firms entering at the free entry equilibrium from part (a), $\boldsymbol{n^c}$, it is clear that there exists excess of entry in the industry.
- The following figure illustrates this result. In particular, the figure depicts n^c and n^* as a function of F in the horizontal axis.. Two important features of the figure are noteworthy:
    - First, note that both n^c and n^* decrease in the entry costs, F.
    - Second, the equilibrium number of firms entering the industry, n^c, lies above the socially optimal number of firms, n^*, for any given level of F; reflecting an excessive entry in the industry when entry is unregulated.



~~For the UN to be content to randomize over its three pure strategies, it must be the case that~~

$$4 + 5x + 5y = 9 - 5y,$$
$$4 + 5x + 5y = 9 - 5x.$$
$$9 - 5x = 9 - 5y.$$

~~The last condition implies $x = y$. Letting this common value be denoted w, then the first condition becomes~~

$$4 + 5w + 5w = 9 - 5w \Rightarrow w = \frac{1}{3}.$$

~~Therefore, Nash equilibrium has Saddam uniformly randomize over its three pure strategies; assigning $\frac{1}{3}$ to each of them.~~

CHAPTER 7, EX. 12

12. Consider the two-player game below. Find all of the mixed-strategy Nash equilibria.

Player 2

Player 1		*Slow*	*Fast*
	Small	2,0	3,8
	Medium	3,7	2,1
	Large	3,4	5,6

ANSWER: Since Large strictly dominates Small then we know that all Nash equilibria assign probability zero to Small. Thus, the Nash Equilibria of this game is equivalent to the Nash equilibria of:

2

1		*Slow*	*Fast*
	Medium	3,7	2,1
	Large	3,4	5,6

Let m denote the probability that 1 assigns to Medium and s denote the probability that 2 assigns to Slow. Let us derive each player's best reply. Note that Large weakly dominates Medium. Hence, if 2 assigns any probability to Fast then 1 strictly prefers Large. If s denotes the probability that 2 assigns to Slow then 1's expected payoff from Medium is

$$s \times 3 + (1 - s) \times 2 = 2 + s,$$

and her expected payoff from Large is

$$s \times 3 + (1 - s) \times 5 = 5 - 2s.$$

Large is strictly preferred when

$$5 - 2s > 2 + s \text{ or } 1 > s.$$

Hence, if $s < 1$ then 1's best reply is $m = 0$; that is, the pure strategy Large. If $s = 1$ then either pure strategy gives a payoff of 3 and, in addition, any mixed strategy gives a payoff of 3. Thus, if $s = 1$ then m is a best reply for all values of m, $0 \leq m \leq 1$. Now consider player 2. Given m, player 2's expected payoff from pure strategy Slow is

