# EconS 594 - Industrial Organization
# Homework #2 - Answer Key

1. **Merger that generate economies of scope.** Consider two firms selling differentiated goods and competing à la Cournot. The inverse demand function of every firm $i$ is

$$p(q_i) = a - bq_i - dq_j,$$

where $i \in \{1, 2\}$, $j \neq i$, and $b > d \geq 0$ indicates that own-price effects dominate cross-price effects. When $d = 0$ products are completely differentiated, whereas when $d \to b$ products are completely homogeneous. The total cost function of every unmerged firm $i$ is

$$C(q_i) = \frac{c}{2}q_i^2$$

When firms merge and coordinate output, total cost of the merged firm becomes

$$C(q_1, q_2) = \frac{c}{2}\left(q_1^2 + q_2^2\right) - \beta q_1 q_2$$

where $a > c > \beta$. When $\beta > 0$, the merged firm enjoys cost efficiencies in jointly producing the goods, which is commonly known as "economies of scope." However, when $\beta < 0$, the merged firm suffers cost inefficiencies, which is referred to as "dis-economies of scope." Finally, when $\beta = 0$, the merged firm does not enjoy any cost complementarities.

   (a) *Before the merger.* Set up firm $i$'s profit maximization problem and solve for equilibrium output and profits. How do they change in parameters $a$, $b$, $c$, and $d$? Explain.

   - Every firm $i$ chooses $q_i$ to solve the following profit maximization problem

$$\max_{q_i \geq 0} \pi(q_i) = (a - bq_i - dq_j)\, q_i - C(q_i)$$

Differentiating with respect to $q_i$, and assuming interior solutions, we obtain

$$a - 2bq_i - dq_j = cq_i$$

Rearranging and solving for $q_i$, the best response function of firm $i$ becomes

$$q_i(q_j) = \frac{a}{2b + c} - \frac{d}{2b + c}q_j$$

which originates at an intercept of $\frac{a}{2b+c}$ and decreases in the output of firm $i$'s rival at a rate of $\frac{d}{2b+c}$.
In a symmetric equilibrium, both firms produce the same output level, $q_1 = q_2$. Inserting this property in the above best response function yields

$$q_1 = \frac{a}{2b + c} - \frac{d}{2b + c}q_1$$

Rearranging and solving for $q$, equilibrium output becomes

$$q_i^* = \frac{a}{2b+c+d}$$

which is increasing in market size $a$ as every firm $i$ produces more units when facing a larger demand. Equilibrium output, however, is decreasing in own-price effect $b$ when consumers become more price-sensitive, and in cross-price effect $d$ when goods are more homogeneous to each other. Finally, equilibrium output decreases in $c$ as every unit becomes more costly to produce.

- Substituting $q_i^* = \frac{a}{2b+c+d}$ into the unmerged firm's profit function, yields

$$\begin{aligned}\pi_i^* &= \left[a - (b+d)\frac{a}{2b+c+d}\right]\frac{a}{2b+c+d} - \frac{c}{2}\left(\frac{a}{2b+c+d}\right)^2 \\ &= \frac{a^2(2b+c)}{2(2b+c+d)^2}\end{aligned}$$

which is increasing in $a$ as every firm $i$ generates more profits in a larger market, but decreasing in cross-price effect $d$ as competition becomes more intense when goods are more homogeneous. In addition, we obtain that

$$\frac{\partial \pi_i^*}{\partial b} = -\frac{a^2(2b+c-d)}{(2b+c+d)^3}$$

which is negative since $b > d$ by assumption, indicating that when own-price effect becomes stronger ($b$ increases), firm $i$ earns less profits in equilibrium. Finally, differentiating the equilibrium profits with respect to $c$ yields

$$\frac{\partial \pi_i^*}{\partial c} = -\frac{a^2(2b+c-d)}{2(2b+c+d)^3}$$

which is negative since $b > d$ by definition, so that when it becomes more costly to produce every unit of the good, firm $i$ earns less profits in equilibrium.

(b) *After the merger.* Find equilibrium output and profits when firms merge into a monopoly. Compared to $\beta = 0$, how are your results affected when $\beta < 0$ and when $\beta > 0$?

