# EconS 503 - Microeconomic Theory II
# Homework #1 - Answer key

1. **Strict dominance, IDSDS, and IDWDS.** Consider the following two-player game, adapted from Tadelis (2013).

| | | Player 2 | | |
|---|---|---|---|---|
| | | $L$ | $C$ | $R$ |
| | $U$ | 3, 3 | 5, 1 | 6, 2 |
| Player 1 | $M$ | 4, 1 | 8, 4 | 3, 6 |
| | $D$ | 4, 0 | 9, 6 | 6, 8 |

   (a) Find the strict dominant equilibrium of this game.

   - This game does not have a strict dominant equilibrium because none of the players has a strictly dominant strategy. For a strict dominant equilibrium to exist, we need all players to use a strictly dominant strategy (or strategies). In this context, we need that, every player $i$, strategy $s_i$ satisfies

$$u_i(s_i, s_j) \geq u_i\left(s_i', s_j\right)$$

     for every $s_i' \neq s_i$ and for all $s_j \in S_j$.

   (b) Which strategy profile/s that survive IDSDS?

   - Let us start with player 1, who does not have strictly dominated strategy. To see this, note that:
     - $u_1(U, s_2) < u_1\left(M, s_2\right)$ when player 2 selects $s_2 = L$ (in the left-hand column) and when he selects $s_2 = C$ (in the center column), but
     - $u_1(U, s_2) > u_1\left(M, s_2\right)$ when player 2 selects $s_2 = R$ in the right-hand column.
   - A similar argument applies when comparing player 1's payoffs from choosing $M$ and $D$:
     - $u_1(M, L) = u_1\left(D, L\right)$ when player 2 chooses $L$,
     - $u_1(M, C) < u_1\left(D, C\right)$ when player 2 chooses $C$, and
     - $u_1(M, R) < u_1\left(D, R\right)$ when player 2 chooses $R$.
   - We can now move to player 2, where $C$ is strictly dominated by $R$ since $u_2(C, s_1) < u_2\left(R, s_1\right)$ for every strategy $s_1$ chosen by player 1. We can then reproduce the remaining matrix after the first two rounds of IDSDS, i.e., after deleting nothing for player 1 and strategy $C$ for player 2.

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $R$ |
| | $U$ | 3, 3 | 6, 2 |
| Player 1 | $M$ | 4, 1 | 3, 6 |
| | $D$ | 4, 0 | 6, 8 |

- We cannot find any more strictly dominated strategies relying on pure strategies. (As a practice, check that allowing for player 1 to randomize would not help us to further reduce the set of strategy profiles surviving IDSDS.) Then, the set of strategies surviving IDSDS is the six strategy profiles in the reduced matrix:

$$\{(U, L), (U, R), (M, L), (M, R), (D, L), (D, R)\}$$

(c) Which strategy profile/s survive IDWDS?

- From part (b), we know that player 2 finds that $C$ is strictly dominated by $R$ since $u_2(C, s_1) < u_2(R, s_1)$ for every strategy $s_1$ chosen by player 1, leaving us with the following reduced matrix.

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $R$ |
| | $U$ | 3, 3 | 6, 2 |
| Player 1 | $M$ | 4, 1 | 3, 6 |
| | $D$ | 4, 0 | 6, 8 |

- Turning to player 1, he finds that $D$ weakly dominates $M$ because:
  - $u_1(M, L) = u_1(D, L)$ when player 2 chooses $L$,
  - $u_1(M, R) < u_1(D, R)$ when player 2 chooses $R$.

  We can then delete $M$ from the previous payoff matrix, leaving us with the following reduced matrix.

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $R$ |
| | $U$ | 3, 3 | 6, 2 |
| Player 1 | $D$ | 4, 0 | 6, 8 |

- We can keep analyzing player 1, finding that $D$ weakly dominates $U$ ($D$ yields a higher payoff than $U$ when player 2 chooses $L$, but they both yield the same payoff when player 2 chooses $R$). Deleting row $U$ from the previous matrix, we obtain the following reduced matrix.

