EconS 503 - Microeconomic Theory II
Homework #8 - Answer key

1. **First and second welfare theorems - An example.** Consider two individuals who trade two goods, x and y, in an economy without production. Every individual i's utility function is $u^i(x_1^i, x_2^i) = x_1^i x_2^i$. Endowments are $\mathbf{e}^A = (500, 100)$ for consumer A and $\mathbf{e}^B = (100, 350)$ for consumer B.

   (a) Find the WEA.

   - From consumer A's UMP, we have that $MRS_{1,2}^A = \frac{p_1}{p_2}$, which yields $\frac{x_2^A}{x_1^A} = \frac{p_1}{p_2}$, or $p_2 x_2^A = p_1 x_1^A$. From his budget constraint, we have that

   $$p_1 x_1^A + p_2 x_2^A = p_1 500 + p_2 100,$$

   or $p_1 x_1^A = p_1 500 + p_2 100 - p_2 x_2^A$. Inserting this result into his tangency condition, yields

   $$p_2 x_2^A = p_1 500 + p_2 100 - p_2 x_2^A$$

   which simplifies to $2p_2 x_2^A = p_1 500 + p_2 100$, or $x_2^A = \frac{p_1}{p_2} 250 + 50$.

   - Similarly, from consumer B's UMP, we have that $MRS_{1,2}^B = \frac{p_1}{p_2}$, which yields $\frac{x_2^B}{x_1^B} = \frac{p_1}{p_2}$, or $p_2 x_2^B = p_1 x_1^B$. From his budget constraint, we have that

   $$p_1 x_1^B + p_2 x_2^B = p_1 100 + p_2 350,$$

   or $p_1 x_1^A = p_1 500 + p_2 100 - p_2 x_2^A$. Inserting this result into his tangency condition, yields

   $$p_2 x_2^B = p_1 100 + p_2 350 - p_2 x_2^B$$

   which simplifies to $2p_2 x_2^B = p_1 100 + p_2 350$, or $x_2^B = \frac{p_1}{p_2} 50 + 175$.

   - Inserting our results from the two UMPs into the feasibility requirement for good 2, we obtain that

   $$\underbrace{\left(\frac{p_1}{p_2} 250 + 50\right)}_{x_2^A} + \underbrace{\left(\frac{p_1}{p_2} 50 + 175\right)}_{x_2^B} = 450$$

   Solving for $\frac{p_1}{p_2}$, we obtain the equilibrium price ratio, $\frac{p_1}{p_2} = \frac{3}{4} = 0.75$.

   - We can finally insert this equilibrium price ratio into our above results, to find the equilibrium amounts of good 2 that each individual consumes, that is,

   $$\begin{aligned} x_2^A &= \frac{p_1}{p_2} 250 + 50 = \frac{475}{2} \simeq 237.5, \text{ and} \\ x_2^B &= \frac{p_1}{p_2} 50 + 175 = \frac{425}{2} \simeq 212.5. \end{aligned}$$

To find the amounts of good 1 that they consume, we can use the tangency condition $p_2 x_2^B = p_1 x_1^B$, or $\frac{p_2}{p_1} x_2^B = x_1^B$, which yields $x_1^B = \frac{4}{3}\frac{425}{2} = \frac{850}{3}$ units. Using the feasibility requirement for good 1, $x_1^A + x_1^B = 600$, we obtain that $x_1^A + \frac{850}{3} = 600$, yielding $x_1^A = \frac{950}{3}$ units.

- Summarizing, the WEA is

$$\left(x_1^A, x_2^A, x_1^B, x_2^B; \frac{p_1}{p_2}\right) = \left(\frac{950}{3}, \frac{475}{2}, \frac{850}{3}, \frac{425}{2}; \frac{3}{4}\right).$$

(b) Find the set of PEAs.

- Setting the tangency condition $MRS_{1,2}^A = MRS_{1,2}^B$ yields $\frac{x_2^A}{x_1^A} = \frac{x_2^B}{x_1^B}$, or after cross-multiplying, $x_2^A x_1^B = x_2^B x_1^A$. The feasibility requirement for good 1 says $x_1^A + x_1^B = 600$, or $x_1^B = 600 - x_1^A$, and similarly the feasibility requirement for good 2 says $x_2^A + x_2^B = 450$, or $x_2^B = 450 - x_2^A$. Inserting these feasibility equations into the tangency condition $x_2^A x_1^B = x_2^B x_1^A$, yields

$$x_2^A \underbrace{(600 - x_1^A)}_{x_1^B} = \underbrace{(450 - x_2^A)}_{x_2^B} x_1^A,$$

which simplifies to $600x_2^A - x_1^A x_2^A = 450x_1^A - x_2^A x_1^A$, and finally to

$$x_2^A = \frac{450}{600} x_2^A = \frac{3}{4} x_1^A.$$

Therefore, efficient allocations satisfy

$$x_2^A = \frac{3}{4} x_1^A \quad \text{where } x_1^A \in [0, 600].$$

For example, bundle $x_1^A = 200$ and $x_2^A = 150$ leaves consumer B with $x_1^B = 600 - 200 = 400$ units of good 1, and $x_2^B = 450 - 150 = 300$ units of good 2.

(c) *First welfare theorem.* Show that the WEA found in part (a) is a PEA.

- The WEA found in part (a) satisfies the efficiency condition $x_2^A = \frac{3}{4} x_1^A$, since $x_2^A = \frac{3}{4} \cdot \frac{950}{3} = \frac{472}{2}$.

(d) *Second welfare theorem.* Consider that the social planner seeks to implement an allocation where individual A enjoys $x_1^A = 200$ and $x_2^A = 150$, while individual B enjoys $x_1^B = 400$ and $x_2^B = 300$. How could this allocation be implemented by a social planner? [*Hint*: Find the pair of taxes and subsidies (t_A, t_B) such that $t_A + t_B = 0$.]