- The merged firm chooses $q_1$ and $q_2$ to solve the joint profit maximization problem

$$\max_{q_1,q_2\geq 0} \pi(q_1,q_2) = (a - bq_1 - dq_2)\,q_1 + (a - bq_2 - dq_1)\,q_2 - C(q_1,q_2)$$

Differentiating with respect to $q_1$, and assuming interior solutions, we find that

$$a - 2bq_1 - 2dq_2 - cq_1 + \beta q_2 = 0$$

Differentiating with respect to $q_2$, and assuming interior solutions, we obtain

$$a - 2bq_2 - 2dq_1 - cq_2 + \beta q_1 = 0$$

Invoking symmetry, we obtain that $q_1 = q_2 = q^C$, yielding

$$a - 2bq - 2dq - cq + \beta q = 0$$

and, solving for $q$, we find that

$$q^C = \frac{a}{2(b+d) + c - \beta}$$

Inserting $q^C = \frac{a}{2(b+d)+c-\beta}$ into the merged firm's profit function, yields

$$\begin{aligned}\pi^C &= \left(a - bq^C - dq^C\right) q^C + \left(a - bq^C - dq^C\right) q^C - \left[\frac{c}{2}\left(\left(q^C\right)^2 + \left(q^C\right)^2\right) - \beta q^C q^C\right] \\ &= 2\left[a - (b+d)\, q^C\right] q^C - (c - \beta)\left(q^C\right)^2 \\ &= \frac{a^2}{2(b+d) + c - \beta}\end{aligned}$$

- *Comparison.*
  - When $\beta = 0$, equilibrium output becomes

    $$q^C = \frac{a}{2(b+d) + c}$$

    which coincides with that in mergers of differentiated duopolists with symmetric costs. In this context, equilibrium profits become $\pi^C = \frac{a^2}{2(b+d)+c}$.
  - When $\beta$ increases, we find that equilibrium output $q^C$ increases since

    $$\frac{\partial q^C}{\partial \beta} = \frac{a}{\left[2(b+d) + c - \beta\right]^2}$$

    is unambiguously positive. Therefore, when $\beta > 0$, the merged firm enjoys economies of scope, so that it produces more units of output and earns higher profits. In contrast, when $\beta < 0$, the merged firm suffers diseconomies of scope, produces fewer units of output and earns lower profits.

(c) *Merger incentives.* Do firms have incentives to merge into a monopoly? Explain.

- Firms have incentives to merge if and only if their joint profits exceed the sum of unmerged firms' profits, that is,

  $$\pi^C \geq \pi_i^* + \pi_j^*$$

  where, given cost symmetry, $\pi_i^* + \pi_j^* = 2\pi_i^*$. Substituting the results from parts (a) and (b), we obtain

  $$\frac{a^2}{2(b+d) + c - \beta} \geq \frac{a^2(2b+c)}{(2b+c+d)^2}$$

  which is rearranged to yield

  $$(2b+c)^2 + 2d(2b+c) + d^2 \geq (2b+c)^2 + (2b+c)(2d - \beta)$$

  Simplifying, we obtain

  $$\beta(2b+c) + d^2 \geq 0.$$

- To interpret this result, it is convenient to separately consider economies and diseconomies of score:
  - When firms enjoy economies of scope ($\beta > 0$), the above inequality unambiguously holds since $b, c, d > 0$ by assumption.
  - However, when firms suffer diseconomies of scope ($\beta < 0$), firms have incentives to merge only if these diseconomies are not too significant, namely,

$$\beta \geq -\frac{d^2}{2b+c}$$

    Informally, this condition means that parameter $\beta$ cannot be too negative. Otherwise, remaining unmerged would a more profitable option, as firms would avoid the signficant cost inefficiencies that arise from the merger. For instance, if $b = 1$, $d = c = \frac{1}{2}$, this condition becomes $\beta > -\frac{\left(\frac{1}{2}\right)^2}{2+\frac{1}{2}} = -\frac{1}{10}$.
- Finally, ratio $-\frac{d^2}{2b+c}$ decreases (becoming more negative) when $d$ increases and when $b$ and $c$ decrease. Graphically, the region of $\beta$ that satisfies the condition $\beta > -\frac{d^2}{2b+c}$ expands, making firms more likely to merge: (i) when their products become more homogeneous (higher $d$); (ii) when sales become less price sensitive (lower $b$); and (iii) when their original costs decrease (lower $c$).

2. **Mergers between firms selling differentiated products, based on Gelves (2014).**[1] Consider two firms producing differentiated products facing an inverse demand function

$$p_i(q_i) = 1 - q_i - \gamma q_j,$$

where $\gamma \in (0, 1)$ represents the degree of product differentiation. When $\gamma = 0$, products are completely differentiated, but when $\gamma = 1$, products are completely homogeneous. Assume that firm 1 has a higher marginal production cost than firm 2, that is, $c_1 > c_2 \geq 0$.

