EconS 594 - Industrial Organization
Homework #1 - Answer Key

1. **Socially Optimal Location in Hotelling model.** Consider mass one of identical consumers uniformly distributed on the unit line, $x \in U[0,1]$. Each consumer buys one unit of the good from firm 1 or firm 2 located at location l_1 and l_2, respectively, where $0 \leq l_1 \leq l_2 \leq 1$. Assume that the price is regulated by the government at $\overline{p}$ and the unit transportation cost is τ.

   Consumer x derives utility from consuming product i, where $i \in \{1, 2\}$, of

   $$v_i(x) = r - \tau |x - l_i| - \overline{p}$$

   where r is the reservation utility of the consumer located at l_i.

   (a) Identify the indifferent consumer $\hat{x}$, and solve for the demands of firm 1 and firm 2.

   - The indifferent consumer $\hat{x}$ solves $v_1(x) = v_2(x)$, yielding

     $$r - \tau |x - l_1| - \overline{p} = r - \tau |x - l_2| - \overline{p}$$

     Since consumer $\hat{x}$ is to the right (left) of firm 1 (firm 2), $l_1 \leq x \leq 1_2$, we have

     $$x - l_1 = l_2 - x$$

     Rearranging and solving for $\hat{x}$, we obtain

     $$\hat{x} = \frac{l_1 + l_2}{2}$$

   - Therefore, the demand of firm 1 is

     $$Q_1(l_1, l_2) = \hat{x} = \frac{l_1 + l_2}{2}$$

     Similarly, the demand of firm 2 is

     $$Q_2(l_1, l_2) = 1 - \hat{x} = \frac{2 - l_1 - l_2}{2}$$

   (b) Write the firm's profit maximization problem and solve for the equilibrium location.

   - Firm i chooses l_i to solve

     $$\pi_i(l_i, l_j) = \begin{cases} \frac{(\overline{p}-c)(l_i+l_j)}{2} & \text{if } l_i < l_j \\ \frac{\overline{p}-c}{2} & \text{if } l_i = l_j \\ \frac{(\overline{p}-c)(2-l_i-l_j)}{2} & \text{if } l_i > l_j \end{cases}$$

- Suppose $l_i < l_j < 1/2$, firm i has incentives to deviate to $l_i > l_j$ if

$$\frac{(\overline{p}-c)(2-l_i-l_j)}{2} > \frac{(\overline{p}-c)(l_i+l_j)}{2}$$

which simplifies to $2-l_i-l_j > l_i+l_j$, or $l_i+l_j < 1$ that holds by assumption.
- Suppose $l_i > l_j > 1/2$, firm i has incentives to deviate to $l_i < l_j$ if

$$\frac{(\overline{p}-c)(l_i+l_j)}{2} > \frac{(\overline{p}-c)(2-l_i-l_j)}{2}$$

which simplifies to $l_i+l_j > 2-l_i-l_j$, or $l_i+l_j > 1$ that holds by assumption.
- Suppose $l_i < 1/2 < l_j$, firm i has incentives to deviate to $l_i = 1/2$ since

$$\frac{(\overline{p}-c)(1/2+l_j)}{2} > \frac{(\overline{p}-c)(l_i+l_j)}{2}$$

which simplifies to $l_i < 1/2$, and a similar argument applies to $l_j < 1/2 < l_i$.
- Therefore, in equilibrium, we must have $l_1^* = l_2^* = 1/2$ in which both firms do not have incentives to deviate. It is straightforward to show that deviating will cause this firm to lose market share to the other firm, thus entailing profit losses.

(c) Find the socially optimal location l_1^{SO} and l_2^{SO}, and compare your results to part (b).

- The total transportation cost is

$$\tau\int_0^{l_1}(l_1-x)\,dx + \tau\int_{l_1}^{\hat{x}}(x-l_1)\,dx + \tau\int_{\hat{x}}^{l_2}(l_2-x)\,dx + \tau\int_{l_2}^{1}(x-l_2)\,dx$$

The first term integrates the transportation cost of consumers from 0 to where firm 1 is located, and the second term is for consumers from the location of firm 1 to the indifferent consumer $\hat{x}$, and analogously for the third and fourth terms.