- *Consumer A.* Now we need to find the redistribution of initial endowment that can lead to such an allocation emerging in equilibrium. We know that for consumer A that $p_1 x_1^A = p_2 x_2^A$ and for consumer B that $p_1 x_1^B = p_2 x_2^B$. We want to now tax consumer B, $t_B < 0$, with the amount being transferred to consumer A, $t_A > 0$ so that $t_A = -t_B$. Therefore, consumer A's budget constraint after including t_A is

$$p_1 x_1^A + p_2 x_2^A = p_1 e_1^A + p_2 e_2^A + t_A,$$

which, after substituting her original endowment $(e_1^A, e_2^A) = (500, 100)$ and that $p_1 x_1^A = p_2 x_2^A$, becomes

$$2p_1 x_1^A = 500p_1 + 100p_2 + t_A$$

Solving for x_1^A, we obtain

$$x_1^A = 250 + 50\frac{p_2}{p_1} + \frac{t_A}{2p_1}.$$

We take this expression for consumer A and insert the specific efficient allocation that we seek to implement; that is, $(x_1^A, x_2^A, x_1^B, x_2^B) = (200, 150, 400, 300)$, insert the price ratio we found in the previous problem; that is, $\frac{p_2}{p_1} = \frac{4}{3}$, and normalize the price of good 2, so that $p_2 = 1$ and $p_1 = \frac{3}{4}$ in equilibrium. Doing this, we obtain:

$$200 = 250 + 50\frac{4}{3} + \frac{t_A}{2\frac{3}{4}},$$

and now we want to solve for t_A:

$$\begin{aligned} 200 &= 250 + 50\frac{4}{3} + \frac{t_A}{2\frac{3}{4}} \\ -50 &= 66.67 + \frac{t_A}{\frac{3}{2}} \\ -116.67 &= \frac{2}{3}t_A \\ -175 &= t_A, \end{aligned}$$

which means we tax consumer A, since $t_A < 0$.

- *Consumer B*. We apply a similar argument to consumer B, so her budget constraint as a function of the tax t_B she faces is

$$p_1 x_1^B + p_2 x_2^B = p_1 e_1^B + p_2 e_2^B + t_B,$$

which, after substituting her endowment, and $p_1 x_1^B = p_2 x_2^B$, we get that

$$2p_1 x_1^B = p_1 100 + p_2 350 + t_B.$$

Solving for x_1^B, we get that

$$x_1^B = 50 + 175\frac{p_2}{p_1} + \frac{t_B}{2p_1},$$

and then inserting the specific efficient allocation that we seek to implement; that is, $(x_1^A, x_2^A, x_1^B, x_2^B) = (200, 150, 400, 300)$, insert the price ratio we found in the previous problem; that is, $\frac{p_2}{p_1} = \frac{4}{3}$, and normalize the price of good 2, so that $p_2 = 1$ and $p_1 = \frac{3}{4}$ in equilibrium. Doing this, we obtain:

$$400 = 50 + 175\frac{4}{3} + \frac{t_B}{2\frac{3}{4}},$$

and solving for t_B, we get that

$$\begin{aligned} 350 &= 233.33 + \frac{t_B}{\frac{3}{2}} \\ 116.76 &= \frac{2}{3} t_B \\ 175 &= t_B. \end{aligned}$$

which is subsidy to consumer B. This coincides with the tax imposed on consumer A.

2. **Exercises 14 from Munoz-Garcia (2017).** Consider an exchange economy with two consumers, A and B, whose utility functions are

$$\begin{aligned} u_A(x_1^A, x_2^A) &= x_1^A x_2^A \\ u_B(x_1^B, x_2^B) &= x_1^B (x_2^B)^2 \end{aligned}$$

with endowments $e^A = (80, 150)$ and $e^B = (210, 180)$ respectively. Assume that consumer A is price setter, i.e., he makes a take-it-or-leave-it price offer to consumer B.

(a) Find the Walrasian Equilibrium allocation (WEA) in this economy.

- Consumer B takes the price ratio announced by consumer A as given, and solves his UMP

$$\begin{aligned} \max_{x_1^B, x_2^B} \quad u_B(x_1^B, x_2^B) &= x_1^B (x_2^B)^2 \\ \text{subject to } p_1 x_1^B + p_2 x_2^B &\leq 210 p_1 + 180 p_2 \end{aligned}$$

His Lagrangian is

$$\mathcal{L} = x_1^B (x_2^B)^2 - \lambda [p_1 x_1^B + p_2 x_2^B - 210 p_1 - 180 p_2]$$

Taking FOCs yields

$$\begin{aligned} \frac{d\mathcal{L}}{dx_1^B} &= (x_2^B)^2 - \lambda p_1 = 0 \;\Rightarrow \lambda = \frac{(x_2^B)^2}{p_1} \\ \frac{d\mathcal{L}}{dx_2^B} &= 2 x_1^B x_2^B - \lambda p_2 = 0 \;\Rightarrow\; \lambda = \frac{2 x_1^B x_2^B}{p_2} \end{aligned}$$

Combining the FOCs, we obtain

$$\frac{p_1}{p_2} = \frac{x_2^B}{2 x_1^B}$$

Before plugging this result into consumer's budget constraint, $p_1 x_1^B + p_2 x_2^B = 210 p_1 + 180 p_2$, we can divide such a constraint by p_2 to obtain

$$\frac{p_1}{p_2} x_1^B + x_2^B = 210 \frac{p_1}{p_2} + 180 \Rightarrow \frac{p_1}{p_2}(x_1^B - 210) = 180 - x_2^B$$

We can now substitute $\frac{p_1}{p_2} = \frac{x_2^B}{2x_1^B}$ in the left term,

$$\frac{x_2^B}{2x_1^B}(x_1^B - 210) = 180 - x_2^B$$

which, solving for x_2^B, yields

$$x_2^B = \frac{360x_1^B}{3x_1^B - 210}$$

which constitutes the offer curve of consumer B.