(a) Find equilibrium output, profits, and social welfare if firms compete à la Cournot.

- Every firm $i$ chooses $q_i$ to solve the profit maximization problem.

$$\max_{q_i \geq 0} \pi_i(q_i) = (1 - q_i - \gamma q_j) q_i - c_i q_i$$

Differentiating with respect to $q_i$, and assuming interior solutions,

$$1 - 2q_i - \gamma q_j - c_i = 0$$

Solving for $q_i$, we obtain the best response function of firm $i$,

$$q_i(q_j) = \frac{1 - c_i}{2} - \frac{\gamma}{2} q_j$$

[1] Gelves, J. A. (2014). Differentiation and Cost Asymmetry: Solving the Merger Paradox. *International Journal of the Economics of Business, 21*(3), 321-340.

which originates at a vertical intercept $\frac{1-c_i}{2}$ and decreases in the output of firm $j$ at a rate of $\frac{\gamma}{2}$. When products become more homogeneous ($\gamma$ increases), firm $i$'s output decreases at a faster rate in firm $j$'s output.
Substituting firm 2's best response function into firm 1's, we obtain

$$q_1 = \frac{1-c_1}{2} - \frac{\gamma}{2}\overbrace{\left(\frac{1-c_2}{2} - \frac{\gamma}{2}q_1\right)}^{q_2}$$

which is simplified to yield

$$\left(1-\frac{\gamma^2}{4}\right)q_1 = \frac{2\left(1-c_1\right)-\gamma\left(1-c_2\right)}{4}$$

Rearranging and solving for $q_1$, we find

$$q_1^{NM} = \frac{2-\gamma-2c_1+\gamma c_2}{4-\gamma^2}$$

where the superscript $NM$ denotes no-merger.
Plugging $q_1^{NM}$ into firm 2's best response function, we find

$$\begin{aligned} q_2^{NM} &= \frac{1-c_2}{2} - \frac{\gamma}{2}\overbrace{\left(\frac{2-\gamma-2c_1+\gamma c_2}{4-\gamma^2}\right)}^{q_1^{NM}} \\ &= \frac{2-\gamma-2c_2+\gamma c_1}{4-\gamma^2} \end{aligned}$$

- Substituting equilibrium output into the firm's profit function, we find

$$\begin{aligned} \pi_1^{NM} &= \left(1-q_1^{NM}-\gamma q_2^{NM}-c_1\right)q_1^{NM} \\ &= \left(\frac{2-\gamma-2c_1+\gamma c_2}{4-\gamma^2}\right)^2 \\ &= \left(q_1^{NM}\right)^2 \\ \pi_2^{NM} &= \left(1-q_2^{NM}-\gamma q_1^{NM}-c_2\right)q_2^{NM} \\ &= \left(\frac{2-\gamma-2c_2+\gamma c_1}{4-\gamma^2}\right)^2 \\ &= \left(q_2^{NM}\right)^2 \end{aligned}$$

- From Belleflamme and Peitz (2015, p. 23), consumer surplus is the area under the demand curve and above price, which is consumers' willingness-to-pay for how much they paid for the goods, i.e.,

$$\begin{aligned} CS &= q_1+q_2-\frac{q_1^2+2\gamma q_1 q_2+q_2^2}{2} - \left(1-q_1-\gamma q_2\right)q_1 - \left(1-q_2-\gamma q_1\right)q_2 \\ &= \frac{q_1^2+2\gamma q_1 q_2+q_2^2}{2} \end{aligned}$$

Finally, social welfare is the sum of consumer and producer surplus, given by

$$\begin{aligned} SW^{NM} &= CS + PS \\ &= \frac{\left(q_1^{NM}\right)^2 + 2\gamma q_1^{NM} q_2^{NM} + \left(q_2^{NM}\right)^2}{2} + \left(q_1^{NM}\right)^2 + \left(q_2^{NM}\right)^2 \\ &= \frac{3\left(q_1^{NM}\right)^2 + 2\gamma q_1^{NM} q_2^{NM} + 3\left(q_2^{NM}\right)^2}{2} \end{aligned}$$

(b) Find equilibrium output, profits, and social welfare if firms merge.

