EconS 503 – Homework #7

Answer Key

Exercise #1 - Nonlinear pricing

1. The monopoly choses p to maximize $\pi = (p - c)(1 - p)$. The profit-maximizing price is easily found as $p^U = \frac{1+c}{2}$. The corresponding profit is $\pi^U = \frac{(1-c)^2}{4}$.

2. Facing a tariff (m, p), the participation constraint of a consumer is $CS(p) - m \geq 0$, where $CS(p)$ is the consumer surplus price p. With demand $Q(p) = 1 - p$, we have $CS(p) = \frac{(1-p)^2}{2}$. As the monopolist's best interest is to set $m = CS(p)$, its problem is thus to set p so as to maximize

$$\pi = CS(p) + (p - c)(1 - p) = \left(\frac{1}{2}\right)(1 - p)(1 + p - 2c).$$

The first-order condition yields $p - c = 0$. Hence, the optimal two-part tariff is $(m, p) = \left(\frac{(1-c)^2}{2}, c\right)$. The optimal two-part tariff consists of selling the good at marginal cost, so as to generate the largest consumer surplus, and in capturing this surplus fully through the fixed fee. The corresponding profit is $\pi^{TP} = m = \frac{(1-c)^2}{2}$. Welfare is maximized under the two-part tariff as it involves marginal cost pricing (whereas uniform pricing results in a deadweight loss).

3. Consumer surplus for agents of type 2 at any price $p \leq 2$ is computed as $CS_2(p) = \frac{(2-p)^2}{4}$. We recall from part (2) of the exercise that $CS_1(p) = \frac{(1-p)^2}{2}$ for any $p \leq 1$. For any price where the two types of consumer buy (i.e., $p \leq 1$), we check that $CS_2(p) \geq CS_1(p)$. One option for the monopolist is to make sure that consumers of both types buy; for this, $m = CS_1(p)$ and $p \leq 1$. We then have the same problem as in part (2), with $c = \frac{1}{2}$: $(m, p) = \left(\frac{1}{8}, \frac{1}{2}\right)$ and $\pi_1 = \frac{1}{8}$. The alternative option is to sell only to type-2 consumers with $m = CS_2(p)$ and $p \leq 2$.

The monopolist's problem is then to choose p to maximize

$$\pi_2 = (1/2)\frac{(2-p)^2}{4} + (p - 1/2)(1 - p/2).$$

The first-order condition yields $p = 1/2$. As $p = 1/2$ (i.e., marginal cost pricing violates the constraint), the monopolist chooses a price of $p = 1$, so that $m = 1/4$. The resulting profit is $\pi_2 = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$. Therefore, $\pi_2 > \pi_1$, implying that the monopolist prefers to sell to type-2 consumers only by setting a price above marginal cost.

Homework #2 – Moral Hazard

a) The contract specifies the payment w_S you receive when you deliver the beer, and the payment w_F you receive when you do not deliver the beer. The two inequalities are the incentive compatibility constraint and your friend's individual rationality constraint:

$$0.9(-e^{-0.2w_S}) + 0.1(-e^{-0.2w_F}) \geq -e^{-0.2(w_F+5)} \qquad \text{(IC)}$$

$$0.9(8 - w_S) + 0.1(-w_F) \geq 3 \qquad \text{(IR)}$$

The solution to these equations (when they hold with equality) is $w_S = 4.81$ and $w_F = -1.25$.

b) The two inequalities are the incentive compatibility constraint and your own individual rationality constraint:

$$0.9(-e^{-0.2w_S}) + 0.1(-e^{-0.2w_F}) \geq -e^{-0.2(w_F+5)} \qquad \text{(IC)}$$

$$0.9(-e^{-0.2w_S}) + 0.1(-e^{-0.2w_F}) \geq -1 \qquad \text{(IR)}$$

We make these equalities, and solve the two equations. The two imply that $-e^{-.2(w_F+5)} = -1$, and so (taking logs $0.2(w_F + 5) = 0$. Hence, $w_F = -5$. From (IR),

$$0.9(-e^{-0.2w_S}) + 0.1\left(-e^{-0.2(-5)}\right) = -1$$

$$e^{-0.2w_S} = \frac{1 - 0.1e}{0.9} = 0.809$$

$$-0.2w_S = \log(0.809) = -0.212$$

$$w_S = 1.06$$

In the first-best contract, it is observable whether the beer is drunken or confiscated, and so that agreement can require that the beer is delivered. The payment is constant because the risk-neutral party should bear all the risk. The payment is the solution to your individual rationality constraint:

$$-e^{-0.2(w)} = 1.$$

Hence, $w = 0$.
Your utility is -1 in both the first-best and second-best contracts, because it is assumed here that your friend gets all the gains from trade. Your friend's utility is higher for the first-best than for the second-best agreements:

