Competition for status acquisition in public good games (Online Appendix)

Félix Muñoz-García
School of Economic Sciences
Washington State University
Pullman, WA 99164, USA
E-mail: fmunoz@wsu.edu

Appendix 2

Proof of Lemma 1

Both players are asked to simultaneously submit their voluntary contributions to the public good. Fixing subject j's contribution, g_j, we obtain player i's best response function

$$g_i(g_j) = \begin{cases} 1 \text{ if } g_j = 0 \\ 1 + \frac{\alpha_i - m}{\alpha_i + m} g_j \text{ if } g_j \in \left(0, \frac{m+\alpha_i}{m-\alpha_i}\right] \\ 0 \text{ if } g_j \in \left(\frac{m+\alpha_i}{m-\alpha_i}, +\infty\right) \end{cases}$$

if $\alpha_i < m$. Note that $0 \geq 1 + \frac{\alpha_i - m}{\alpha_i + m} g_j$ holds if, after solving for g_j, $g_j \geq \frac{m+\alpha_i}{m-\alpha_i}$. Threshold $\frac{m+\alpha_i}{m-\alpha_i}$ is positive if $\alpha_i < m$; see figure 1(a) in the paper. In contrast, when $\alpha_i > m$ this threshold is never binding for any positive g_j, i.e., g_i does not become zero or negative for any positive value of g_j, see figure 1(b).

Then, the corresponding best response function for player i in this case is

$$g_i(g_j) = \begin{cases} 1 \text{ if } g_j = 0 \\ 1 + \frac{\alpha_i - m}{\alpha_i + m} g_j \text{ if } g_j > 0 \end{cases}$$

□

Proof of Proposition 1

First, take a given player i's best response function, $g_i(g_j)$. Then, $g_i^{Sm} = 1$ only when: (1) the slope of player j's best response function, $g_j(g_i)$, is smaller than -1, and (2) the horizontal intercept of player i's best response function, $g_i(g_j)$, is higher than 1. Otherwise, both players' best response functions would cross each other in an interior point. That is, $g_i^{Sm} = 1$ if and only if $\frac{\alpha_j - m}{\alpha_j + m} \leq -1$, which holds if $\alpha_j \leq 0$. And $\frac{m+\alpha_i}{m-\alpha_i} \geq 1$ if and only if $\alpha_i > 0$

Since $\alpha_i, \alpha_j \geq 0$, the above conditions on player i and j's concerns about status are $\alpha_i \geq 0$ and $\alpha_j = 0$. Hence, $g_i^{Sm} = 1$ if and only if $\alpha_i \geq 0$ and $\alpha_j = 0$. Secondly, $g_i^{Sm} = 0$ only when the opposite happens. That is, when $\alpha_i = 0$ and $\alpha_j \geq 0$. Finally, when none of the above cases is satisfied, i.e., when $\alpha_i > 0$ and $\alpha_j > 0$, then we have an interior solution. Solving for g_i and g_j in a system of two equations, we obtain $g_i^{Sm} = \frac{\alpha_i(\alpha_j+m)}{(\alpha_i+\alpha_j)m}$, as the interior Nash equilibrium contribution level.

Sufficiency. Let us now check that the second order conditions of incentive compatibility are satisfied. Suppose all but player i submit a contribution to the public good according to the above equilibrium prediction. I next show that, for any α_i, contributor i maximizes his utility by following g_i^{Sm}. Let

$$U(g, \alpha_i) = w - g_i + \ln\left[m(g_i + g_j^{Sm}) + \alpha_i\left(g_i - g_j^{Sm}\right)\right]$$

be the utility level of player i when contributing g to the public good, and having a concern α_i about status acquisition. We must now show that the derivative $U_g(g, \alpha_i) \geq 0$ for all $g < g_i^{Sm}$, and $U_g(g, \alpha_i) \leq 0$ for all $g > g_i^{Sm}$, which imply that $U(g, \alpha_i)$ is indeed maximized at exactly $g = g_i^{Sm}$. Differentiating $U(g, \alpha_i)$ with respect to g,

$$U_g(g, \alpha_i) = -1 + \frac{\alpha_i + m}{\alpha_i\left(g - g_j^{Sm}\right) + m(g + g_j^{Sm})}$$