- If firms merge, they solve the following joint profit maximization problem

$$\max_{q_1, q_2 \geq 0} \pi(q_1, q_2) = (1 - q_1 - \gamma q_2 - c_1) q_1 + (1 - q_2 - \gamma q_1 - c_2) q_2$$

Differentiating with respect to $q_1$, and assuming interior solutions, yields

$$1 - 2q_1 - 2\gamma q_2 - c_1 = 0$$

which, after solving for $q_1$, we obtain

$$q_1 = \frac{1 - c_1}{2} - \gamma q_2.$$

Similarly, differentiating with respect to $q_2$, we find that

$$1 - 2q_2 - 2\gamma q_1 - c_2 = 0$$

which, after solving for $q_2$, yields

$$q_2 = \frac{1 - c_2}{2} - \gamma q_1.$$

Inserting this result, $q_2 = \frac{1-c_2}{2} - \gamma q_1$, into $q_1 = \frac{1-c_1}{2} - \gamma q_2$, we obtain

$$q_1 = \frac{1 - c_1}{2} - \gamma \overbrace{\left(\frac{1 - c_2}{2} - \gamma q_1\right)}^{q_2}$$

which simplifies to

$$\left(1 - \gamma^2\right) q_1 = \frac{1 - c_1 - \gamma(1 - c_2)}{2}$$

Rearranging, and solving for $q_1$, yields the equilibrium output of firm 1 under the merger

$$q_1^M = \frac{1 - \gamma - c_1 + \gamma c_2}{2(1 - \gamma^2)}$$

where the superscript $M$ denotes merger.
Inserting this output level in $q_2 = \frac{1-c_2}{2} - \gamma q_1$, we obtain firm 2's equilibrium output under the merger

$$\begin{aligned} q_2^M &= \frac{1 - c_2}{2} - \gamma \overbrace{\left(\frac{1 - \gamma - c_1 + \gamma c_2}{2(1 - \gamma^2)}\right)}^{q_1^M} \\ &= \frac{1 - \gamma - c_2 + \gamma c_1}{2(1 - \gamma^2)} \end{aligned}$$

- Substituting equilibrium output into the merged firm's profit function, we find

$$\begin{aligned}
\pi_1^M &= \left(1 - q_1^M - \gamma q_2^M - c_1\right) q_1^M \\
&= \frac{1-c_1}{2}\frac{1-\gamma-c_1+\gamma c_2}{2\left(1-\gamma^2\right)} \\
\pi_2^M &= \left(1 - q_2^M - \gamma q_1^M - c_2\right) q_2^M \\
&= \frac{1-c_2}{2}\frac{1-\gamma-c_2+\gamma c_1}{2\left(1-\gamma^2\right)}
\end{aligned}$$

- Finally, substituting $q_1^M$ and $q_2^M$ into the expression for social welfare, we have

$$CS = \frac{\left(q_1^M\right)^2 + 2\gamma q_1^M q_2^M + \left(q_2^M\right)^2}{2}$$

  Thus, social welfare is the sum of consumer and producer surplus, given by

$$\begin{aligned}
SW^M &= CS + PS \\
&= \frac{\left(q_1^M\right)^2 + 2\gamma q_1^M q_2^M + \left(q_2^M\right)^2}{2} + \frac{1-c_1}{2}q_1^M + \frac{1-c_2}{2}q_2^M
\end{aligned}$$

(c) *Cost symmetry.* Assume that costs satisfy $c_1 = c_2 = 0$. How does welfare change with $\gamma$? Is the merger welfare improving? Explain.

- *No merger.* When $c_1 = c_2 = 0$, firms are symmetric, so output under no merger becomes

$$q_1^{NM} = q_2^{NM} = \frac{1}{2+\gamma}$$

  yielding welfare under no merger of

$$\begin{aligned}
SW^{NM} &= \frac{1}{2}\left[3\left(\frac{1}{2+\gamma}\right)^2 + 2\gamma\frac{1}{2+\gamma}\frac{1}{2+\gamma} + 3\left(\frac{1}{2+\gamma}\right)^2\right] \\
&= \frac{3+\gamma}{\left(2+\gamma\right)^2}
\end{aligned}$$

  Differentiating $SW^{NM}$ with respect to $\gamma$, we obtain

$$\begin{aligned}
\frac{\partial SW^{NM}}{\partial \gamma} &= \frac{\left(2+\gamma\right)^2 - 2\left(2+\gamma\right)\left(3+\gamma\right)}{\left(2+\gamma\right)^4} \\
&= -\frac{4+\gamma}{\left(2+\gamma\right)^3} < 0
\end{aligned}$$

  so that welfare under no merger decreases as goods become more homogeneous.