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $R$ |
| Player 1 | $D$ | 4, 0 | 6, 8 |

  Finally, turning to player 2, we can say that $R$ strictly dominates $L$, which leaves us with the following reduced matrix, entailing that $(D, R)$ is the unique strategy profile that survives IDWDS.

| | | Player 2 | |
|---|---|---|---|
| | | $R$ | |
| Player 1 | $D$ | 6, 8 | |

  That is, IDWDS= $(D, R)$.

2. **NE and rationalizability.** In this exercise, we formally show the relationship between NE strategy profiles and rationalizability.

   (a) Show that if strategy profile $s^* = (s_1^*, \ldots, s_N^*)$ is a Nash equilibrium of a $N$-player game, it must also survive rationalizability.

   - We first recall the definition of Nash equilibrium we discussed in class: Strategy profile $s^* = (s_1^*, \ldots, s_N^*)$ is a Nash equilibrium if every player $i$ finds that his equilibrium strategy $s_i^* \in S_i$ satisfies

     $$u_i\left(s_i^*, s_{-i}\right) \geq u_i\left(s_i, s_{-i}\right)$$

     for all $s_i \in S_i$, $s_i \neq s_i^*$, and $s_{-i} \in S_{-i}$.

   - *NE survives rationalizability.* Operating by contradiction, consider that $s_i^*$ is a Nash equilibrium strategy for player $i$, but suppose that $s_i^*$ is not rationalizable. Then there must be another strategy, $s_i' \neq s_i^*$, where $s_i' \in S_i$, such that

     $$u_i\left(s_i', s_{-i}\right) > u_i\left(s_i^*, s_{-i}\right)$$

     In words, $s_i^*$ is never a best response (i.e., $NBR$) to *any* strategy profile $s_{-i} \in S_{-i}$ chosen by player $i$'s opponents. In particular, when $s_{-i} = s_{-i}^*$, the above inequality implies that $s_i^*$ is not a best response to $s_{-i}^*$, thus contradicting that $s_i^*$ being a Nash equilibrium strategy.

   (b) Show that the converse of part (a) is not necessarily true. (For this part, an example suffices.)

   - *Rationalizable strategies are not necessarily NE.* Consider the following example:

|  |  | Player 2 |  |
|---|---|---|---|
|  |  | $L$ | $R$ |
| Player 1 | $U$ | 5, 1 | 0, 0 |
|  | $D$ | 0, 0 | 3, 2 |

     The best responses of both players are

     $$BR_1\left(L\right) = U \qquad BR_2\left(U\right) = L$$
     $$BR_1\left(R\right) = D \qquad BR_2\left(D\right) = R$$

     For illustrative purposes, the next matrix underlines best response payoffs for players 1 and 2:

|  |  | Player 2 |  |
|---|---|---|---|
|  |  | $L$ | $R$ |
| Player 1 | $U$ | $\underline{5}$, $\underline{1}$ | 0, 0 |
|  | $D$ | 0, 0 | $\underline{3}$, $\underline{2}$ |

     According to rationalizability, there is no strategy that is never played by both players, that is, $NBR_1 = \varnothing$ for player 1 and $NBR_2 = \varnothing$ for player 2. Intuitively, player 1 responds with either $U$ or $D$ to player 2's strategy choice ($U$ after $L$, and $D$ after $R$), leaving no strategy of player 1 unused. A similar argument applies for player 2, who uses $L$ to respond to $U$, and $R$ to respond to $D$, leaving him with no strategy unused.

- Therefore, the set of rationalizable strategies is

  $$Rationalizable = \{(U,L),(U,R),(D,L),(D,R)\}$$

  while the set of Nash equilibria is

  $$NE = \{(U,L),(D,R)\}$$

  Therefore, the set of rationalizable strategies contains strategy profiles, in particular, $(U,R),(D,L)$, that are *not* Nash equilibria. Graphically, the set of Nash equilibrium strategies is a subset of those strategy profiles surviving rationalizability.

3. **IDSDS implies IDWDS, but the converse is not true.** Consider strategy profile $s = (s_1, s_2, ..., s_N)$.