Let us now suppose that $g < g_i^{Sm}(\alpha_i)$, and denote $\tilde{\alpha}_i$ to be the concern about status for which the equilibrium contribution is exactly g, i.e., $g_i^{Sm}(\tilde{\alpha}_i) = g$. Since $g_i^{Sm}(\alpha_i)$ is strictly increasing in α_i (as one can check from the suggested equilibrium contribution g_i^{Sm}, and confirmed in lemma 4) this implies that $g_i^{Sm}(\alpha_i) > g_i^{Sm}(\tilde{\alpha}_i)$ if and only $\alpha_i > \tilde{\alpha}_i$. Then, $U_g(g, \tilde{\alpha}_i) \leq U_g(g, \alpha_i)$. Since by definition, $g_i^{Sm}(\tilde{\alpha}_i) = g$, it implies that $U_g(g, \tilde{\alpha}_i) = 0$. Hence, $U_g(g, \alpha_i) \geq 0$ for all $g < g_i^{Sm}$. By a similar argument, $U_g(g, \alpha_i) \leq 0$ for all $g > g_i^{Sm}$. Therefore, $U(g, \alpha_i)$ is maximized at $g = g_i^{Sm}$. □

Proof of Lemma 2

Differentiating g_i^{Sm} with respect to α_i, we obtain

$$\frac{\partial g_i^{Sm}}{\partial \alpha_i} = \begin{cases} 0 \text{ if } \alpha_i > 0 \text{ and } \alpha_j = 0 \\ \frac{\alpha_j(\alpha_j+m)}{(\alpha_i+\alpha_j)^2 m} \text{ if } \alpha_i > 0 \text{ and } \alpha_j > 0 \\ 0 \text{ if } \alpha_i = 0 \text{ and } \alpha_j > 0 \end{cases}$$

which is weakly positive for all parameter values. On the other hand, differentiating g_i^{Sm} with respect to α_j, we obtain

$$\frac{\partial g_i^{Sm}}{\partial \alpha_j} = \begin{cases} 0 \text{ if } \alpha_i > 0 \text{ and } \alpha_j = 0 \\ \frac{\alpha_i(\alpha_i - m)}{(\alpha_i+\alpha_j)^2 m} \text{ if } \alpha_i > 0 \text{ and } \alpha_j > 0 \\ 0 \text{ if } \alpha_i = 0 \text{ and } \alpha_j > 0 \end{cases}$$

which is weakly positive for all parameter values if $\alpha_i \geq m$. Finally, differentiating g_i^{Sm} with respect to α_j, we obtain

$$\frac{\partial g_i^{Sm}}{\partial \alpha_j} = \begin{cases} 0 \text{ if } \alpha_i > 0 \text{ and } \alpha_j = 0 \\ -\frac{\alpha_i \alpha_i}{(\alpha_i+\alpha_j)m^2} \text{ if } \alpha_i > 0 \text{ and } \alpha_j > 0 \\ 0 \text{ if } \alpha_i = 0 \text{ and } \alpha_j > 0 \end{cases}$$

which is weakly negative for all parameter values. □

Proof of Lemma 3

If $\alpha_i > 0$ and $\alpha_j = 0$, then from proposition 1 we know that $g_i^{Sm} = 1$ and $g_j^{Sm} = 0$. Hence, $G^{Sm} = 1$. If, on the contrary, $\alpha_i = 0$ and $\alpha_j \geq 0$, then from proposition 1 we also know that $g_i^{Sm} = 0$ and $g_j^{Sm} = 1$. Hence, $G^{Sm} = 1$ as well. Finally, if $\alpha_i > 0$ and $\alpha_j = 0$, then

$$g_i^{Sm} = \frac{\alpha_i\left(\alpha_j + m\right)}{\left(\alpha_i + \alpha_j\right)m},$$

and similarly for player j, which yields

$$G^{Sm} = 1 + \frac{2\alpha_i \alpha_j}{(\alpha_i + \alpha_j)m}.$$

Note that if status concerns (α_i, α_j) are chosen in order to maximize G^{Sm}, $\max\limits_{\alpha_i,\alpha_j \geq 0} G^{Sm}$, we obtain the following first order condition for every α_i,

$$\frac{2\alpha_j^2}{(\alpha_i + \alpha_j)^2 m} \leq 0,$$

and for α_j,

$$\frac{2\alpha_i^2}{(\alpha_i + \alpha_j)^2 m} \leq 0.$$

This gives a continuum of (α_i, α_j) pairs for which G^{Sm} is maximal at $\alpha_i = \alpha_j = \alpha$, and increasing both in α_i and in α_j. □