- Consumer A anticipates this offer curve of consumer B, along with the following feasibility conditions

$$\begin{aligned} x_1^B &= 290 - x_1^A \quad \text{for good 1, and} \\ x_2^B &= 330 - x_2^A \quad \text{for good 2} \end{aligned}$$

From our above result of the offer curve of consumer B, $x_2^B = \frac{360x_1^B}{3x_1^B - 210}$, the feasibility condition for good 2 can be rewritten as

$$\frac{360x_1^B}{3x_1^B - 210} = 330 - x_2^A$$

which can be rearranged as

$$360x_1^B - 990x_1^B + 3x_1^B x_2^A + 69300 - 210x_2^A = 0$$

Substituting the feasibility condition for good 1, we obtain,

$$360(290 - x_1^A) - 990(290 - x_1^A) + 3(290 - x_1^A)x_2^A + 69300 - 210x_2^A = 0$$

and simplifying, yields an expression that is a function of x_1^A and x_2^A alone, that is,

$$630x_1^A - 3x_2^A x_1^A + 660x_2^A - 113,400 = 0$$

Hence, consumer A's problem becomes

$$\max_{x_1^A, x_2^A} \quad u_A(x_1^A, x_2^A) = x_1^A x_2^A$$
$$\text{subject to } 630x_1^A - 3x_2^A x_1^A + 660x_2^A - 113,400 = 0$$

with associated Lagrangian

$$\mathcal{L} = x_1^A x_2^A - \lambda[630x_1^A - 3x_2^A x_1^A + 660x_2^A - 113,400]$$

Taking FOCs yields

$$\begin{aligned} \frac{d\mathcal{L}}{dx_1^A} &= x_2^A - 630\lambda + 3x_2^A\lambda = 0 \;\Leftrightarrow \lambda = \frac{x_2^A}{630 - 3x_2^A} \\ \frac{d\mathcal{L}}{dx_2^A} &= x_1^A - 660\lambda + 3x_1^A\lambda = 0 \;\Longleftrightarrow \lambda = \frac{x_1^A}{660 - 3x_1^A} \end{aligned}$$

Setting the above FOCs equal to each other, we obtain

$$x_2^A(660 - 3x_1^A) = x_1^A(630 - 3x_2^A)$$
$$\implies x_2^A = \frac{630}{660}x_1^A$$

Plugging this result in the constraint of consumer A, we find that

$$630x_1^A - 2.86(x_1^A)^2 + 630x_1^A - 113,400 = 0$$

Finally, solving for x_1^A yields two roots, $x_1^A = 126.08$ and $x_1^A = 314.48$, but the second root is infeasible since it exceeds the total endowment of the good. Hence, $x_1^A = 126.08$ implying that the amount of good 2 for this consumer is

$$x_2^A = \frac{630}{660}x_1^A = \frac{630}{660} \times 126.09 = 120.35$$

Using the feasibility conditions, we can obtain the equilibrium consumption bundle of individual B,

$$x_1^B = 290 - x_1^A \Rightarrow x_1^B = 163.92$$

and

$$x_2^B = 330 - x_2^A = 209.65$$

In summary, the WEA is

$$\left(x_1^A, x_2^A; x_1^B, x_2^B\right) = (126.08, 120.35; 163.92, 209.65)\,.$$

(b) Find the Pareto optimal allocation (PEA) in this economy, and check if the WEA from part (a) is a PEA.

- For a PEA, we need

$$MRS_{1,2}^A = MRS_{1,2}^B$$

which in this setting entails

$$\frac{x_2^A}{x_1^A} = \frac{x_2^B}{2x_1^B}$$

Using the feasibility conditions,

$$\begin{aligned} x_1^B &= 290 - x_1^A \\ x_2^B &= 330 - x_2^A \end{aligned}$$

Plugging x_1^B and x_2^B in terms of x_1^A and x_1^A in the $MRS_{1,2}^A = MRS_{1,2}^B$ condition, we obtain

$$\frac{x_2^A}{x_1^A} = \frac{330 - x_2^A}{2(290 - x_1^A)}$$

Rearranging, we find that the contract curve describing all PEAs is given by

$$580x_2^A - 330x_1^A - x_1^A x_2^A = 0$$

Plugging the WEA found in part (a) in this equation, we find that

$$580x_2^A - 330x_1^A - x_1^A x_2^A = 13,022.87 \neq 0$$

entailing that the WEA is not Pareto optimal, i.e., the WEA does not lie on the contract curve. Hence, the presence of market power (with one individual being the price setter) prevents the First Theorem of Welfare Economics from holding.

3. **Exercises 22 from Munoz-Garcia (2017).** Consider a two-commodity economy where the price of commodity 1 is normalized in terms of commodity 2, whereby $\frac{p_1}{p_2} = p$. Suppose the excess demand function for commodity 1 is given by

$$z_1(p) = 1 - 4p + 5p^2 - 2p^3$$

(a) How many equilibria can you find?

- The excess demand for commdity 1 at relative price ρ can be written

$$z_1(p) = 1 - 4p + 5p^2 - 2p^3 = (1-p)^2(1-2p)$$

so that $z_1(p) = 0$ holds at two equilibrium prices, $p_1 = 1$ and $p_2 = \frac{1}{2}$, i.e., the two roots of $(1-p)^2(1-2p) = 0$. Figure 6.17 plots $z_1(p)$ as a function of price ratio p.