   (a) Show that if $s$ survives IDWDS, it must also survive IDSDS. [*Hint*: Use contradiction, by considering a strategy profile that, despite surviving IDWDS, violates IDSDS.]

   - We start by reproducing the definition of IDWDS and IDSDS.
   - *IDWDS.* Strategy profile $\hat{s} = (\hat{s}_1, \ldots, \hat{s}_N)$ survives IDWDS if, at any round of deletion, player $i$'s strategy $\hat{s}_i$ satisfies

     $$u_i(\hat{s}_i, s_{-i}) \geq u_i(s_i, s_{-i})$$

     where $\hat{s}_i \neq s_i \in S_i$ and for all $s_{-i} \in S_{-i}$, and satisfies $u_i(\hat{s}_i, s_{-i}) \geq u_i(s_i, s_{-i})$ for at least one strategy profile of player $i$'s opponents $s_{-i}$.
   - *IDSDS.* Strategy profile $s' = (s'_1, \ldots, s'_N)$ survives IDSDS if, at every round of deletion, player $i$'s strategy $s'_i$ satisfies

     $$u_i(\hat{s}_i, s_{-i}) > u_i(s_i, s_{-i})$$

     where $\hat{s}_i \neq s_i \in S_i$ and for all $s_{-i} \in S_{-i}$.
   - Operating by contradiction, assume that strategy $\hat{s}_i$ survives IDWDS but does not survive IDSDS. Therefore, there must be another strategy $\overline{s}_i \neq \hat{s}_i$ that survives IDSDS, yielding a strictly higher payoff than $\hat{s}_i$ in at least one round of deletion, that is,

     $$u_i(\overline{s}_i, s_{-i}) > u_i(\hat{s}_i, s_{-i})$$

     for all $s_{-i} \in S_{-i}$, which contradicts the above definition of IDWDS. Then, if a strategy profile survives IDWDS, it must also survive IDSDS.

   (b) Show that if $s$ survives IDSDS, it does not need to survive IDWDS. (An example suffices.)

   - Consider the following two-player game from Tadelis (2013).

|  |  | Player 2 |  |  |
|---|---|---|---|---|
|  |  | $L$ | $C$ | $R$ |
|  | $U$ | 3, 3 | 5, 1 | 6, 2 |
| Player 1 | $M$ | 4, 1 | 8, 4 | 3, 6 |
|  | $D$ | 4, 0 | 9, 6 | 6, 8 |

- **Applying IDSDS.** We first find which strategy profiles survive IDSDS.
- Let us start with player 1, who does not have strictly dominated strategy. To see this, note that:
  - $u_1(U, s_2) < u_1(M, s_2)$ when player 2 selects $s_2 = L$ (in the left-hand column) and when he selects $s_2 = C$ (in the center column), but
  - $u_1(U, s_2) > u_1(M, s_2)$ when player 2 selects $s_2 = R$ in the right-hand column.
- A similar argument applies when comparing player 1's payoffs from choosing $M$ and $D$:
  - $u_1(M, L) = u_1(D, L)$ when player 2 chooses $L$,
  - $u_1(M, C) < u_1(D, C)$ when player 2 chooses $C$, and
  - $u_1(M, R) < u_1(D, R)$ when player 2 chooses $R$.
- We can now move to player 2, where $C$ is strictly dominated by $R$ since $u_2(C, s_1) < u_2(R, s_1)$ for every strategy $s_1$ chosen by player 1. We can then reproduce the remaining matrix after the first two rounds of IDSDS, i.e., after deleting nothing for player 1 and strategy $C$ for player 2.

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $R$ |
| Player 1 | $U$ | 3, 3 | 6, 2 |
| | $M$ | 4, 1 | 3, 6 |
| | $D$ | 4, 0 | 6, 8 |

- We cannot find any more strictly dominated strategies relying on pure strategies. (As a practice, check that allowing for player 1 to randomize would not help us to further reduce the set of strategy profiles surviving IDSDS.) Then, the set of strategies surviving IDSDS is the six strategy profiles in the reduced matrix:

$$IDSDS = \{(U, L), (U, R), (M, L), (M, R), (D, L), (D, R)\}.$$

- **Applying IDWDS.** We now find strategy profiles survive IDWDS.
- From our above discussion, we know that player 2 finds that $C$ is strictly dominated by $R$ since $u_2(C, s_1) < u_2(R, s_1)$ for every strategy $s_1$ chosen by player 1, leaving us with the following reduced matrix.