Proof of Proposition 2

Operating by sequential rationality, player i inserts the follower's best response function into his utility function,

$$U_i = w - g_i + \ln\left[m(g_i + g_j(g_i)) + \alpha_i\left(g_i - g_j(g_i)\right)\right],$$

which is maximized at

$$g_i^{Seq} = \begin{cases} 0 \;\text{ if }\; \alpha_i \in [0, \bar{\alpha}_i] \\ \frac{\alpha_i\alpha_j + 3\alpha_i m + \alpha_j m - m^2}{2m(\alpha_i+\alpha_j)} \text{ if } \alpha_i \in (\bar{\alpha}_i, +\infty) \end{cases}$$

where $\bar{\alpha}_i = \frac{m(m-\alpha_j)}{3m+\alpha_j}$ Given the above contribution of the first donor and $g_j(g_i)$ specified above, player j submits

$$g_j^{Seq} = \begin{cases} 1 \;\text{ if }\; \alpha_i \in [0, \bar{\alpha}_i) \\ \frac{1}{2}\left(\frac{\alpha_i\alpha_j}{(\alpha_i+\alpha_j)m} + \frac{m}{\alpha_i+\alpha_j} + \frac{4\alpha_j}{\alpha_j+m} - 1\right) \text{ if } \alpha_i \in [\bar{\alpha}_i, \hat{\alpha}_i) \\ 0 \text{ if } \alpha_i \in [\hat{\alpha}_i, +\infty) \end{cases}$$

if $\alpha_j < m$. Clearly, note that when player j's best response function is negative, i.e., $\alpha_j < m$, player j submits no positive contribution if $1 - \frac{\alpha_i - m}{\alpha_i + m} g_j \geq \frac{m+\alpha_j}{m-\alpha_j}$, or in equilibrium, when $\alpha_i \geq \hat{\alpha}_i$, where

$$\hat{\alpha}_i = \frac{m\left(3\alpha_j^2 + m^2\right)}{-\alpha_j^2 - 4\alpha_j m + m^2}$$

On the other hand, if player j's best response function is positive, $\alpha_j > m$, player j submits

$$g_j^{Seq} = \begin{cases} 1 \;\text{ if }\; \alpha_i \in [0, \bar{\alpha}_i) \\ \frac{1}{2}\left(\frac{\alpha_i\alpha_j}{(\alpha_i+\alpha_j)m} + \frac{m}{\alpha_i+\alpha_j} + \frac{4\alpha_j}{\alpha_j+m} - 1\right) \text{ if } \alpha_i \in [\bar{\alpha}_i, +\infty) \end{cases}$$

Clearly, the above two expressions for g_j^{Seq} can be simplified to

$$g_j^{Seq} = \begin{cases} 1 \;\text{ if }\; \alpha_i \in [0, \bar{\alpha}_i) \\ \frac{1}{2}\left(\frac{\alpha_i\alpha_j}{(\alpha_i+\alpha_j)m} + \frac{m}{\alpha_i+\alpha_j} + \frac{4\alpha_j}{\alpha_j+m} - 1\right) \text{ if } \alpha_j < m \text{ and } \alpha_i \in [\bar{\alpha}_i, \hat{\alpha}_i), \text{ or if } \alpha_j > m \text{ and } \alpha_i \in [\bar{\alpha}_i, +\infty) \\ 0 \text{ if } \alpha_j < m \text{ and } \alpha_i \in [\hat{\alpha}_i, +\infty) \end{cases}$$

□

Proof of Corollary 1

First result: From proposition 2, we know that player i submits strictly positive contributions if and only if

$$\alpha_i > \frac{m\left(m - \alpha_j\right)}{3m + \alpha_j}.$$

Then, if $\alpha_i = 0$, the former condition can only be satisfied if $0 > \frac{m(m-\alpha_j)}{3m+\alpha_j}$, or $\alpha_j > m$.