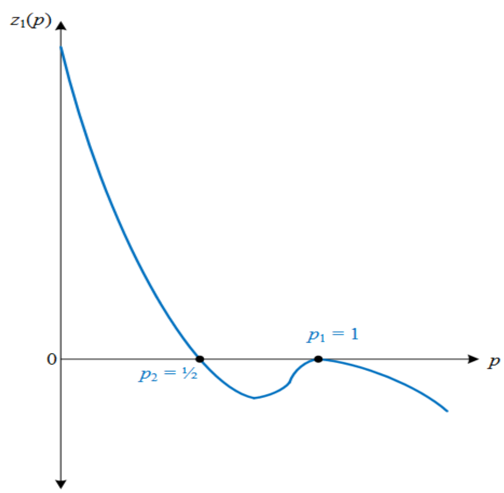

Figure 6.17. Excess demand $z_1(p)$.

(b) Which of the equilibrium price ratios you found are stable?

- In order to test whether an equilibrium price p is stable, we need that $\frac{\partial z_1(p)}{\partial p} < 0$ so the excess demand function crosses the horizontal axis from above. Intuitively, for $p < p_i^*$ there is excess demand while for $p > p^*$ there is excess supply. Since $\frac{\partial z_1(p)}{\partial p} = -4 + 10p - 6p^2$, evaluating this derivative at each of the equilibrium prices p_1 and p_2 we obtain that
  - At $p_1 = 1$, $\left.\frac{\partial z_1(p)}{\partial p}\right|_{p_1=1} = -4 + 10 - 6 = 0$. As depicted in the figure of part (a), this equilibrium is unstable. (To be precise, it is only locally stable from above.)

– At $p_2 = \frac{1}{2}$, $\left.\frac{\partial z_1(p)}{\partial p}\right|_{p_2=\frac{1}{2}} = -4 + 10\left(\frac{1}{2}\right) - 6\left(\frac{1}{2}\right)^2 = -\frac{1}{2} < 0$; and thus the equilibrium is stable, i.e., the excess demand function crosses the horizontal axis from above, as depicted in the figure.

(c) Consider now that the aggregate endowment of good 1 increases. How are your results from parts (a) and (b) affected?

- An increase in the aggregate endowment of good 1, ω_1, reduces the excess demand of this good, $z_1(p) = x_1(p) - \omega_1$, where $x_1(p)$ denotes aggregate demand for good 1. Hence, $z_1(p)$ experiences a downward shift, as depicted in Figure 6.18. While equilibrium price $p_2 = \frac{1}{2}$ moves leftward, the equilibrium price $p_1 = 1$ is absent. Therefore, the only equilibrium is stable.

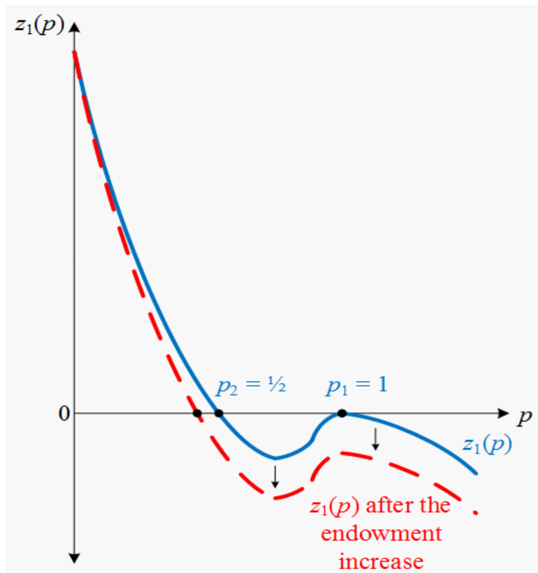

Figure 6.18. Excess demand and Stability.

4. **Exercises from MWG:**

   (a) Chapter 23 (mechanism design): Exercise 23.C.10.

5. **Public Good Provision - Different mechanisms.** Suppose that you and your colleagues want to buy a coffee machine for your office, with some of you willing to pay more for the machine than others. However, willingness to pay is privately observed. The cost of the machine is C. We would like to find a decision rule in which: (i) each individual reports a valuation (i.e., direct mechanism); and (ii) the coffee maker is purchased if and only if it is efficient to do so. Let us next analyze if it is possible to find a cost-sharing rule which gives incentive for everyone to truthfully report his valuation.

   In particular, assume n individuals, each with private valuation θ_i, where $\theta_i \sim U(0,1)$. The allocation function is binary $y \in \{0,1\}$, i.e., the coffee machine is purchased or not. Let t_i be the transfer from individual i, implying a utility of

$$u_i(y, \theta_i, t_i) = y\theta_i - t_i$$

Let $i \in \{1, ..., n\}$ denote the individuals, and let $i = 0$ denote the original owner of the good (the store).

(a) What is the efficient assignment rule, $y^*(\theta_1, ..., \theta_n)$?

- The efficient assignment rule is

$$y^*(\theta_1, ..., \theta_n) = \begin{cases} 1 & \text{if } \sum_{j=1}^{n} \theta_j \geq C \\ 0 & \text{otherwise} \end{cases}$$

In words, the coffee machine is purchased if and only if the sum of all valuations exceeds its total cost.

(b) *Equal-share rule.* Consider the following equal-share rule: When the public good is provided, the cost is equally divided by all n individuals.