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $R$ |
| Player 1 | $U$ | 3, 3 | 6, 2 |
| | $M$ | 4, 1 | 3, 6 |
| | $D$ | 4, 0 | 6, 8 |

- Turning to player 1, he finds that $D$ weakly dominates $M$ because:
  - $u_1(M, L) = u_1(D, L)$ when player 2 chooses $L$,
  - $u_1(M, R) < u_1(D, R)$ when player 2 chooses $R$.

We can then delete $M$ from the previous payoff matrix, leaving us with the following reduced matrix.

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $R$ |
| | $U$ | 3, 3 | 6, 2 |
| Player 1 | $D$ | 4, 0 | 6, 8 |

- We can keep analyzing player 1, finding that $D$ weakly dominates $U$ ($D$ yields a higher payoff than $U$ when player 2 chooses $L$, but they both yield the same payoff when player 2 chooses $R$). Deleting row $U$ from the previous matrix, we obtain the following reduced matrix.

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $R$ |
| Player 1 | $D$ | 4, 0 | 6, 8 |

Finally, turning to player 2, we can say that $R$ strictly dominates $L$, which leaves us with the following reduced matrix, entailing that $(D, R)$ is the unique strategy profile that survives IDWDS.

| | | Player 2 | |
|---|---|---|---|
| | | $R$ | |
| Player 1 | $D$ | 6, 8 | |

That is,

$$IDWDS = (D, R).$$

- **Conclusion:** Strategy profiles such as $(U, L), (U, R), (M, L), (M, R), (D, L)$, which survived IDSDS, do not need to survive IDWDS.

4. **If a game has a NE, it must survive IDSDS.** Consider the two-player game in the following matrix, where payoffs satisfy $a \neq b \neq c \neq d$. If the game has a unique psNE, show that it must be the unique strategy profile surviving IDSDS.

| | | *Player 2* | |
|---|---|---|---|
| | | $A$ | $B$ |
| *Player 1* | $A$ | $a$, $a$ | $c$, $c$ |
| | $B$ | $b$, $b$ | $d$, $d$ |

- Assume without loss of generality that strategy profile $(A, A)$ is the unique psNE. Then, two things must simultaneously hold:
  - First, because $(A, A)$ is a NE and no two payoffs are the same, then we must have that $s_1^A$ is a best response for player 1, i.e., $a > b$, and $s_2^A$ is a best response for player 2, i.e., $a > c$.
  - Second, because $(B, B)$ is *not* a Nash equilibrium then its payoff pair $(d, d)$ cannot satisfy $d > c$ and $d > b$. (Otherwise it would have been another psNE). Then, $c > d$ or $b > d$, or both.

- These two statements imply that either:
  - $a > b$ and $c > d$, which means that $s_1^B$ is strictly dominated by $s_1^A$; or
  - $a > c$ and $b > d$ which means that $s_2^B$ is strictly dominated by $s_2^A$.
- This implies that strategy $B$ will be deleted for players 1 and 2 after applying one round of IDSDS (or two rounds, if necessary), thus leaving us with a unique strategy profile, $(A, A)$, that survives IDSDS.

5. **Exercises from Tadelis:**

   (a) Chapter 4: Exercise 4.3.

   (b) Chapter 5: Exercises 5.1 and 5.9.

■

(b) Provide an example of a game in which there is more than one weakly dominant strategy equilibrium.

**Answer:** In the following game each player is indifferent between his strategies and so each one is weakly dominated by the other. This means that any outcome is a weakly dominant strategy equilibrium.