Second result: Since $\bar{\alpha} = \frac{m(m-\alpha_j)}{3m+\alpha_j} < m$, for any $\alpha_j \geq 0$, then if $m < \alpha_i$ we must have $\bar{\alpha} < m < \alpha_i$ for any $\alpha_j \geq 0$. Therefore, $\bar{\alpha} < \alpha_i$, and player i submits a strictly positive contribution for any concern about status player j may have, $\alpha_j \geq 0$. $\square$

Proof of Lemma 4

Differentiating g_i^{Seq} with respect to α_i, we obtain

$$\frac{\partial g_i^{Seq}}{\partial \alpha_i} = \begin{cases} 0 \text{ if } \alpha_i \in [0, \bar{\alpha}_i] \\ \frac{(\alpha_j+m)^2}{2(\alpha_i+\alpha_j)^2 m} \text{ if } \alpha_i > \bar{\alpha}_i \end{cases}$$

which is weakly positive for any parameter values. On the other hand, differentiating g_i^{Seq} with respect to α_j, we obtain$\frac{(\alpha_i-m)^2}{2(\alpha_i+\alpha_j)^2 m}$

$$\frac{\partial g_i^{Seq}}{\partial \alpha_j} = \begin{cases} 0 \text{ if } \alpha_i \in [0, \bar{\alpha}_i] \\ \frac{(\alpha_i-m)^2}{2(\alpha_i+\alpha_j)^2 m} \text{ if } \alpha_i > \bar{\alpha}_i \end{cases}$$

which is weakly positive for any parameter values. $\square$

Proof of Lemma 5

When $\alpha_i < \bar{\alpha}_i$, we know from proposition 2 that player i does not contribute, but player j responds submitting a contribution of $g_j^{Seq} = 1$. This is valid both when $\alpha_j < m$ and when $\alpha_j < m$. Then, $G^{Seq} = 1$.

In contrast, when $\alpha_i \in [\bar{\alpha}_i \hat{\alpha}_i)$ and $\alpha_j < m$ (or when $\alpha_i \in [\bar{\alpha}_i + \infty)$ and $\alpha_j > m$) from proposition 2 we know that player i submits

$$g_i^{Seq} = \frac{\alpha_i \alpha_j + 3\alpha_i m + \alpha_j m - m^2}{2m\left(\alpha_i + \alpha_j\right)}$$

while player j responds by submitting

$$g_j^{Seq} = \frac{1}{2}\left(\frac{\alpha_i \alpha_j}{(\alpha_i + \alpha_j)m} + \frac{m}{\alpha_i + \alpha_j} + \frac{4\alpha_j}{\alpha_j + m} - 1\right)$$

Then, the total contributions when $\alpha_i > \bar{\alpha}_i$ adds up to

$$G^{Seq} = \frac{2\alpha_j}{\alpha_j + m} + \frac{\alpha_i(\alpha_j + m)}{(\alpha_i + \alpha_j)m}.$$

Finally, if $\alpha_i \in [\bar{\alpha}_i + \infty)$ and $\alpha_j < m$, from proposition 2 we know that player i submits

$$g_i^{Seq} = \frac{\alpha_i\alpha_j + 3\alpha_i m + \alpha_j m - m^2}{2m\left(\alpha_i + \alpha_j\right)}$$

and player j does not submit any positive contribution (since his best response function is positively sloped and, for these parameter values, it crosses the g_i-axis), what implies

$$G^{Seq} = \frac{\alpha_i\alpha_j + 3\alpha_i m + \alpha_j m - m^2}{2m\left(\alpha_i + \alpha_j\right)}.$$

□

Proof of Lemma 6

Regarding player i, the difference between his equilibrium contribution in the simultaneous and sequential game is $\frac{(\alpha_i - m)(\alpha_j - m)}{2(\alpha_i + \alpha_j)m}$ which is positive if either $\alpha_i > m$ and $\alpha_j > m$, or if $\alpha_i < m$ and $\alpha_j < m$. Hence, if $\alpha_i > m$ and $\alpha_j > m$ (or if $\alpha_i < m$ and $\alpha_j < m$), then $g_i^{Sm} > g_i^{Seq}$. Regarding player j, the difference between his equilibrium contribution in the simultaneous and sequential game is

$$\frac{\left(\alpha_i - m\right)\left(\alpha_j - m\right)^2}{2\left(\alpha_i + \alpha_j\right)m\left(\alpha_j - m\right)}$$

which is positive if and only if $\alpha_i > m$. Hence, if $\alpha_i > m$, $g_j^{Sm} > g_j^{Seq}$. □