1. Before starting any computation, what would you expect - whether each individual would overstate or understate their valuation?
   - Because of free-rider incentives, each individual may have an incentive to understate his valuation. The equal-share payment rule, however, makes transfers independent of his report.
2. Confirm that the transfer rule is written by:

$$t_i(\theta) = \frac{C}{n} y^*(\theta)$$

   - By the equal-share rule, each individual will pay $\frac{C}{n}$ if the project happens, and 0 otherwise. Hence, the transfer rule is

$$t_i(\theta) = \frac{C}{n} y^*(\theta)$$

3. Let $V_i(\tilde{\theta}_i|\theta_i, \theta_{-i})$ be individual i's payoff when i reports $\tilde{\theta}_i$ instead of his true valuation θ_i, while the others truthfully report their valuations θ_{-i}. Show that

$$V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = \left(\theta_i - \frac{C}{n}\right) y^*(\tilde{\theta}_i, \theta_{-1})$$

   - Using the definition of player i's utility function, we can intert in the above equal-share transfer rule to obtain

$$\begin{aligned} V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) &= \theta_i y^*(\tilde{\theta}_i, \theta_{-1}) - t_i^*\left(\tilde{\theta}_i, \theta_{-i}\right) \\ &= \theta_i y^*(\tilde{\theta}_i, \theta_{-1}) - \frac{C}{n} y^*(\tilde{\theta}_i, \theta_{-1}) \\ &= \left(\theta_i - \frac{C}{n}\right) y^*(\tilde{\theta}_i, \theta_{-1}) \end{aligned}$$

4. Let $U_i\left(\tilde{\theta}_i|\theta_i\right)$ be individual i's expected payof when he reports $\tilde{\theta}_i$ instead of the true valuation θ_i. Show that

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = \left(\theta_i - \frac{C}{n}\right) E_{\theta_{-i}}\left[y^*(\tilde{\theta}_i, \theta_{-1})\right]$$

- Player i's expected payoff for misreporting $\tilde{\theta}_i \neq \theta_i$ is just the expected value of the utility found above, that is

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = E_{\theta_{-i}}\left[V_i(\tilde{\theta}_i|\theta_i, \theta_{-i})\right] = \left(\theta_i - \frac{C}{n}\right) E_{\theta_{-i}}\left[y^*(\tilde{\theta}_i, \theta_{-1})\right]$$

5. Suppose that i's private valuation θ_i satisfies $\theta_i > \frac{C}{n}$. Assuming that the others are telling the truth, what is the best response for i? What if $\theta_i < \frac{C}{n}$? Is this mechanism strategy-proof? Is this mechanism Bayesian incentive compatible?
   - If player i's valuation θ_i satisfies $\theta_i > \frac{C}{n}$, $U_i\left(\tilde{\theta}_i|\theta_i\right)$ is maximized when $E_{\theta_{-i}}\left[y^*(\tilde{\theta}_i, \theta_{-1})\right]$ is maximized. Hence, individual i would report $\tilde{\theta}_i$ as large as possible, i.e., $\tilde{\theta}_i = 1$. In contrast, if θ_i satisfies $\theta_i < \frac{C}{n}$, individual i would report $\tilde{\theta}_i$ as small as possible, i.e., $\tilde{\theta}_i = 0$. The mechanism is neither strategy-proof, nor Bayesian incentive compatible.

(c) *Proportional payment rule.* Consider now the proportional payment rule:

$$t_i(\theta) = \frac{\theta_i C}{\sum_j \theta_j} y^*(\theta)$$

where every individual i pays a share of the total cost equal to the proportion that his reported valuation signifies out of the total reported valuations.

1. Under this rule, what would you expect - whether each individual would overstate or understate the valuation?
   - Now the payment is a function of the report. Notice that this cost-sharing rule is balanced-budget. Hence, you may expect that the agents have incentive to free-ride.
2. Show that the utility of reporting $\tilde{\theta}_i$ is now

$$V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = \left(\theta_i - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-1})$$

   - The payoff to each individual will be their actual valuation, less the amount the have to pay based on what they report if the project happens, and 0 otherwise. That is,

$$\begin{aligned} V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) &= \theta_i y^*(\tilde{\theta}_i, \theta_{-1}) - t_i^*\left(\tilde{\theta}_i, \theta_{-i}\right) \\ &= \theta_i y^*(\tilde{\theta}_i, \theta_{-1}) - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j} y^*(\tilde{\theta}_i, \theta_{-1}) \\ &= \left(\theta_i - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-1}) \end{aligned}$$

3. For simplicity, suppose two individuals, $n = 2$ and a total cost of $C = 1$. Show that

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = \tilde{\theta}_i\left(\theta_i - \log\left(\tilde{\theta}_i + 1\right)\right)$$

- Again, by definition, the expected utility of misreporting $\tilde{\theta}_i$ is

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = E_{\theta_{-i}}\left[\left(\theta_i - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-1})\right]$$

Suppose now that $n = 2$ and $C = 1$. Then the above expression becomes

$$\begin{aligned} U_i\left(\tilde{\theta}_i|\theta_i\right) &= E_{\theta_{-i}}\left[\left(\theta_i - \frac{\tilde{\theta}_i}{\tilde{\theta}_i + \theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-1})\right] \\ &= \int_{1-\tilde{\theta}_i}^{1}\left(\theta_i - \frac{\tilde{\theta}_i}{\tilde{\theta}_i + \theta_j}\right) d\theta_j \\ &= \left[\theta_i\theta_j - \tilde{\theta}_i \log(\tilde{\theta}_i + \theta_j)\right]_{1-\tilde{\theta}_i}^{1} \\ &= \theta_i - \tilde{\theta}_i \log(\tilde{\theta}_i + 1) - \left[\theta_i(1-\tilde{\theta}_i) - \tilde{\theta}_i \log(\tilde{\theta}_i + (1-\tilde{\theta}_i))\right] \\ &= \tilde{\theta}_i\left(\theta_i - \log(\tilde{\theta}_i + 1)\right) \end{aligned}$$

4. Is this mechanism strategy-proof? Is it Bayesian incentive compatible?
   - It is straightforward to show that the expected utility of reporting $\tilde{\theta}_i$ is decreasing in player i's report $\tilde{\theta}_i$, since

   $$\frac{\partial}{\partial\tilde{\theta}_i} U_i\left(\tilde{\theta}_i|\theta_i\right)\Big|_{\tilde{\theta}_i=\theta_i} = \frac{\theta_i^2}{1+\theta_i} - \log(\theta_1 + 1) < 0 \text{ for all } \theta_i \in (0,1]$$

   implying that every player i has incentives to underreport his true valuation θ_i as much as possible, i.e., $\tilde{\theta}_i = 0$. Hence, this mechanism is neither strategy-proof nor Bayesian incentive compatible.
5. Which way is everyone biased, overstate or understate? What is the intuition?
   - The negative sign in part (iv) suggests that $U_i\left(\tilde{\theta}_i|\theta_i\right)$ is maximized at $\tilde{\theta}_i$ smaller than θ_i. Each individual has an incentive to understate the valuation.