Case	Utility
1st-best	$0.9 (8) + 0.1 (0) = 7.2$
2nd-best	$0.9 (8 - 1.06) + 0.1 (0 - (-5)) = 6.75$

Maximizing with respect to $R(x)$ for every positive x leads to the following first order conditions

$$\frac{\partial \mathcal{L}}{\partial R(x)} = 0 \Leftrightarrow \frac{1}{u'(W_0 + R(x) - x)} = \frac{1}{\lambda} + \frac{\mu}{\lambda} \frac{p'(a)}{p(a)} \text{ for } \forall x > 0$$

$$\frac{\partial \mathcal{L}}{\partial R(0)} = 0 \Leftrightarrow \frac{1}{u'(W_0 + R(0))} = \frac{1}{\lambda} - \frac{\mu}{\lambda} \frac{p'(a)}{1 - p(a)}$$

where we assumed that $p(a)$ never equals to 0 or 1. In other words, we assume that it is extremely costly for the agent to ensure there will be no loss whatsoever, but extremely cheap to make sure that the loss does not occur with probability 1. Given the positive Lagrange multipliers and $p'(a) < 0$, we find for $\forall x > 0$

$$\frac{1}{u'(W_0 + R(0))} > \frac{1}{\lambda} > \frac{1}{u'(W_0 + R(x) - x)}$$

$\Leftrightarrow$

$$R(0) > R(x) - x.$$

So the optimal contract is such that the insuree is punished when a loss occurs, but the punishment is not related to the size of the loss x since the probability of a larger loss is independent of the agent's effort. Therefore, the second-best contract will induce variation in the insuree's payoff only between 0 and $x > 0$. The exact shape of the contract will be fully determined by the first order condition with respect to effort and the zero-profit condition.

4.3 Question 12

Consider a principal-agent problem with three exogenous states of nature, θ_1, θ_2, and θ_3; two effort levels, a_L and a_H; and two output levels, distributed as follows as a function of the state of nature and the effort level:

State of nature	θ_1	θ_2	θ_3
Probability	0.25	0.5	0.25
Output under a_H	18	18	1
Output under a_L	18	1	1

The principal is risk neutral while the agent has utility function $\sqrt{w}$ when receiving monetary compensation w, minus the cost of effort, which is normalized to 0 for a_L and to 0.1 for a_H. The agent's reservation expected utility is 0.1.

1. Derive the first-best contract.

2. Derive the second-best contract when only output levels are observable.

3. Assume the principal can buy for a price of 0.1 an information system that allows the parties to verify whether state of nature θ_3 happened or not. Will the principal buy this information system? Discuss.

4.3.1 First-Best Contract

The probabilities are as follows:

$$\Pr\left(q = 18|a_H\right) = 0.75$$
$$\Pr\left(q = 18|a_L\right) = 0.25.$$

Thus, when the principal can contract on the level of effort, he will solve

$$\max_{a_i,w} 0.25q(a_i;\theta_1) + 0.5q(a_i;\theta_2) + 0.25q(a_i;\theta_3) - w$$

subject to

$$\sqrt{w} - \psi(a_i) \geq 0.1$$

Under action choice a_H,

$$\sqrt{w_H^*} - 0.1 = 0.1 \Rightarrow w_H = 0.04.$$

Therefore,

$$E\pi_P^*(a_H) = 18\,(0.75) + 1\,(0.25) - 0.04 = 13.71.$$

Under action choice a_L,

$$\sqrt{w_L^*} = 0.1 \Rightarrow w_L^* = 0.01.$$

Therefore,

$$E\pi_P^*(a_L) = 18\,(0.25) + 1\,(0.75) - 0.01 = 5.24.$$

Thus, the optimal contract specifies $a^* = a_H$ and $w^* = 0.04$.