Proof of Proposition 3

Applying proposition 1 of Romano and Yildirim (2001), we know that whenever $1 + \frac{\partial g_j(g_i)}{\partial g_i} > 0$, the sign of $\frac{\partial U_i}{\partial g_j}\frac{\partial g_j(g_i)}{\partial g_i}$ and $G^{Seq} - G^{Sm}$ coincide. We first find $1 + \frac{\partial g_j(g_i)}{\partial g_i}$. In particular,

$$1 + \frac{\partial g_j(g_i)}{\partial g_i} = 1 + \frac{\alpha_j - m}{\alpha_j + m}$$

which is positive for any $\alpha_j > 0$. On the other hand, from corollary 1, we know that for any $i, j = \{1, 2\}$ where $j \neq i$

$$\frac{\partial U_i}{\partial g_j}\frac{\partial g_j(g_i)}{\partial g_i} = \begin{cases} > 0 \text{ if } \alpha_i < m \text{ and } \alpha_j > m, \\ < 0 \quad \text{otherwise} \end{cases}$$

Therefore, if $\alpha_i < m$ and $\alpha_j > m$ for all $i, j = \{1, 2\}$ and $j \neq i$, then $\frac{\partial U_i}{\partial g_j}\frac{\partial g_j(g_i)}{\partial g_i} > 0$ and $G^{Seq} > G^{Sm}$; and if $\alpha_i > m$ and $\alpha_j > m$ (or if $\alpha_i < m$ and $\alpha_j < m$), we have that $\frac{\partial U_i}{\partial g_j}\frac{\partial g_j(g_i)}{\partial g_i} < 0$

and $G^{Seq} < G^{Sm}$. Finally, note that if $\alpha_i = \alpha_j = 0$, then

$$\frac{\partial g_j(g_i)}{\partial g_i} = 1 + \frac{-m}{m} = 0,$$

which implies that $\frac{\partial U_i}{\partial g_j}\frac{\partial g_j(g_i)}{\partial g_i}$ becomes zero, and therefore $G^{Seq} = G^{Sm}$. This result is consistent with that in Varian (1994) where, for public good games with players without preferences for social status, he shows that $G^{Seq} \leq G^{Sm}$. $\square$

Proof of Corollary 2

For $N = 2$ players, simultaneous contribution mechanisms generate a larger total revenue than sequential mechanisms (see proposition 3). For $N \geq 3$, note that total contributions in the simultaneous-move game are $G^{Sm} = \frac{m+\alpha}{m}$ and hence are constant in the population size N. In the sequential game, total donations in the interior equilibrium (when $\alpha > \widetilde{\alpha}$) are

$$G^{Seq} = \frac{m(N-1)(4+N^2)\alpha + (N^2-2)\alpha^2 - 2m^2(N-1)^2}{(N-2)[m^2(N-1)^2 - \alpha^2]},$$

which approach $\frac{\alpha}{m}$ as $N \to \infty$, implying that $G^{Sm} > G^{Seq}$. In the corner solution (when $\alpha \leq \widetilde{\alpha}$), total contributions in the sequential game are

$$G^{Seq} = \frac{(N-1)(m+\alpha)}{[\alpha + (N-1)m]},$$

which lie weakly below $G^{Sm} = \frac{m+\alpha}{m}$ for all N . We can hence conclude that $G^{Sm} \geq G^{Seq}$. (Note that this corner solution embodies the case in which $\alpha = 0$, since $\alpha \leq \widetilde{\alpha}$, where total contributions under both the simultaneous and sequential mechanism, G^{Sm} and G^{Seq}, become $G^{Sm} = G^{Seq} = 1$ and are hence constant in the population size N). $\square$

Proof of Proposition 4

First, take a given player i's best response function, $g_i(g_j)$. Then, $g_i^{Sm,Sen} = 1 - \frac{\alpha_i S_i}{\alpha_i + m}$ only when:

1. The slope of player j's best response function, $g_j(g_i)$, is smaller than -1, and
2. The horizontal intercept of player i's best response function, $g_i(g_j)$, is higher than $1 - \frac{\alpha_j S_j}{\alpha_j + m}$.