$$\begin{aligned}
&= \tau\left|l_1x-\frac{x^2}{2}\right|_0^{l_1} + \tau\left|\frac{x^2}{2}-l_1x\right|_{l_1}^{\hat{x}} + \tau\left|l_2x-\frac{x^2}{2}\right|_{\hat{x}}^{l_2} + \tau\left|\frac{x^2}{2}-l_2x\right|_{l_2}^{1}\\
&= \tau\left(l_1^2-\frac{l_1^2}{2}\right) + \tau\left(\frac{\hat{x}^2}{2}-l_1\hat{x}\right) - \tau\left(\frac{l_1^2}{2}-l_1^2\right) + \tau\left(l_2^2-\frac{l_2^2}{2}\right)\\
&\quad - \tau\left(l_2\hat{x}-\frac{\hat{x}^2}{2}\right) + \tau\left(\frac{1}{2}-l_2\right) - \tau\left(\frac{l_2^2}{2}-l_2^2\right)\\
&= \tau\left[\hat{x}^2-(l_1+l_2)\,\hat{x}+l_1^2+l_2^2-l_2+\frac{1}{2}\right]\\
&= \frac{\tau}{4}\left[2+4l_1^2+4l_2^2-4l_2-(l_1+l_2)^2\right]
\end{aligned}$$

where we substitute $\hat{x} = \frac{l_1+l_2}{2}$ into the last line of expression.

Differentiating the transportation cost function with respect to l_1, we find

$$8l_1 - 2(l_1 + l_2) = 0$$

Rearranging and solving for l_1, we have

$$l_1(l_2) = \frac{l_2}{3}$$

Differentiating the transportation cost function with respect to l_2, we find

$$8l_2 - 4 - 2(l_1 + l_2) = 0$$

Rearranging and solving for l_2, we have

$$l_2(l_1) = \frac{l_1 + 2}{3}$$

Substituting $l_1(l_2) = \frac{l_2}{3}$ into the above expression, we obtain

$$l_2 = \frac{l_2}{9} + \frac{2}{3}$$

Rearranging and solving for l_2, yields

$$l_2^{SO} = \frac{3}{4}$$

Substituting $l_2^{SO} = \frac{3}{4}$ into $l_1(l_2) = \frac{l_2}{3}$, yields

$$l_1^{SO} = \frac{1}{4}$$

- Comparing with the results in part (b), we find that the socially efficient location of firm 1 is $l_1^{SO} = \frac{1}{4}$ that is halfway between point 0 and its equilibrium position $l_1^* = \frac{1}{2}$. Similarly, the socially efficient location of firm 2 is $l_2^{SO} = \frac{3}{4}$ that is halfway between and its equilibrium position $l_2^* = \frac{1}{2}$ and point 1.

2. **Hotelling with non-uniformly distributed consumers.** Consider mass one of identical consumers distributed on the unit line according to the cumulative distribution function $F(x) = x^2(\alpha - \beta x)$, where $\alpha, \beta > 0$ and $x \in [0, 1]$. Each consumer buys one unit of the homogenous good from either firm 1 or firm 2 located at the end points on the unit line, and suffers a unit travel cost of $\tau > 0$. The indirect utility that consumer x derives from consuming the good is

$$\begin{aligned} v_1(x) &= r - \tau x - p_1 \quad \text{from firm 1} \\ v_2(x) &= r - \tau(1 - x) - p_2 \quad \text{from firm 2} \end{aligned}$$

where $r > 0$ measures consumers' valuation on the quality of the homogeneous product.

(a) Use the properties of symmetric distribution function, where $F\left(\frac{1}{2}\right) = \frac{1}{2}$ and $F(1) = 1$ to characterize the cumulative distribution function $F(x)$.

- Evaluating $F(x) = x^2(\alpha - \beta x)$ at $x = \frac{1}{2}$, we obtain

$$F\left(\frac{1}{2}\right) = \frac{1}{2^2}\left(\alpha - \frac{\beta}{2}\right) = \frac{1}{2},$$

since $F\left(\frac{1}{2}\right) = \frac{1}{2}$. Simplifying this expression, yields

$$\beta = 2(\alpha - 2)$$

Substituting $\beta = 2(\alpha - 2)$ into $F(1) = 1$, we find that

$$1^2[\alpha - 2(\alpha - 2)] = 1$$

or

$$\alpha = 2(\alpha - 2)$$

Rearranging, we find $\alpha = 3$. Inserting this result into $\beta = 2(\alpha - 2)$, yields $\beta = 2$, so that the distribution function is

$$F(x) = x^2(3 - 2x)$$

(b) *Finding demands.* Using the cumulative distribution function you found in part (a), identify the indifferent consumer $\hat{x}$, and solve for the demands of firms 1 and 2.