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{\text{Player 2}} \\
 & & H & T \\
\cline{3-4}
\text{Player 1} & H & 1,1 & 1,1 \\
\cline{3-4}
 & T & 1,1 & 1,1 \\
\cline{3-4}
\end{array}$$

■

4.3

3. **Discrete first-price auction:** An item is up for auction. Player 1 values the item at 3 while player 2 values the item at 5. Each player can bid either 0, 1 or 2. If player $i$ bids more than player $j$ then $i$ win's the good and pays his bid, while the loser does not pay. If both players bid the same amount then a coin is tossed to determine who the winner is, who gets the good and pays his bid while the loser pays nothing.

   (a) Write down the game in matrix form.

   **Answer:** We need to determine what the payoffs are if the bidders tie. The one who wins the coin toss bids his bid and the loser gets and pays nothing. Hence, we can just calculate the expected payoff as a 50:50 lottery between getting nothing and winning. For example, if both players bid 2 then player 1 gets $3-2=1$ unit of payoff with probability $\frac{1}{2}$ and player 2 gets $5-2=3$ units of payoff with probability $\frac{1}{2}$, so the pair of payoffs is $(\frac{1}{2}, \frac{3}{2})$.

Player 2

| | | 0 | 1 | 2 |
|---|---|---|---|---|
| Player 1 | 0 | $\frac{3}{2}, \frac{5}{2}$ | $0, 4$ | $0, 3$ |
| | 1 | $2, 0$ | $1, 2$ | $0, 3$ |
| | 2 | $1, 0$ | $1, 0$ | $\frac{1}{2}, \frac{3}{2}$ |

(b) Does any player have a strictly dominated strategy?

**Answer:** Yes - for player 2 bidding 0 is strictly dominated by bidding 2. ■

(c) Which strategies survive IESDS?

**Answer:** After removing the strategy 0 of player 2, player 1's strategy of 0 is dominated by 2, so we can remove that too. But then, in the remaining $2 \times 2$ game where both players can choose 1 or 2, bidding 1 is strictly dominated by bidding 2 for player 2, and after this round, bidding 1 is strictly dominated by bidding 2 for player 1. Hence, the unique strategy that survives IESDS is $(2, 2)$ yielding expected payoffs of $(\frac{1}{2}, \frac{3}{2})$. ■

4. ~~**eBay's recommendation:** It is hard to imagine that anyone is not familiar with eBay©, the most popular auction website by far. The way a typical eBay auction works is that a good is placed for sale, and each bidder places a "proxy bid", which eBay keeps in memory. If you enter a proxy bid that is lower than the current highest bid, then your bid is ignored. If, however, it is higher, then the current bid increases up to one increment (say, 1 cent) above the *second highest* proxy bid. For example, imagine that three people placed bids on a used laptop of \$55, \$98 and \$112. The current price will be at \$98.01, and if the auction ended the player who bid \$112 would win at a price of \$98.01. If you were to place a bid of \$103.45 then the who bid \$112 would still win, but at a price of \$103.46, while if your bid was \$123.12 then~~

# 5

# Pinning Down Beliefs: Nash Equilibrium

5.1

1. Prove Proposition ??.

   **Answer:** (1) Assume that $s^*$ is a strict dominant strategy equilibrium. This implies that for any player $i$, $s_i^*$ is a best response to any choice of his opponents including $s_{-i}^*$, which in turn implies that $s^*$ is a Nash equilibrium.
   (2) Assume that $s^*$ is the unique survivor of IESDS. By construction of the IESDS procedure, there is no round in which $s_i^*$ is strictly dominated against the surviving strategies of $i$'s opponents, an in particular, against $s_{-i}^*$, implying that $s_i^*$ is a best response to $s_{-i}^*$, which in turn implies that $s^*$ is a Nash equilibrium.
   (3) Assume that $s^*$ is the unique Rationalizable strategy profile. By construction of the Rationalizability procedure, any strategy of player $i$ that survives a round of rationalizability can be a best response to some strategy of $i$'s opponents that survives that round. Hence, by definition, $s_i^*$ is a best response to $s_{-i}^*$, which in turn implies that $s^*$ is a Nash equilibrium. ■

2. ~~A strategy $s^W \in S$ is a **weakly dominant strategy equilibrium** if $s_i^W \in S_i$ is a weakly dominant strategy for all $i \in N$. That is if $v_i(s_i^W, s_{-i}) \geq v_i(s_i', s_{-i})$ for all $s_i' \in S_i$ and for all $s_{-i} \in S_{-i}$. Provide an example of a game~~

for which there is a weakly dominant strategy equilibrium, as well as another Nash equilibrium.