(d) *VCG mechanism.* Let us consider now the VCG mechanism. Recall that the efficient rule $y^*(\theta)$ determines that the coffee machine is bought if and only if total valuations satisfy $\sum_i \theta_i \geq C$. Remember that we need to include the original owner of the public good; $i = 0$. Then, the total surplus when the valuation of individual i is considered in $\theta = (\theta_1, \theta_2, ..., \theta_n)$ is

$$\sum_{j\neq i} v_j\left(y^*(\theta), \theta_j\right) = \begin{cases} \sum_{j\neq i}\theta_j & \text{if } \sum_j \theta_j \geq C \\ C & \text{if } \sum_j \theta_j < C \end{cases}$$

while total surplus when the valuation of individual i is ignored, θ_{-i}, is

$$\sum_{j\neq i} v_j\left(y^*(\theta_{-i}), \theta_j\right) = \begin{cases} \sum_{j\neq i}\theta_j & \text{if } \sum_{j\neq i} \theta_j \geq C \\ C & \text{if } \sum_{j\neq i} \theta_j < C \end{cases}$$

The only difference in total surplus arises from the allocation rule which specifies that, when θ_i is considered, the good is purchased if and only if $\sum_j \theta_j \geq C$, whereas when θ_i is ignored, the good is bought if and only if $\sum_{j \neq i} \theta_j \geq C$. Hence, the VCG transfer is

$$\begin{aligned} t_i^*(\theta) &= -\left(\sum_{j \neq i} v_j(y^*(\theta), \theta_j) - \sum_{j \neq i} v_j(y^*(\theta_{-i}), \theta_j) \right) \\ &= \begin{cases} C - \sum_{j \neq i} \theta_j & \text{if } \sum_{j \neq i} \theta_j < C \leq \sum_j \theta_j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Intuitively, player i pays the difference between everyone else's valuations, $\sum_{j \neq i} \theta_j$, and the total cost of the good, C. Such a payment, however, only occurs when aggregate valuations exceed the total cost, $\sum_j \theta_j \geq C$, and thus the good is purchased, and when the valuations of all other players do not yet exceed the total cost of the good, $\sum_{j \neq i} \theta_j < C$, so the difference $C - \sum_{j \neq i} \theta_j$ is paid by player i in his transfer.

1. Show that in this mechanism player i's utility from reporting a valuation $\tilde{\theta}_i \neq \theta_i$ is

$$\begin{aligned} V_i(\tilde{\theta}_i | \theta_i, \theta_{-i}) &= v_i\left(y^*\left(\tilde{\theta}_i, \theta_{-i}\right), \theta_i\right) - t_i^*\left(\tilde{\theta}_i, \theta_{-i}\right) \\ &= \begin{cases} 0 & \text{if } \tilde{\theta}_i + \sum_{j \neq i} \theta_j < C \\ \sum_j \theta_j - C & \text{if } \sum_{j \neq i} \theta_j < C \leq \tilde{\theta}_i + \sum_{j \neq i} \theta_i \\ \theta_i & \text{if } C \leq \sum_{j \neq i} \theta_j \end{cases} \end{aligned}$$

   - This is just the definition of the payoff function for the VCG.

2. Is this mechanism strategy-proof? Is this Bayesian incentive compatible?
   - In order to test if this direct revelation mechanism is strategy-proof,
   - Suppose that $C \leq \sum_{j \neq i} \theta_j$, i.e., the public good will be purchased regardless of individual i's reported valuation. Then $V_i(\tilde{\theta}_i | \theta_i, \theta_{-i}) = \theta_i$, which is independent of player i's reported valuation, $\tilde{\theta}_i$. Hence, telling the truth is player i's best response.
   - Now suppose that $\sum_{j \neq i} \theta_j < C \leq \sum_j \theta_j$, i.e., individual i's valuation is pivotal. Then by reporting a valutation $\tilde{\theta}_i$ such that $\tilde{\theta}_i \geq C - \sum_{j \neq i} \theta_j$, his utility becomes $V_i(\tilde{\theta}_i | \theta_i, \theta_{-i}) = \sum_j \theta_j - C \geq 0$. This includes the case of telling the truth; $\tilde{\theta}_i = \theta_i \geq C - \sum_{j \neq i} \theta_j$. If, instead, individual i lies by reporting $\tilde{\theta}_i < C - \sum_{j \neq i} \theta_j$, then his utility becomes $V_i(\tilde{\theta}_i | \theta_i, \theta_{-i}) = 0$ since the good is not purchased given that $\tilde{\theta}_i < C - \sum_{j \neq i} \theta_j$ entails $\tilde{\theta}_i + \sum_{i \neq j} \theta_j < C$. Hence, misreporting his valuation cannot be profitable.
   - Finally, suppose that $\sum_j \theta_j < C$, i.e., the public good will not be purchased regardless of individual i's valuation. Then, by honestly revealing his valuation, $\tilde{\theta}_i = \theta_i < C - \sum_{j \neq i} \theta_j$, his payoff is $V_i(\tilde{\theta}_i | \theta_i, \theta_{-i}) = 0$ since the good is not purchased. By lying, $\tilde{\theta}_i \geq C - \sum_{j \neq i} \theta_j$, his payoff is $V_i(\tilde{\theta}_i | \theta_i, \theta_{-i}) = \sum_j \theta_j - C < 0$. Telling a lie is then not profitable. Hence,

truth-telling is the best strategy for i, regardless of the values of θ_{-i}. The VCG mechanism is thus strategy-proof, and also Bayesian incentive compatible.