- The indifferent consumer solves

$$r - \tau\hat{x} - p_1 = r - \tau(1 - \hat{x}) - p_2$$

which, after rearranging, yields

$$\hat{x} = \frac{1}{2} + \frac{p_2 - p_1}{2\tau}$$

- The demand for firm 1 is

$$D_1(p_1, p_2) = F(\hat{x}) = \frac{(\tau + p_2 - p_1)^2(2\tau - p_2 + p_1)}{4\tau^3}$$

- The demand for firm 2 is

$$D_2(p_1, p_2) = 1 - F(\hat{x}) = \frac{4\tau^3 - (\tau + p_2 - p_1)^2(2\tau - p_2 + p_1)}{4\tau^3}$$

(c) *Equilibrium prices.* Suppose that production costs are zero. Set up firm i's profit maximization problem, derive the first-order conditions, and then invoke symmetry to find equilibrium prices.

- Firm 1 chooses p_1 to solve the following profit maximization problem,

$$\begin{aligned}\max_{p_1>0} \pi_1(p_1) &= p_1 D_1(p_1, p_2) \\ &= \frac{p_1(\tau + p_2 - p_1)^2(2\tau - p_2 + p_1)}{4\tau^3}\end{aligned}$$

Taking the first order condition with respect to p_1, yields

$$\frac{(\tau + p_2 - p_1)\left[(\tau + p_2 - p_1)(2\tau - p_2 + 2p_1) - 2p_1(2\tau - p_2 + p_1)\right]}{4\tau^3} = 0$$

which best response function, $p_1(p_2)$, implicitly solves

$$4p_1^2 + (4\tau - 5p_2)p_1 - (\tau + p_2)(2\tau - p_2) = 0$$

- Firm 2 chooses p_2 to solve the following profit maximization problem,

$$\begin{aligned}\max_{p_2>0} \pi_2(p_2) &= p_2 D_2(p_1, p_2) \\ &= \frac{p_2\left[4\tau^3 - (\tau + p_2 - p_1)^2(2\tau - p_2 + p_1)\right]}{4\tau^3}\end{aligned}$$

Taking the first order condition with respect to p_2, yields

$$\frac{4\tau^3 - (\tau + p_2 - p_1)\left[(\tau + p_2 - p_1)(2\tau - 2p_2 + p_1) + 2p_2(2\tau - p_2 + p_1)\right]}{4\tau^3} = 0$$

which best response function, $p_2(p_1)$, implicitly solves

$$4p_2^3 - 9p_1p_2^2 - 6(\tau - p_1)(\tau + p_1)p_2 - (\tau - p_1)^2(2\tau + p_1) + 4\tau^3 = 0$$

- Since firms are symmetric and the distribution is symmetric around the mean, in equilibrium we have $p^* = p_1^* = p_2^*$, such that equilibrium prices become

$$p^* = \frac{2\tau}{3}$$

(d) Find the equilibrium demand and profits of every firm.

- Substituting $p^* = \frac{2\tau}{3}$ into $D_1(p_1, p_2)$, firm 1's equilibrium demand becomes

$$D_1^* = \frac{1}{2}$$

with associated profits of

$$\pi_1^* = \frac{\tau}{3}$$

- Substituting $p^* = \frac{2\tau}{3}$ into $D_2(p_1, p_2)$, firm 2's equilibrium demand becomes

$$D_2^* = \frac{1}{2}$$

with associated profits of

$$\pi_2^* = \frac{\tau}{3}$$

(e) *Comparison with uniformly distributed consumers.* Compare your results (i.e., firms' equilibrium prices, demand, and profits) to the setting where consumers are uniformly distributed in $[0, 1]$. Explain.

- When consumers are uniformly distributed, recall from previous exercises that

$$p^U = \tau$$
$$D^U = \frac{1}{2}$$
$$\pi^U = \frac{\tau}{2}$$

where the superscript U stands for the setting of uniform distribution.

- Comparing to the setting where consumers are distributed according to $F(x)$, it is straightforward to verify that every firm charges a lower price under $F(x)$, that is,

$$p^* = \frac{2\tau}{3} < \tau = p^U$$

in order to attract the mass of consumers concentrated around the mean 1/2. Furthermore, since firms equally spilt the consumers, where $D_1^* = D_2^* = D^U = 1/2$, but charging a lower price, every firm generates lower profits in equilibrium, that is,

$$\pi_1^* = \pi_2^* = \frac{\tau}{3} < \frac{\tau}{2} = \pi^U$$

(f) *Socially optimal location.* Assume that the social planner decides the firms' locations to minimize the total transportation costs of all consumers that are distributed according to $F(x)$. Find the socially optimal location of firm 1, l_1, and of firm 2, l_2, and compare your results to the setting of uniform distribution. Interpret.