- *Merger.* Similarly, when $c_1 = c_2 = 0$ output under merger becomes

$$q_1^M = q_2^M = \frac{1}{2\left(1+\gamma\right)}$$

yielding welfare under merger of

$$SW^M = \frac{\left(\frac{1}{2(1+\gamma)}\right)^2 + 2\gamma\frac{1}{2(1+\gamma)}\frac{1}{2(1+\gamma)} + \left(\frac{1}{2(1+\gamma)}\right)^2}{2} + \frac{1}{2}\frac{1}{2(1+\gamma)} + \frac{1}{2}\frac{1}{2(1+\gamma)}$$
$$= \frac{3}{4(1+\gamma)}$$

Differentiating $SW^M$ with respect to $\gamma$, we obtain

$$\frac{\partial SW^M}{\partial \gamma} = -\frac{3}{4(1+\gamma)^2} < 0$$

so that welfare under merger decreases as goods become more homogeneous.

- *Comparison.* It is straightforward to verify that $SW^{NM} > SW^M$ since

$$\frac{3+\gamma}{(2+\gamma)^2} \geq \frac{3}{4(1+\gamma)}$$

simplifies to

$$\gamma(\gamma+4) > 0$$

which holds for all $\gamma \in [0,1]$, so that mergers between cost-symmetric firms always reduce welfare. Importantly, this occurs regardless of how homogenous or differentiated their products are.

(d) Assume $\gamma = 1$ and $c_2 = 0$. When will the merger be welfare improving? Explain.

- *No merger.* When $\gamma = 1$, goods are homogeneous, so output under no merger is

$$q_1^{NM} = \frac{1-2c_1+c_2}{3}$$
$$q_2^{NM} = \frac{1-2c_2+c_1}{3}$$

where $q_1^{NM} \geq 0$ if $c_1 \leq \frac{1+c_2}{2}$, and $q_2^{NM} \geq 0$ if $1+c_1-2c_2 > 1-c_2 > 0$ that is always satisfied. In this context, social welfare under no merger of

$$SW^{NM} = \frac{3\left(\frac{1-2c_1+c_2}{3}\right)^2 + 2\frac{1-2c_1+c_2}{3}\frac{1-2c_2+c_1}{3} + 3\left(\frac{1-2c_2+c_1}{3}\right)^2}{2}$$
$$= \frac{3(1-2c_1+c_2)^2 + 2\gamma(1-2c_1+c_2)(1-2c_2+c_1) + 3(1-2c_2+c_1)^2}{18}$$
$$= \frac{3(1-c_1)^2 + 3(1-c_2)^2 + 2(1-c_1)(1-c_2) + 8(c_1-c_2)^2}{18}$$

- *Merger.* The merged firm produces all output at the low-cost firm 2, so that

$$q_1^M = 0$$
$$q_2^M = \frac{1-c_2}{2}$$

yielding social welfare under merger of

$$SW^M = \overbrace{\frac{1}{2}\left(q_2^M\right)^2}^{CS} + \overbrace{\left(1 - q_2^M - c_2\right) q_2^M}^{PS}$$
$$= \frac{1}{2}\left(\frac{1-c_2}{2}\right)^2 + \left(1 - \frac{1-c_2}{2} - c_2\right)\frac{1-c_2}{2}$$
$$= \frac{3\left(1-c_2\right)^2}{8}$$

- *Comparison.* Mergers reduce welfare if and only if $SW^{NM} \geq SW^M$, where

$$\frac{3\left(1-c_1\right)^2 + 3\left(1-c_2\right)^2 + 2\left(1-c_1\right)\left(1-c_2\right) + 8\left(c_1-c_2\right)^2}{18} \geq \frac{3\left(1-c_2\right)^2}{8}$$

or, after rearranging,

$$12\left(1-c_1\right)^2 - 15\left(1-c_2\right)^2 + 8\left(1-c_1\right)\left(1-c_2\right) + 32\left(c_1-c_2\right)^2 \geq 0$$

For example, when $c_2 = 0$, the above inequality simplifies to

$$\left(22c_1 - 5\right)\left(2c_1 - 1\right) \geq 0$$

Since $q_1^{NM} \geq 0$ requires $c_1 \leq \frac{1}{2}$, we find that mergers reduce (enhance) welfare if $0 < c_1 \leq \frac{5}{22}$ ($\frac{5}{22} < c_1 \leq \frac{1}{2}$). Intuitively, when firms are significantly cost-asymmetric, mergers that shut down the less efficient firm and shift all output to the more efficient firm can enhance welfare. In contrast, if firms are relatively symmetric in costs, mergers unambiguously reduce welfare.