$$m \times 7 + (1 - m) \times 4 = 4 + 3m,$$

player 2's expected payoff from pure strategy Fast is

$$m \times 1 + (1 - m) \times 6 = 6 - 5m.$$

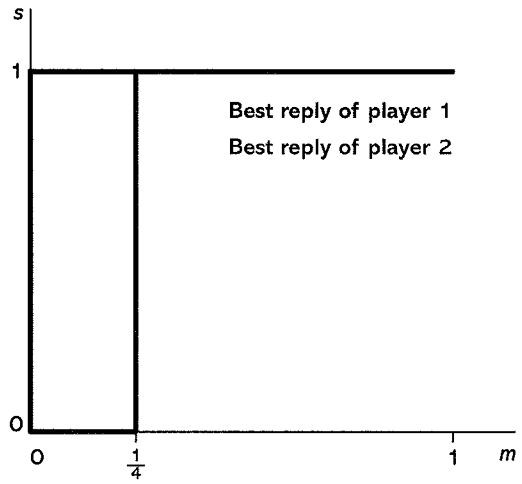
Slow is strictly preferred to Fast when

$$4 + 3m > 6 - 5m \text{ or } m > \frac{1}{4},$$

and Fast is strictly preferred to Slow when

$$4 + 3m < 6 - 5m \text{ or } m < \frac{1}{4}.$$

Player 2 is indifferent between Fast and Slow when $m = \frac{1}{4}$ and, furthermore, all mixed strategies give the same payoff of 4.75. Thus, player 2's best reply is $s = 0$ when $m < \frac{1}{4}$, all values of s when $m = \frac{1}{4}$, and $s = 1$ when $m > \frac{1}{4}$. The best reply functions are plotted in the figure below, and we can see that (m, s) is a Nash equilibrium if $(m, s) = (0, 0)$ or $(m, s) = (m, 1)$ and $m \geq \frac{1}{4}$. Alternatively stated, if $s < 1$ then there is a unique best reply for player 1 of $m = 0$ and the best reply of player 2 is $s = 0$. Hence, $(m, s) = (0, 0)$ is a Nash equilibrium. If $s = 1$ then any mixed strategy for player 1 is a best reply; however, $s = 1$ is a best reply for player 1 if and only if $m \geq \frac{1}{4}$. Thus, $(m, s) = (m, 1)$ is a Nash equilibrium and $m \geq \frac{1}{4}$.



~~13. Consider the two-player game below. Find all of the mixed-strategy Nash equilibria.~~

		Player 2	
		Left	*Right*
Player 1	*Top*	1,2	0,2
	Bottom	1,0	3,4

~~**ANSWER:** There are two pure-strategy Nash equilibrium: (T,L) and (B,R), where T refers to Top, B to Bottom, L to Left, and R to Right. In fact, these are the only mixed-strategy Nash equilibria.~~

Melekhin

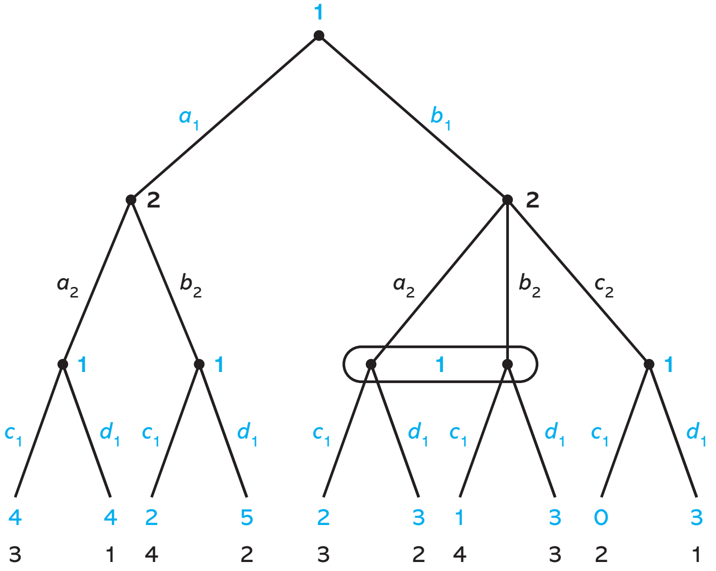
Borodin	*Defect*	*Renege*
Defect	8,8	6,5
Renege	5,6	7,7

Now turn to the subgame that is the game itself. If the Nash equilibrium is (*defect*, *defect*), at both final subgames the captain prefers to not send the letter—as it results in a payoff of 8—while sending the letter means a lower payoff of 7. This gives us one subgame perfect Nash equilibrium of (*do not send letter, defect/defect*, *defect/defect*). Now suppose the Nash equilibrium at the subgame in which the letter is not sent is (*renege*, *renege*). If the letter is sent the captain again earns a payoff of 7, and if the letter is not sent his payoff is 4, as both officers renege. We then find that (*send letter, defect/renege*, *defect/renege*) is a second subgame perfect Nash equilibrium.

b. Explain why the captain would send the letter.