- The total transportation cost in this setting is given by

$$TC(l_1, l_2) = \underbrace{\overbrace{\tau \int_0^{l_1} (l_1 - x)\, dF(x)}^{\text{Consumers to the left of } l_1} + \overbrace{\tau \int_{l_1}^{\hat{x}} (x - l_1)\, dF(x)}^{\text{Consumers to the right of } l_1}}_{\text{Transportation cost of consumers buying from firm 1}}$$
$$+ \underbrace{\overbrace{\tau \int_{\hat{x}}^{l_2} (l_2 - x)\, dF(x)}^{\text{Consumers to the left of } l_2} + \overbrace{\tau \int_{l_2}^{1} (x - l_2)\, dF(x)}^{\text{Consumers to the right of } l_2}}_{\text{Transportation cost of consumers buying from firm 2}}$$

The first term integrates the transportation cost of consumers from 0 to firm 1, which is located at l_1, and the second term represents the transportation cost of consumers located between firm 1 and the indifferent consumer located at $\hat{x}$. The third and fourth terms are defined analogously for firm 2, which is located at l_2.

- Since $F(x) = x^2(3-2x)$, we obtain that $f(x) = 6x(1-x)$, yielding

$$\begin{aligned} TC(l_1, l_2) = \quad & 6\tau \int_0^{l_1} (l_1 - x) x (1-x)\, dx + 6\tau \int_{l_1}^{\hat{x}} (x - l_1) x (1-x)\, dx \\ & + 6\tau \int_{\hat{x}}^{l_2} (l_2 - x) x (1-x)\, dx + 6\tau \int_{l_2}^{1} (x - l_2) x (1-x)\, dx \\ & = 6\tau \left[\frac{l_1}{2} x^2 - \frac{1+l_1}{3} x^3 + \frac{1}{4} x^4 \right]_0^{l_1} - 6\tau \left[\frac{l_1}{2} x^2 - \frac{1+l_1}{3} x^3 + \frac{1}{4} x^4 \right]_{l_1}^{\hat{x}} \\ & + 6\tau \left[\frac{l_2}{2} x^2 - \frac{1+l_2}{3} x^3 + \frac{1}{4} x^4 \right]_{\hat{x}}^{l_2} - 6\tau \left[\frac{l_2}{2} x^2 - \frac{1+l_2}{3} x^3 + \frac{1}{4} x^4 \right]_{l_2}^{1} \end{aligned}$$

- Furthermore, since the distribution is symmetric, we have that the indifferent consumer is located at $\hat{x} = \frac{l_1+l_2}{2}$, simplifying the above expression into

$$TC(l_1, l_2) = \frac{\tau}{16} \begin{bmatrix} 16 l_1^3 (2 - l_1) + 16 l_2^3 (2 - l_2) - 8 (2 l_2 - 1) \\ - (l_1 + l_2)^2 \left(4 l_1 - 2 l_1 l_2 + 4 l_2 - l_1^2 - l_2^2 \right) \end{bmatrix}$$

Differentiating the total cost function with respect to l_1, we obtain

$$30 l_1^3 - 42 l_1^2 - 6 l_1^2 l_2 + 12 l_1 l_2 - 6 l_1 l_2^2 + 6 l_2^2 - 2 l_2^3 = 0$$

Similarly, differentiating the total cost function with respect to l_2, yields

$$30 l_2^3 - 42 l_2^2 - 6 l_2^2 l_1 + 12 l_1 l_2 - 6 l_1^2 l_2 + 6 l_1^2 - 2 l_1^3 + 8 = 0$$