Otherwise, both players' best response functions would cross each other in an interior point. Therefore, $g_i^{Sm,Sen} = 1 - \frac{\alpha_i S_i}{\alpha_i + m}$ if and only if

$$\begin{aligned}
\frac{\alpha_j - m}{\alpha_j + m} &\leq -1 \text{ implies } \alpha_j \leq 0, \text{ and} \\
\frac{\alpha_i S_i - \alpha_i - m}{\alpha_i - m} &\geq 1 - \frac{\alpha_j S_j}{\alpha_j + m} \text{ implies } \alpha_i \geq \frac{\alpha_j S_j m}{\alpha_j\left(S_i + S_j - 2\right) + \left(S_i - 2\right) m}
\end{aligned}$$

Since $\alpha_i, \alpha_j \geq 0$, the above conditions on player i and j's concerns about status are

$$\alpha_i \geq \frac{\alpha_j S_j m}{\alpha_j (S_i + S_j - 2) + (S_i - 2) m} \text{ and } \alpha_j = 0.$$

Secondly, $g_i^{Sm,Sen} = 0$ when the opposite happens. That is, when $\alpha_i = 0$ and $\alpha_j \geq \frac{\alpha_i S_i m}{\alpha_i (S_j + S_i - 2) + (S_j - 2) m}$. Finally, when both

$$\alpha_i \geq \frac{\alpha_j S_j m}{\alpha_j (S_i + S_j - 2) + (S_i - 2) m} \text{ and } \alpha_j \geq \frac{\alpha_i S_i m}{\alpha_i (S_j + S_i - 2) + (S_j - 2) m},$$

we have an interior solution. Solving for g_i and g_j in a system of two equations, we obtain interior solutions, and therefore,

$$g_i^{Sm,Sen} = \begin{cases} 1 - \frac{\alpha_i S_i}{\alpha_i + m} \text{ if } \alpha_i \geq \tilde{\alpha}_i \text{ and } \alpha_j \geq 0 \\ \frac{\alpha_j S_j m - \alpha_i [\alpha_j (S_i + S_j - 2) + m (S_i - 2)]}{2(\alpha_i + \alpha_j) m} \text{ if } \alpha_i \geq \tilde{\alpha}_i \text{ and } \alpha_j \geq \tilde{\alpha}_j \\ 0 \text{ if } \alpha_i \geq 0 \text{ and } \alpha_j \geq \tilde{\alpha}_j \end{cases}$$

where $\tilde{\alpha}_i = \frac{\alpha_j S_j m}{\alpha_j (S_i + S_j - 2) + (S_i - 2) m}$ and $\tilde{\alpha}_j = \frac{\alpha_i S_i m}{\alpha_i (S_j + S_i - 2) + (S_j - 2) m}$. Differentiating $g_i^{Sm,Sen}$ with respect to S_i, we find that

$$\frac{\partial g_i^{Sm,Sen}}{\partial S_i} = -\frac{\alpha_i (\alpha_j + m)}{2 (\alpha_i + \alpha_j) m}$$

which is negative for all parameter values. Similarly, differentiating $g_i^{Sm,Sen}$ with respect to S_j, yields

$$\frac{\partial g_i^{Sm,Sen}}{\partial S_j} = \frac{\alpha_j (m - \alpha_i)}{2 (\alpha_i + \alpha_j) m}$$

which is negative if and only if $m < \alpha_i$. Using the second mover's best response function, $g_j(g_i)$, from lemma 10, we know that

$$g_i(g_j) = \begin{cases} 1 - \frac{\alpha_j S_j}{\alpha_j + m} \text{ if } g_i = 0 \\ 1 - \frac{\alpha_j S_j}{\alpha_j + m} + \frac{\alpha_j - m}{\alpha_j - m} g_j \text{ if } g_i \in [0, \frac{\alpha_j S_j - \alpha_j - m}{\alpha_j - m}] \\ 0 \text{ if } g_i > \frac{\alpha_j S_j - \alpha_j - m}{\alpha_j - m} \end{cases}$$

Regarding player i, we know that he inserts the above best response function into his utility function,

$$U_i = w - g_i + \ln [m(g_i + g_j(g_i)) + \alpha_i (S_i + g_i - g_j(g_i))]$$

and differentiating with respect to g_i, and solving for g_i we obtain the following optimal contribution

$$g_i^{Seq,Sen} = \begin{cases} 0 \text{ if } \alpha_i \in [0, \alpha_i^A] \\ \frac{(\alpha_j + \alpha_j S_j - m) m - \alpha_i [\alpha_j (S_i + S_j - 1) + (S_i - 3) m]}{2(\alpha_j + \alpha_j) m} \text{ if } \alpha_i > \alpha_i^A \end{cases}$$

where $\alpha_i^A = \frac{(\alpha_j+\alpha_j S_j - m)m}{\alpha_j(S_i+S_j-1)+(S_i-3)m}$. Given the above contribution of the first mover, we can now use $g_j(g_i)$ to find player j's equilibrium contribution.

