Example #3 from Chapter 9 in JR

Player *i*'s reported type	1	2	3	4	5	6	7	8	9
Expected payment	10/9	2/3	1/3	1/9	0	0	1/9	1/3	2/3

Player *i* is pivotal for building the swimming pool

With *N*=2 players, we know that individual *i* is pivotal for the swimming pool (S) if aggregate surplus when we consider his preferences satisfies

$$(\theta_i - 5) + (\theta_j - 5) \leq 0 \Leftrightarrow \theta_i \leq 10 - \theta_j,$$

but the surplus of all other individuals (which in this case is only the utility of individual *j*) satisfies

$$\theta_j - 5 > 0 \Leftrightarrow \theta_j > 5.$$

Intuitively, the above conditions say that society would choose S when considering *i*'s preferences but the bridge (B) when ignoring his preferences. Note that conditions $\theta_i \leq 10 - \theta_j$ and $\theta_j > 5$ are compatible when $\theta_i \leq 5$. To see this point, depict both equations in the (θ_i, θ_j)-quadrant, shading the areas that each condition identifies, and then see for which region of (θ_i, θ_j)-pairs both areas superimpose.

We analyze each entry of the table separately:

- $\theta_i = 1$, player *i* can be pivotal only for S, the possible values of θ_j that satisfy the above two conditions are $\{6,7,8,9\}$. Therefore, individual *i*'s expected externality (and his payment to *j*) is
$$\bar{t}_i^{VCG} = \frac{1}{9}(6-5) + \frac{1}{9}(7-5) + \frac{1}{9}(8-5) + \frac{1}{9}(9-5) = \frac{10}{9}$$
- $\theta_i = 2$, player *i* can be pivotal only for S, the possible values of θ_j that satisfy the above two conditions are $\{6,7,8\}$. Therefore, individual *i*'s expected externality (and his payment to *j*) is
$$\bar{t}_i^{VCG} = \frac{1}{9}(6-5) + \frac{1}{9}(7-5) + \frac{1}{9}(8-5) = \frac{2}{3}$$
- $\theta_i = 3$, player *i* can be pivotal only for S, the possible values of θ_j that satisfy the above two conditions are $\{6,7\}$. Thus, individual *i*'s expected externality (and his payment to *j*) is
$$\bar{t}_i^{VCG} = \frac{1}{9}(6-5) + \frac{1}{9}(7-5) = \frac{1}{3}$$
- $\theta_i = 4$, player *i* can be pivotal only for S, the possible values of θ_j that satisfy the above two conditions is $\{6\}$. Therefore, individual *i*'s expected externality (and his payment to *j*) is
$$\bar{t}_i^{VCG} = \frac{1}{9}(6-5) = \frac{1}{9}$$

Player *i* is pivotal for building the bridge

Similarly, individual *i* is pivotal for the bridge (B) if aggregate surplus when we consider his preferences satisfies

$$(\theta_i - 5) + (\theta_j - 5) > 0 \Leftrightarrow \theta_i > 10 - \theta_j,$$

but the surplus of all other individuals (which in this case is only the utility of individual *j*) satisfies

$$\theta_j - 5 \leq 0 \Leftrightarrow \theta_j \leq 5.$$

Intuitively, the above conditions say that society would choose S when considering *i*'s preferences but the bridge (B) when ignoring his preferences. Note that conditions $\theta_i > 10 - \theta_j$ and $\theta_j \leq 5$ are compatible when $\theta_i \geq 5$. To see this point, depict both equations in the (θ_i, θ_j)-quadrant, shading the areas that each condition identifies, and then see for which region of (θ_i, θ_j)-pairs both areas superimpose.

We analyze each entry of the table separately:

- $\theta_i = 5$, player *i* can be pivotal only for B, but there is no possible value of θ_j that satisfies both above conditions. Therefore, individual *i*'s expected externality (and payment to player *j*) is

$$\bar{t}_i^{VCG} = 0$$

- $\theta_i = 6$, player *i* can be pivotal only for B. The only possible value of θ_j that satisfies the above two conditions is $\{5\}$. Therefore, individual *i*'s expected externality is

$$\bar{t}_i^{VCG} = \frac{1}{9}(5-5) = 0$$

- $\theta_i = 7$, player *i* can be pivotal only for B. In this case, the only possible values of θ_j that satisfy the above two conditions are $\{4,5\}$. Therefore, individual *i*'s expected externality is

$$\bar{t}_i^{VCG} = \frac{1}{9}(5-5) + \frac{1}{9}(5-4) = \frac{1}{9}$$

- $\theta_i = 8$, player *i* can be pivotal only for B. In this setting, the only possible values of θ_j that satisfy the above two conditions are $\{3,4,5\}$. Therefore, individual *i*'s expected externality is

$$\bar{t}_i^{VCG} = \frac{1}{9}(5-5) + \frac{1}{9}(5-4) + \frac{1}{9}(5-3) = \frac{1}{3}$$

- $\theta_i = 9$, player *i* can be pivotal only for B. In this setting, the only possible values of θ_j that satisfy the above two conditions are $\{2,3,4,5\}$. Therefore, individual *i*'s expected externality is

$$\bar{t}_i^{VCG} = \frac{1}{9}(6-5) + \frac{1}{9}(5-4) + \frac{1}{9}(5-3) + \frac{1}{9}(5-3) = \frac{2}{3}$$