Simultaneously solving for l_1 and l_2 in the two first-order conditions above, we find that the socially optimal locations are

$$l_1^{SO,F} = 0.326 \qquad \text{and} \qquad l_2^{SO,F} = 0.674$$

where the superscript SO,F stands for socially optimal location under $F(x)$. Comparing the above results those when consumers are uniformly distributed, we see that

$$l_1^{SO,U} = 0.25 < 0.326 = l_1^{SO,F}$$

where SO,U denotes socially optimal location under uniform distribution, and

$$l_2^{SO,U} = 0.75 > 0.674 = l_2^{SO,F}$$

- To understand this result, first note that both $F(x)$ and $U[0,1]$ have the same mean of $1/2$, where

$$\begin{aligned} E_F[x] &= 6 \int_0^1 x^2 (1-x)\, dx \\ &= 6 \left[\frac{x^3}{3} \right]_0^1 - 6 \left[\frac{x^4}{4} \right]_0^1 \\ &= \frac{1}{2}, \text{ and} \\ E_U[x] &= \int_0^1 x dx = \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2} \end{aligned}$$

but consumers are less dispersed under $F(x)$ than under $U[0,1]$, since

$$\begin{aligned} E_F\left[x^2\right] &= 6\int_0^1 x^3(1-x)\,dx \\ &= 6\left[\frac{x^4}{4}\right]_0^1 - 6\left[\frac{x^5}{5}\right]_0^1 \\ &= \frac{3}{10}, \text{ and} \\ E_U\left[x^2\right] &= \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \end{aligned}$$

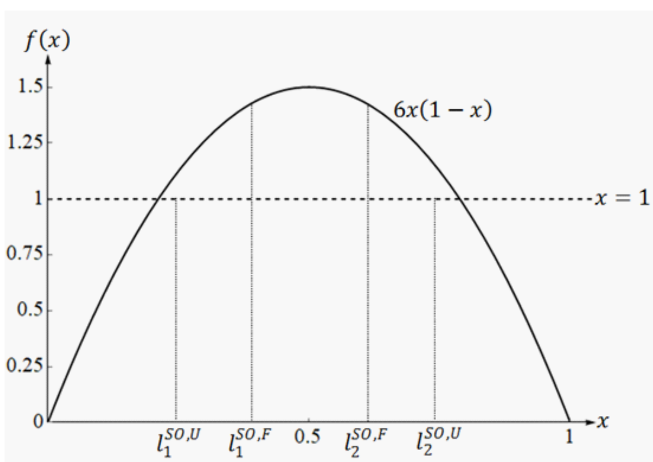
yielding the following variances for each distribution function

$$\begin{aligned} Var_F(x) &= E_F\left[x^2\right] - \left(E_F[x]\right)^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20} = 0.05 \\ Var_U(x) &= E_U\left[x^2\right] - \left(E_U[x]\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \simeq 0.08 \end{aligned}$$

Since $Var_F(x) < Var_U(x)$, consumers are more concentrated around the mean under $F(x)$ than under $U[0,1]$, such that firms are located closer to the mean, that is,

$$l_1^{SO,U} < l_1^{SO,F} < \frac{1}{2} < l_2^{SO,F} < l_2^{SO,U},$$

under $F(x)$ than under $U[0,1]$, as the following figure indicates, which plots the density function of $F(x) = x^2(3-2x)$, $f(x) = 6x(1-x)$, and that of the uniform $F(x) = x$, $f(x) = 1$.



3. **Salop circle with quadratic transportation costs.** Consider the Salop circle we presented in class, but assume that transportation costs are now τd^2, where d denotes the distance that the consumer travels to his selected shop.

   (a) Find the equilibrium price in this setting. How does it differ from that under linear transportation costs?

- Assume that there are n firms. Consider firm i's choice of price p_i given that the other firms charge p. A consumer located at distance $x < \frac{1}{n}$ from firm i is indifferent between firm i and the nearest competitor (firm j) if

$$p_i + \tau x^2 = p_j + \tau \left(\frac{1}{n} - x \right)^2 .$$

Solving for x, we obtain

$$x = \frac{n(p_j - p_i) + \frac{\tau}{n}}{2\tau}$$

which implies that firm i's demand is twice this amount (consumers to the left and right of the indifferent consumer), that is,

$$Q_i(p_i, p) = 2x = \frac{1}{n} - \frac{n(p_i - p_j)}{\tau}.$$

The firm then solves the following profit-maximization problem

$$\max_{p_i} \ (p_i - c)Q_i(p_i, p).$$

Differentiating with respect to p_i, yields

$$\frac{1}{n} - \frac{n(2p_i - p_j - c)}{\tau} = 0$$

In a symmetric equilibrium, $p_i = p_j = p$. Inserting this property in the above first-order condition yields

$$\frac{1}{n} = \frac{n(p-c)}{\tau}$$

Solving for p, we obtain the equilibrium price

$$p^* = c + \frac{\tau}{n^2}.$$

(b) Find the equilibrium number of firms entering the industry, n^e, when entry cost is $e > 0$.