~~**Answer:** Consider the following game:~~

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\text{Player 2}} \\ & & L & R \\ \cline{3-4} \text{Player 1} & U & 1,1 & 1,1 \\ \cline{3-4} & D & 1,1 & 2,2 \\ \cline{3-4} \end{array}$$

~~In this game, $(D, R)$ is a weakly dominant strategy equilibrium (and of course, a Nash equilibrium), yet $(U, L)$ is a Nash equilibrium that is not a weakly dominant strategy equilibrium.~~ ■

5.3

3. Consider a 2 player game with $m$ pure strategies for each player that can be represented by a $m \times m$ matrix. Assume that for each player no two payoffs in the matrix are the same.

   (a) Show that if $m = 2$ and the game has a unique pure strategy Nash equilibrium then this is the unique strategy profile that survives IESDS.

   **Answer:** Consider a general $2 \times 2$ game as follows,

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\text{Player 2}} \\ & & s_{2a} & s_{2b} \\ \cline{3-4} \text{Player 1} & s_{1a} & v_1^{aa}, v_2^{aa} & v_1^{ab}, v_2^{ab} \\ \cline{3-4} & s_{1b} & v_1^{ba}, v_2^{ba} & v_1^{bb}, v_2^{bb} \\ \cline{3-4} \end{array}$$

   and assume without loss of generality that $(s_{1a}, s_{2a})$ is the unique pure strategy Nash equilibrium.[1] Two statements are true: first, because $(s_{1a}, s_{2a})$ is a Nash equilibrium and no two payoffs are the same for each player then $v_1^{aa} > v_1^{ba}$ and $v_2^{aa} > v_2^{ab}$. Second, because $(s_{1b}, s_{2b})$ is *not* a Nash equilibrium then $v_1^{bb} > v_1^{ab}$ and $v_2^{bb} > v_2^{ba}$ cannot hold together (otherwise it would have been another Nash equilibrium). These

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[1] The term "without loss of generality" means that we are choosing one particular strategy profile but there is nothing special about it and we could have chosen any one of the others using the same argument.

two statements imply that either $(i)$ $v_1^{aa} > v_1^{ba}$ *and* $v_1^{ab} > v_1^{bb}\}$ in which case $s_{1b}$ is dominated by $s_{1a}$, or $(ii)$ $v_2^{aa} > v_2^{ab}$ *and* $v_2^{ba} > v_2^{bb}$ in which case $s_{2b}$ is strictly dominated by $s_{2a}$. This implies that either $s_{1b}$ or $s_{2b}$ (or both) will be eliminated in the first round of IESDS, and from the fact that $v_1^{aa} > v_1^{ba}$ and $v_2^{aa} > v_2^{ab}$ it follows that if only one of the strategies was removed in the first round of IESDS then the remaining one will be removed in the second and final round, leaving $(s_{1a}, s_{2a})$ as the unique strategy that survives IESDS. ■

(b) Show that if $m = 3$ and the game has a unique pure strategy equilibrium then it may not be the only strategy profile that survives IESDS.

**Answer:** Consider this following game:

| | | | | |
|---|---|---|---|---|
| | | Player 2 | | |
| | | $L$ | $M$ | $R$ |
| Player 1 | $U$ | 7, 6 | 3, 0 | 6, 5 |
| | $C$ | 1, 3 | 4, 4 | 0, 2 |
| | $D$ | 8, 7 | 2, 1 | 5, 8 |

Notice that for both players none of the strategies are strictly dominated implying that IESDS does not restrict any strategy profile survives IESDS. However, this game has a unique Nash equilibrium: $(C, M)$. ■