ANSWER: The payoffs are such that if the letter is sent, then it is a dominant strategy for each of the two officers to defect. However, if the letter is not sent, then there are two Nash equilibria for that subgame: one in which both officers defect and one in which both renege. If the letter is not sent—so there is the opportunity to return to the Soviet Union without their plan to defect having been discovered—then an officer wants to renege unless everyone else is planning to defect. By sending the letter, the captain ensures that the officers will continue to defect; while if the letter is not sent, then induced equilibrium play could mean that they renege and the plan to defect fails.

5. Consider the extensive form game shown here. The top payoff at a terminal node is for player 1. Find all subgame perfect Nash equilibria for the game below.

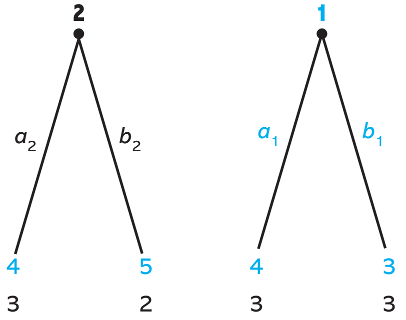


ANSWER: To ease the discussion, we will enumerate the subgames. (1) The subgame associated with player 1 having chosen a_1 and player 2, a_2, has two Nash equilibria: c_1 and d_1. (2) The subgame associated with player 1 having chosen a_1 and player 2, b_2, has a unique Nash equilibrium of d_1. (3) The subgame associated with player 1 having chosen b_1 and player 2, c_2, has a unique Nash equilibrium of d_1. (4) Consider the subgame associated with player 1 having chosen b_1.

Substituting the Nash equilibrium payoffs for subgame (3), the strategic form of this game is shown in the following figure:

	a_2	b_2	c_2
c_1	2,3	1,4	3,1
d_1	3,2	3,3	3,1

where player 1's strategy refers to player 2's choice of a_2 or b_2. The unique Nash equilibrium is (d_1, b_2). Thus, if player 1 chooses b_1, she'll expect a payoff of 3. (5) Consider the subgame in which player 1 chose a_1. As there are two Nash equilibria associated with subgame (1), we need to consider each in turn. Suppose the Nash equilibrium at subgame (1) is c_1. After substituting the Nash equilibrium payoffs for subgames (1) and (2), we have the extensive form shown in the figure below left. The unique Nash equilibrium is player 2 choosing a_2. Next, consider the game itself: the extensive form is as shown in the figure on the below right. It has a unique Nash equilibrium of a_1.



We then have constructed one subgame perfect Nash equilibrium: player 1 chooses a_1 at the initial node, c_1 if player 1 chose a_1 and player 2 chose a_2, d_1 if player 1 chose a_1 and player 2 chose b_2, d_1 if player 1 chose b_1 and player 2 chose a_2 or b_2, and d_1 if player 1 chose b_1 and player 2 chose c_2; player 2 chooses a_2 if player 1 chose a_1 and b_2 if player 1 chose b_1.

Now suppose the Nash equilibrium at subgame (1) is d_1. Consider the subgame associated with player 1 having chosen a_1. After substituting the Nash equilibrium payoffs, we have the extensive form shown in the following figure below left. The unique Nash equilibrium is player 2 choosing b_2. Next consider the game itself. Inserting the equilibrium payoffs, the extensive form is as in the following figure on the below right. It has a unique Nash equilibrium of a_1. We have then constructed a second subgame Nash perfect equilibrium: player 1 chooses a_1 at the initial node, d_1 if player 1 chose a_1 and player 2 chose a_2, d_1 if player 1 chose a_1 and player 2 chose b_2, d_1 if player 1 chose b_1 and player 2 chose a_2 or b_2, and d_1 if player 1 chose b_1 and player 2 chose c_2; player 2 chooses b_2 if player 1 chose a_1 and b_2 if player 1 chose b_1.