- Firm profits in equilibrium are

$$\begin{aligned} \pi^*(e) &= (p^* - c)Q_i(p^*, p^*) - e \\ &= \left(c + \frac{\tau}{n^2} - c \right) \left(\frac{1}{n} - \frac{n(p_i - p_j)}{\tau} \right) - e \\ &= \frac{\tau}{n^3} - e. \end{aligned}$$

Using the zero-profit condition, $\pi^*(e) = 0$, and solving for e, we obtain the equilibrium number of firms entering the industry, as follows

$$n^e = \left(\frac{\tau}{e} \right)^{1/3}$$

which entails an equilibrium price of

$$p^* = c + \frac{\tau}{\left[\left(\frac{\tau}{e} \right)^{1/3} \right]^2} = c + \tau^{1/3} e^{2/3}.$$

(c) Find the socially optimal number of firms entering the industry, n^{SO}, when entry cost is $e > 0$.

- To find the socially optimal number of firms entering the industry, recall that social welfare in this context coincides with total entry costs plus total transportation costs. This is because consumers purchase one unit of the good and all the market is covered, entailing that loss in consumer surplus coincides with the increase in firm revenues.
  Therefore, we only need to minimize the sum of total entry costs and total transportation costs, as follows

$$\min_n ne + 2n\tau \int_0^{\frac{1}{2n}} x^2 dx = \min_n ne + \frac{\tau}{12n^2}$$

  Differentiating with respect to n, yields

$$e - \frac{\tau}{6n^3} = 0$$

  which, solving for n, yields the socially optimal number of firms entering the industry

$$n^{SO} = \left(\frac{\tau}{6e}\right)^{1/3} = \frac{1}{6^{1/3}} n^e \simeq 0.55 n^e.$$

(d) Compare your results in parts (b) and (c). Interpret.

- The socially optimal number of firms is lower than the equilibrium number of firms, $n^{SO} < n^e$, since every firm does not internalize the business-stealing effect that its entry imposes on other firms. The regulator internalizes this external effect, seeking less entry.

(e) Compare your result in part (d) against the case in which transportation costs are linear (check your class notes). Interpret.

- Equilibrium entry with linear transportation costs is $n^e = \left(\frac{\tau}{e}\right)^{1/2}$, thus being larger than under quadratic transportation costs. We can also measure the excessive entry in each context as follows. Under linear transportation costs, excessive entry is given by

$$\text{Excessive Entry} = \left(\frac{\tau}{e}\right)^{1/2} - \frac{1}{2}\left(\frac{\tau}{e}\right)^{1/2} = 0.5\left(\frac{\tau}{e}\right)^{1/2}$$

  while under quadratic transportation costs, we have

$$\text{Excessive Entry} = \left(\frac{\tau}{e}\right)^{1/3} - \left(\frac{\tau}{6e}\right)^{1/3} \simeq 0.45\left(\frac{\tau}{e}\right)^{1/3}.$$

4. **Cournot competition between one public and N private firms.** Consider the setting in Exercise 2.13 of Choi-Dunaway-Munoz (2021), but now allowing for $N \geq 1$ private firms. In this context, aggregate output is $Q \equiv q_0 + \sum_{i=1}^{N} q_i$, where $i = 1, 2, ..., N$ denotes private firms. The private firm still maximizes its profits

$$\pi_i = p(X)q_i - cq_i$$

while the public firm maximizes a combination of social welfare and profits

$$V_0 = \alpha W + (1-\alpha)\pi_0$$

where social welfare is given by $W = \int_0^Q p(y)dy - cQ$, and its profits are $\pi_0 = p(Q)q_0 - cq_0$. Intuitively, parameter α represents the weight that the manager of the public firm assigns to social welfare, while $1-\alpha$ is the weight that he assigns to profits. All firms simultaneously and independently choose their output levels.

(a) Find the best-response function of every private firm, $q_i(q_0)$. How are your results affected by a marginal increase in N? Interpret.