3. For simplicity, suppose two individuals, $n = 2$, and a total cost of $C = 0.5$. Compute y^*, t_1^* and t_2^* for the following (θ_1, θ_2) pairs.

θ_1	θ_2
0.1	0.3
0.3	0.3
0.3	0.8
0.8	0.8

- For the case of $\theta_1 = 0.1$ and $\theta_2 = 0.3$, we have that VCG transfers become

$$t_i^*(\theta) = \begin{cases} -\sum_{j\neq i}\theta_j + C & \text{if } \sum_{j\neq i}\theta_j < C \leq \sum_j \theta_j \\ 0 & \text{otherwise} \end{cases}$$

implying that the transfer player 1 pays is

$$t_1(\theta) = \begin{cases} -0.3 + 0.5 & \text{if } 0.3 < 0.5 \leq 0.4 \\ 0 & \text{otherwise} \end{cases}$$

and the transfer that player 2 pays is

$$t_2(\theta) = \begin{cases} -0.1 + 0.5 & \text{if } 0.1 < 0.5 \leq 0.4 \\ 0 & \text{otherwise} \end{cases}$$

As we can see, the upper inequality does not hold, and thus the good is not purchased, $y^*(\theta) = 0$, and transfers are zero, $t_1^*(\theta) = t_2^*(\theta) = 0$. Following the same steps, the results for valuation pairs $(0.3, 0.3)$, $(0.3, 0.8)$, and $(0.8, 0.8)$ are presented in the following table

θ_1	θ_2	$y^*(\theta)$	$t_1^*(\theta)$	$t_2^*(\theta)$
0.1	0.3	0	0	0
0.3	0.3	1	0.2	0.2
0.3	0.8	1	0	0.2
0.8	0.8	1	0	0

4. Show that the expected revenue from this mechanism is $E\left[t_1^*(\theta_1, \theta_2) + t_2^*(\theta_1, \theta_2)\right] = \frac{1}{6} \simeq 0.167$. Based on what you calculated in part (iii), is this problematic?
   - If $\theta_2 \geq C$, then player 1 doesn't need to pay anything $t_1^* = 0$. If $\theta_2 < C$, then player 1's transfer is $t_1^* = -\theta_2 + C$ if and only if $\theta_1 + \theta_2 \geq C$. Hence, player 1's expected transfer is

$$\begin{aligned} E_\theta\left[t_1^*(\theta_1, \theta_2)\right] &= \int_{\{(\theta_1,\theta_2)|\theta_1+\theta_2 \geq C\}} (-\theta_2 + C)d\theta_1 d\theta_2 \\ &= \int_0^C \int_{-\theta_2+C}^1 (-\theta_2 + C)d\theta_1 d\theta_2 = \frac{1}{12} \end{aligned}$$

By symmetry, $E_\theta\left[t_2^*(\theta_1, \theta_2)\right] = \frac{1}{12}$, entailing that expected revenue becomes

$$E_\theta\left[t_1^*(\theta_1, \theta_2) + t_2^*(\theta_1, \theta_2)\right] = \frac{1}{6} \simeq 0.167$$

This is problematic, because the expected revenue, 0.167, is smaller than the total cost, 0.5, implying a budget deficit. The VCG mechanism has two nice properties: efficiency and incentive compatibility. However, balanced budget condition and participation constraint are not necessarily satisfied.

we know that:

Homework #8 - Answer Key

$$\frac{\partial v(k(r,\theta_{-i}),r)}{\partial k} \geq \frac{\partial v(k(r,\theta_{-i}),\theta_i)}{\partial k} \quad \text{for all} \quad r \geq \theta_i. \qquad \text{(vi)}$$

using (v) and (vi) we get:

$$u(\theta_i,\hat{\theta}_i) - u(\theta_i,\theta_i) \leq \int_{\theta}^{\hat{\theta}} \left[\frac{\partial v(k(r,\theta_{-i}),r)}{\partial k} \cdot \frac{\partial k(r,\theta_{-i})}{\partial r} + \frac{\partial t(r,\theta_{-i})}{\partial r}\right] dr = 0$$

because the bracketed term equals zero for all r (see equation (23.C.12)). This, however, contradicts our negation assumption that $u(\theta_i,\hat{\theta}_i)-u(\theta_i,\theta_i)>0$ so $f(\cdot)$ must be truthfully implementable.

Case 2: Suppose $\hat{\theta}_i<\theta_i$. We can proceed as before, however the inequality in (vi) above will be reversed, and we will have a minus sign before the integral, so we will get the same contradiction.

23.C.10 [First Printing Errata: At the end of the first paragraph insert: "Assume throughout that conditions are such that (23.C.8) holding is a necessary condition for $(k^*(\cdot),t_1(\cdot),\ldots,t(\cdot)_I)$ to be truthfully implementable in dominant strategies." Also, in the second line of part c) insert the word "implementable" before "ex post efficient social choice function".]

a) Sufficiency: Suppose that we can write $V^*(\theta) = \sum_i V_i(\theta_{-i})$. Consider the transfer functions of the form

$$t_i(\theta) = \left[\sum_{j\neq i} v_j(k^*(\theta),\theta_j)\right] + h_i(\theta_{-i}) .$$

where for all i,

$$h_i(\theta_{-i}) = -(I - 1)V_i(\theta_{-i}) \quad \text{for all } \theta_{-i} .$$