4. ~~**Splitting Pizza:** You and a friend are in an Italian restaurant, and the owner offers both of you an 8-slice pizza for free under the following condition. Each of you must simultaneously announce how many slices you would like; that is, each player $i \in \{1, 2\}$ names his desired amount of pizza, $0 \leq s_i \leq 8$. If $s_1 + s_2 \leq 8$ then the players get their demands (and the owner eats any leftover slices). If $s_1 + s_2 > 8$, then the players get nothing. Assume that you each care only about how much pizza you individually consume, and the more the better.~~

   (a) ~~Write out or graph each player's best-response correspondence.~~

~~total hours spent (by everyone) cleaning, minus a number $c$ times the hours spent (individually) cleaning. That is,~~

$$v_i(s_1, s_2, \ldots, s_n) = -c \cdot s_i + \sum_{j=1}^{n} s_j$$

~~Assume everyone chooses simultaneously how much time to spend cleaning.~~

5.9

(a) Find the Nash equilibrium if $c < 1$.

**Answer:** The payoff function is linear in one's own time spent $s_i$ and in the time spent by the other roommates $s_j$, and we can rewrite the payoff function as

$$v_i(s_i, s_{-i}) = s_i - cs_i + \sum_{j \neq i} s_j \ .$$

Considering this payoff function, if $c < 1$ then every additional amount $\varepsilon$ of time that $i$ spends cleaning gives him an extra payoff of $(1-c)\varepsilon > 0$ so that each player $i$ would choose to spend all the 5 hours cleaning. Note that using a first-order condition would not work here because taking the derivative of $v_i(s_i, s_{-i})$ with respect to $s_i$ will just yield $1 - c = 0$ which is not true for $c < 1$. This implies that there is a "corner" solution in the range $s_i \in [0, 5]$, in this case the Nash equilibrium is at the corner $s_i^* = 5$ for all $i = 1, 2, ..., n$. ■

(b) Find the Nash equilibrium if $c > 1$.

**Answer:** Similarly to (a) above, every additional amount $\varepsilon$ of time that $i$ spends cleaning gives him an extra payoff of $(1 - c)\varepsilon < 0$, so that each player $i$ would choose to spend no time cleaning and the Nash equilibrium is $s_i^* = 0$ for all $i = 1, 2, ..., n$. ■

(c) Set $n = 5$ and $c = 2$. Is the Nash equilibrium Pareto efficient? If not, can you find an outcome where everyone is better off than at the Nash equilibrium outcome?

**Answer:** Following the analysis in part (b), the unique Nash equilibrium is where everyone chooses to spend no time cleaning and everyone's payoff is equal to zero. Consider the case where everyone is somehow forced to choose $s_i = 1$. Each player's payoff will be

$$\begin{aligned} v_i(s_i, s_{-i}) &= s_i - cs_i + \sum_{j \neq i} s_j \\ &= 1 - 2 \times 1 + 4 \times 1 = 3 > 0 \ . \end{aligned}$$

so that all the players will be better off if they all chose $s_i = 1$. In fact, each amount of time $\varepsilon > 0$ that player $i$ chooses to clean cause him a personal loss of $\varepsilon - 2\varepsilon = \varepsilon$, but increases the payoff of each of the other players by $\varepsilon$. Hence, if we can get each player to increase his time cleaning by $\varepsilon$, this yields an increase of value for each player that equals his own loss, but the former is multiplied by the number of players. Hence, the best symmetric outcome is when each player chooses $s_i = 5$.

■

~~10. **Synergies:** Two division managers can invest time and effort in creating a better working relationship. Each invests $e_i \geq 0$, and if both invest more then both are better off, but it is costly for each manager to invest. In particular, the payoff function for player $i$ from effort levels $(e_i, e_j)$ is $v_i(e_i, e_j) = (a + e_j)e_i - e_i^2$.~~

~~(a) What is the best response correspondence of each player?~~

~~**Answer:** If player $i$ believes that player $j$ chooses $e_j$ then $i$'s first order optimality condition for maximizing his payoff is,~~

$$a + e_j - 2e_i = 0 \ ,$$

~~yielding the best response function,~~

$$BR_i(e_j) = \frac{a + e_j}{2} \text{ for all } e_j \geq 0.$$

■