- Each private firm i maximizes

$$\max_{q_i \geq 0} \; \pi_i = \left[1 - (q_0 + q_i + \sum_{j \neq i}^{N} q_j)\right] q_i - cq_i$$

Differentiating with respect to q_i, yields

$$1 - q_0 - 2q_i - \sum_{j \neq i}^{N} q_j - c = 0$$

In a symmetric equilibrium, $q_i = q_j$, implying that

$$\sum_{j \neq i}^{N} q_j = (N-1)q_i,$$

so we can rewrite the above first order condition as follows

$$1 - q_0 - (N+1)q_i - c = 0$$

Solving for q_i, we find the best response function of every private firm i

$$q_i(q_0) = \frac{1-c}{N+1} - \frac{1}{N+1}q_0.$$

- *Comparative statics.*
  - As expected, this best response function is unaffected by the weight that the public firm assigns on social welfare, α.
  - However, an increase in the number of private firms (higher N) produces a downward shift in the vertical intercept, $\frac{1-c}{N+1}$, and in the slope, $\frac{1}{N+1}$, entailing a lower output level for every private firm. In the special case where a single private firm competes against a public firm, $N = 1$, this best response function simplifies to

$$q_i(q_0) = \frac{1-c}{2} - \frac{1}{2}q_0,$$

as in standard Cournot duopolies.

(b) Find the best-response function of the public firm, $q_0(q_i)$. How are your results affected by a marginal increase in N? And how are they affected by a marginal increase in α? Interpret.

- The public firm maximizes

$$\begin{aligned}\max_{q_0 \geq 0} V_0 &= \alpha W + (1-\alpha)\pi_0 \\ &= \alpha \left[\int_0^{q_0+\sum_{i=1}^N q_i} p(y)dy - c\left(q_0 + \sum_{i=1}^N q_i\right)\right] + (1-\alpha)\left[\left(1-(q_0+\sum_{i=1}^N q_i)\right) q_0 - \right. \\ &= \alpha \left[\int_0^{q_0+Nq_i} (1-y)dy - c\,(q_0+Nq_i)\right] + (1-\alpha)\left[(1-(q_0+Nq_i))\,q_0 - cq_0\right]\end{aligned}$$

Differentiating with respect to x_0, and using Leibniz's rule, we obtain

$$\frac{\partial V}{\partial q_0} = \alpha\left[1 - q_0 - Nq_i - c\right] + (1-\alpha)\left(1 - 2q_0 - Nq_i - c\right) = 0$$

Solving for q_0, we find the public firm's best response function

$$q_0(q_i) = \frac{1-c}{2-\alpha} - \frac{N}{2-\alpha} q_i$$

- *Comparative statics.*
  - An increase in the weight that the public firm assigns to welfare, α, produces an outward shift in the best response function, as described in part (a) of Exercise 2.13 in the case of $N = 1$. Intuitively, as the public firm assigns more weight on welfare, it produces a larger output q_0 for a given aggregate output of its private rivals, $Q = Nq_i$.
  - An increase in the number of private firms, N, does not affect the vertical intercept of the public firm's best response function, $\frac{1-c}{2-\alpha}$, but makes it steeper, since $\frac{N}{2-\alpha}$ increases. Intuitively, as the public firm faces more private rivals, it decreases its output more significantly for a given increase in q_i, since the simultaneous output competition of more firms would bring the total output closer to the perfectly competitive level (and reaches that level when $N \to \infty$). Needless to say, when $N = 1$, the above best response function for the public firm, $q_0(q_i)$, coincides with that in part (a) of Exercise 2.13, $q_0(q_i) = \frac{1-c}{2-\alpha} - \frac{1}{2-\alpha} q_i$.

(c) Evaluate the above best response functions at $\alpha = 0$, $\alpha = 1/2$, and $\alpha = 1$. Interpret.

- When $\alpha = 0$, the best response function of the public firm, $q_0(q_i)$, is

$$q_0(q_i) = \frac{1-c}{2} - \frac{N}{2} q_i$$

which becomes steeper in the number of private firms N, and coincides with the best response function in standard Cournot oligopoly models where all firms are private ($\alpha = 0$).

- When $\alpha = 1/2$, the best response function of the public firm, $q_0(q_i)$, is

$$q_0(q_i) = \frac{2(1-c)}{3} - \frac{2N}{3} q_i = 2\left[\frac{1-c}{3} - \frac{N}{3} q_i\right]$$

which lies above the best response function evaluated at $\alpha = 0$.
- When $\alpha = 1$, the best response function of the public firm, $q_0(q_i)$, becomes

$$q_0(q_i) = 1 - c - N q_i$$

which is larger than the best response function evaluated at $\alpha = 1/2$.
- In summary, when α increases from 0 to 1 the best-response function of the public firm pivots outward, becoming steeper, with its horizontal intercept being unaffected.[1]

(d) Calculate the equilibrium quantities for the private and public firms. Find the aggregate output in equilibrium. How are your results affected by a marginal increase in N? Interpret.