By proposition 23.C.4, $(k^*(\cdot),t_1(\cdot),\ldots,t(\cdot)_I)$ is truthfully implementable in dominant strategies. Moreover, for all θ we have,

$$\sum_i t_i(\theta) = \sum_i \left[\sum_{j\neq i} v_j(k^*(\theta),\theta_j)\right] + \sum_i h_i(\theta_{-i})$$

$$= (I - 1)V^*(\theta) - (I - 1)\sum_i V_i(\theta_{-i}) = 0$$

Necessity: Suppose $(k^*(\cdot),t_1(\cdot),\ldots,t(\cdot)_I)$ is ex post efficient and is truthfully implementable in dominant strategies. Since (23.C.8) is necessary (by assumption) for truthful implementation, this means that there exist functions $\{h_i(\theta_{-i})\}_{i=1}^I$ such that

$$(I - 1)V^*(\theta) + \sum_i h_i(\theta_{-i}) = \sum_i \left[\sum_{j\neq i} v_j(k^*(\theta),\theta_j)\right] + \sum_i h_i(\theta_{-i})$$

$$= \sum_i t_i(\theta) = 0$$

But this implies that by defining

$$V_i(\theta_{-i}) = \left(\frac{-1}{I - 1}\right)h_i(\theta_{-i}) ,$$

we can then write $V^*(\theta) = \sum_i V_i(\theta_{-i})$.

b) If $v_i(k,\theta_i) = \theta_i k - \frac{1}{2} k^2$ for all i, then, $k^*(\theta) = \text{Argmax}_k(\sum_i\theta_i)k - \frac{3}{2} k^2$ for all θ, and so the FOC implies that $k^*(\theta) = \frac{\sum_i\theta_i}{3}$. Hence,

$$V^*(\theta) = \sum_{i=1}^{3}\left[\theta_i\left(\frac{\sum_i\theta_i}{3} \right) - \frac{1}{2}\left(\frac{\sum_i\theta_i}{3} \right)^2 \right]$$

$$= \left(\frac{\sum_i\theta_i}{3} \right)\sum_i\left[\theta_i - \frac{1}{2}\left(\frac{\sum_i\theta_i}{3} \right) \right]$$

$$= (\theta_1+ \theta_2+ \theta_3)[\theta_1+ \theta_2+ \theta_3 - \tfrac{1}{2}(\theta_1+ \theta_2+ \theta_3)]$$

$$= \tfrac{1}{2}(\textstyle\sum_i\theta_i)^2$$

$$= (\theta_1^2 + \theta_2^2 + \theta_3^2 + 2\theta_1\theta_2 + 2\theta_1\theta_3 + 2\theta_2\theta_3) .$$

We now define,

$$V_1(\theta_2,\theta_3) = \frac{\theta_2^2 + \theta_3^2}{2} + 2\theta_2\theta_3 ,$$

$$V_2(\theta_1,\theta_3) = \frac{\theta_1^2 + \theta_3^2}{2} + 2\theta_1\theta_3 ,$$

$$V_3(\theta_1,\theta_2) = \frac{\theta_1^2 + \theta_2^2}{2} + 2\theta_1\theta_2 \ ,$$

and the result then follows from part a) above since

$$V^*(\theta) = V_1(\theta_2,\theta_3) + V_2(\theta_1,\theta_3) + V_3(\theta_1,\theta_2) \ .$$

c) If $V^*(\theta) = \sum_i V_i(\theta_{-i})$ then clearly $\dfrac{\partial^I V^*(\theta)}{\partial\theta_1\cdots\partial\theta_I} = 0$.

d) In this case, $V^*(\theta_1,\theta_2) = v_1(k^*(\theta),\theta_1) + v_2(k^*(\theta),\theta_2)$, therefore,

$$\frac{\partial V^*}{\partial\theta_1} = \left(\frac{\partial v_1}{\partial k} + \frac{\partial v_2}{\partial k}\right)\frac{\partial k}{\partial\theta_1} + \frac{\partial v_1}{\partial\theta_1} \ ,$$

$$\frac{\partial^2 V^*}{\partial\theta_1\partial\theta_2} = \left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right)\left(\frac{\partial k}{\partial\theta_1}\right)\left(\frac{\partial k}{\partial\theta_2}\right) + \frac{\partial^2 v_2}{\partial k\partial\theta_2}\cdot\frac{\partial k}{\partial\theta_1} + \frac{\partial^2 v_1}{\partial k\partial\theta_1}\cdot\frac{\partial k}{\partial\theta_2} .$$

Since,

$$\frac{\partial v_1}{\partial k} + \frac{\partial v_2}{\partial k} = 0 \ ,$$

we have,

$$\frac{\partial^2 v_i}{\partial k\partial\theta_i} = -\frac{\partial k}{\partial\theta_i}\left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right),$$

which in turn implies that

$$\frac{\partial^2 V^*}{\partial\theta_1\partial\theta_2} = -\left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right)\left(\frac{\partial k}{\partial\theta_1}\right)\left(\frac{\partial k}{\partial\theta_2}\right) \neq 0 \ ,$$

thus proving the statement.

23.C.11 Let agent i's Bernoulli utility function be $u_i(v_i(k,\theta_i) + \bar{m}_i + t_i$ and assume in negation that Proposition 23.C.4 no longer holds. That is, there exists i, θ_i, $\hat{\theta}_i$, and θ_{-i} such that:

$$u_i(v_i(k^*(\hat{\theta}_i,\theta_{-i}),\theta_i) + \bar{m}_i + t_i(\hat{\theta}_i,\theta_{-i}) > u_i(v_i(k^*(\theta),\theta_i) + \bar{m}_i + t_i(\theta)$$

Substituting from (23.C.8) we get:

$$u_i(v_i(k^*(\hat{\theta}_i,\theta_{-i}),\theta_i)+\bar{m}_i+\sum_{j\neq i} v_j(k^*(\hat{\theta}_i,\theta_{-i}),\theta_j) + h_i(\theta_{-i})) >$$