$$g_j^{Seq,Sen} = \begin{cases} 1 - \frac{\alpha_j S_j}{\alpha_j+m} \text{ if } \alpha_i \in [0, \alpha_i^A] \\ \frac{1}{2}\left[\frac{m}{\alpha_i+\alpha_j} + \frac{4\alpha_j}{\alpha_j+m} - \frac{\alpha_i+\alpha_j-\alpha_i S_i+\alpha_j S_j}{(\alpha_i+\alpha_j)m} - \frac{\alpha_i\alpha_j(S_i+S_j-4)}{\alpha_i+\alpha_j}\right] \text{ if } \alpha_i \in \left[\alpha_i^A, \alpha_i^B\right] \\ 0 \text{ if } \alpha_i > \alpha_i^B \end{cases}$$

where $\alpha_i^A = \frac{(\alpha_j+\alpha_j S_j-m)m}{\alpha_j(S_i+S_j-1)+(S_i-3)m}$ and $\alpha_i^B = \frac{m\left(m^2-(S_j-3)\alpha_j^2-\alpha_j S_j m\right)}{\alpha_j^2(S_i+S_j-1)+m\alpha_j(S_j-4)-m^2(S_i-1)}$. Differentiating $g_i^{Seq,Sen}$ and $g_j^{Seq,Sen}$ with respect to S_i and S_j, respectively, we obtain

$$\frac{\partial g_i^{Sm,Sen}}{\partial S_i} = -\frac{\alpha_i(\alpha_j+m)}{2(\alpha_i+\alpha_j)m}, \text{ and } \frac{\partial g_j^{Sm,Sen}}{\partial S_j} = -\frac{\alpha_j(\alpha_i+m)}{2(\alpha_i+\alpha_j)m}$$

which are negative for all parameter values. Similarly, differentiating $g_i^{Seq,Sen}$ and $g_j^{Seq,Sen}$ with respect to S_j and S_i, respectively, yields

$$\frac{\partial g_i^{Sm,Sen}}{\partial S_j} = \frac{\alpha_j(m-\alpha_i)}{2(\alpha_i+\alpha_j)m}, \text{ and } \frac{\partial g_j^{Sm,Sen}}{\partial S_i} = \frac{\alpha_i(m-\alpha_j)}{2(\alpha_i+\alpha_j)m}$$

which are negative if and only if $m < \alpha_i$ and $m < \alpha_j$ respectively.

Applying Romano and Yildirim (2001), we know that whenever $1 + \frac{\partial g_j(g_i)}{\partial g_i} > 0$, the sign of $\frac{\partial U_i}{\partial g_j}\frac{\partial g_j(g_i)}{\partial g_i}$ and $G^{Seq} - G^{Sm}$ coincide. We first find $1 + \frac{\partial g_j(g_i)}{\partial g_i}$. In particular,

$$1 + \frac{\partial g_j(g_i)}{\partial g_i} = 1 + \frac{\alpha_j - m}{\alpha_j + m}$$

which is positive for any $\alpha_j > 0$. On the other hand,

$$\frac{\partial U_i}{\partial g_j} = \frac{-\alpha_i + m}{\alpha_i(S_i + g_i - g_j) + m(g_i + g_j)}$$

which is negative if and only if $\alpha_i > m$. Then, from corollary 1, we know that for all $j \neq i$

$$\frac{\partial U_i}{\partial g_j}\frac{\partial g_j(g_i)}{\partial g_i} = \begin{cases} > 0 \text{ if } \alpha_i < m \text{ and } \alpha_j > m, \\ < 0 \quad \text{otherwise} \end{cases}$$

Therefore, if $\alpha_i < m$ and $\alpha_j > m$ for all $j \neq i$, then $\frac{\partial U_i}{\partial g_j}\frac{\partial g_j(g_i)}{\partial g_i} > 0$ and $G^{Seq} > G^{Sm}$; and if $\alpha_i > m$ and $\alpha_j > m$ (or if $\alpha_i < m$ and $\alpha_j < m$), then $\frac{\partial U_i}{\partial g_j}\frac{\partial g_j(g_i)}{\partial g_i} < 0$ and $G^{Seq} < G^{Sm}$ □