- *Individual output.* The equilibrium quantities solve the system of two equations:

$$q_i(q_0) = \frac{1-c}{N+1} - \frac{1}{N+1} q_0 \quad \text{and}$$
$$q_0(q_i) = \frac{1-c}{2-\alpha} - \frac{N}{2-\alpha} q_i$$

Simultaneously solving for q_0 and q_i, we find the individual output levels:

$$q_0 = \frac{1-c}{N(1-\alpha) - \alpha + 2} \quad \text{and}$$
$$q_i = \frac{(1-c)(1-\alpha)}{N(1-\alpha) - \alpha + 2}$$

Both q_0 and q_i are decreasing in N. Intuitively, as the number of private firms in market increases, the public firm and every private firm i would lower their respective output levels. Differentiating with respect to α, we obtain that

$$\frac{\partial q_0}{\partial \alpha} = \frac{(1-c)(N+1)}{[N(1-\alpha) - \alpha + 2]^2} > 0 \quad \text{and}$$
$$\frac{\partial q_i}{\partial \alpha} = -\frac{1-c}{[N(1-\alpha) - \alpha + 2]^2} < 0$$

implying that, as the public firm assigns a larger weight to welfare, it produces more units while every private firm responds producing fewer units –referred to as the "crowding out" effect in public economics.

[1] Recall that, to find the horizontal intercept, we set the equation of the best response function equal to zero, $\frac{1-c}{2-\alpha} - \frac{N}{2-\alpha} q_i = 0$. Rearranging, we obtain $1 - c = N q_i$, which solving for q_i yields $q_i = \frac{1-c}{N}$ that is independent of α.

- *Aggregate output.* Summing the above output levels, we find that aggregate output is

$$\begin{aligned} Q &= q_0 + \sum_{i=1}^{N} q_i \\ &= q_0 + N q_i \\ &= \frac{1-c}{N(1-\alpha)-\alpha+2} + N\frac{(1-c)(1-\alpha)}{N(1-\alpha)-\alpha+2} \\ &= \frac{(1-c)\left[N(1-\alpha)+1\right]}{N(1-\alpha)-\alpha+2}. \end{aligned}$$

which is increasing in the number of private firms, N, since

$$\frac{\partial Q}{\partial N} = \frac{(1-c)(1-\alpha)^2}{\left[N(1-\alpha)-\alpha+2\right]^2} > 0$$

and in the weight that the public firm assigns to welfare, α, because

$$\frac{\partial Q}{\partial \alpha} = \frac{1-c}{\left[N(1-\alpha)-\alpha+2\right]^2} > 0.$$

Intuitively, aggregate output increases in the total number of private firms N and in the weight that the public firm assigns to welfare, α.

(e) Calculate the socially optimal output level and compare it with the equilibrium outcome you obtained in part (d). How are your results affected by a marginal increase in N? Interpret.

- We know the social welfare is given by

$$W = \int_0^Q p(y)dy - cQ.$$

Differentiating with respect to Q, yields

$$\frac{\partial W}{\partial Q} = p(Q) - c = 0$$

Solving for Q, we obtain

$$p\left(Q\right) = c$$

- Hence, the socially optimal output level happens where the inverse demand function crosses the marginal cost curve, $p(Q) = c$, which in this context implies $1 - Q = c$, i.e., $Q = 1 - c$. The aggregate equilibrium output Q that we found in part (d), $\frac{(1-c)[N(1-\alpha)+1]}{N(1-\alpha)-\alpha+2}$, lies below the socially optimal level, $1 - c$, because the output difference

$$(1-c) - \frac{(1-c)\left[N(1-\alpha)+1\right]}{N(1-\alpha)-\alpha+2} = \frac{(1-c)(1-\alpha)}{N(1-\alpha)-\alpha+2}$$

is positive given that $c, \alpha \in [0,1]$ and $N \geq 1$ by assumption.

However, this output difference, $\frac{(1-c)(1-\alpha)}{N(1-\alpha)-\alpha+2}$, is decreasing in N, approaching zero when $N \to +\infty$ (perfectly competitive private market). Similarly, this output difference is decreasing in α, becoming zero at $\alpha = 1$ (when the public firm assigns complete weight to the consumer surplus and producing $q_0 = 1-c$ units).