4.3.2 Second-Best Contract

When only output levels are observable, the optimal contract to induce the effort level a_H solves a minimization of the wage bill,

$$\min_{w_{18},w_1} 0.75w_{18} + 0.25w_1$$

subject to

$$0.75\sqrt{w_{18}} + 0.25\sqrt{w_1} - 0.1 \geq 0.1 \tag{IR}$$
$$0.75\sqrt{w_{18}} + 0.25\sqrt{w_1} - 0.1 \geq 0.25\sqrt{w_{18}} + 0.75\sqrt{w_1}. \tag{IC}$$

We can prove that in the two outcomes/two actions case both (IR) and (IC) must be binding at the optimum. If (IR) is not binding, the expected wage bill can be decreased by lowering w_1 whereas the (IC) will even be relaxed. If (IC) is not binding, but the (IR) is, then we can decrease w_{18} and increase w_1 such that the (IR) is still binding. In particular, given

$$\frac{dw_1}{dw_{18}} = -3\frac{\sqrt{w_1}}{\sqrt{w_{18}}},$$

this would change the expected wage bill by

$$0.75dw_{18} + 0.25(-3\frac{\sqrt{w_1}}{\sqrt{w_{18}}})dw_{18}.$$

By (IC), we have $w_1 < w_{18}$. Therefore, a decrease in w_{18} leads to a negative change in the wage bill,

$$0.75(1 - \sqrt{\frac{w_1}{w_{18}}})dw_{18} < 0$$

With the binding restrictions (IR) and (IC), it is easy to find the optimal wage schedule

$$\begin{aligned} w_1 &= 0.0025 \\ w_{18} &= 0.0625. \end{aligned}$$

Note that the limited liability constraint implied by the utility function $\sqrt{w}$ is not binding here. The expected utility of the principal is

$$E\pi_P^{**}(a_H) = 13.75 - 0.0625 \times 0.75 - 0.0025 \times 0.25 = 13.7025.$$

So the expected profit of inducing a_H is greater than the expected profit of inducing a_L, regardless of whether the effort level is observable or not.

4.3.3 Information System

For this specific price the answer is trivially no. The principal is not willing to buy the information system, because even with perfect information the highest profit he can attain is 13.71. Therefore, the maximal gain of using the information system will always be smaller than the cost of implementing the system,

$$13.71 - 13.7025 = 0.0075 < 0.1.$$

We now find the profits the principal can make when state θ_3 can be observed. Then we can find for which price the principal would be willing to pay for the information system. Let us denote by y the signal that takes the value 1 when θ_3 is realized and the value 0 otherwise.

If $y = 1$, the principal can still not identify the effort level being exerted. However, when $y = 0$, only the low effort level can lead to an output level of 1. So by penalizing the agent infinitely when $q = 1$ and $y = 0$, the first-best can be attained. Indeed, the agent will choose the high effort level to be sure to avoid the punishment. Given that no variation in received income is needed, the high effort can be implemented at the lowest cost. However, given the liability constraint implied by the utility function, the principal is restricted to positive levels of compensation. Hence, the principal will pay $w_1 = 0$ when the output is 1 and the signal is 0. In addition, w_{18} and w_{θ_3}, the wages paid when $q = 18$ and $y = 1, q = 1$ respectively, are set to solve

$$\min_{w_{18}, w_{\theta_3}} 0.75w_{18} + 0.25w_{\theta_3}$$

subject to

$$0.75\sqrt{w_{18}} + 0.25\sqrt{w_{\theta_3}} - 0.1 \geq 0.1 \tag{IR}$$

$$0.75\sqrt{w_{18}} + 0.25\sqrt{w_{\theta_3}} - 0.1 \geq 0.25\sqrt{w_{18}} + 0.75\left(\frac{2}{3}w_1 + \frac{1}{3}\sqrt{w_{\theta_3}}\right). \tag{IC}$$

The incentive compatibility constraint is binding and simplifies to

$$0.75\sqrt{w_{18}} - 0.1 = 0.25\sqrt{w_{18}}.$$

Hence,
$$w_{18} = 0.04.$$

and
$$w_{\theta_3} = \left(\frac{0.2 - 0.75 \times 0.2}{0.25}\right)^2 = 0.04.$$

So the first-best can be achieved by the contract

$$w_{18} = w_{\theta_3} = 0.04 \text{ and } w_1 = 0.$$

Finally, from the calculation above, the principal will implement the information system if the cost falls below 0.0075.

4.4 Question 13

Consider the modified linear managerial-incentive-scheme problem, where the manager's effort, a affects current profits, $q_1 = a + \varepsilon_{q_1}$, and future profits, $q_2 = a + \varepsilon_{q_2}$, where ε_{q_t} are i.i.d. with normal distribution $N(0, \sigma_q^2)$. The manager retires at the end of the first period, and the manager's compensation cannot be based on q_2. However, her compensation can depend on the stock price $P = 2a + \varepsilon_P$, where $\varepsilon_P \sim N(0, \sigma_P^2)$. Derive the optimal compensation contract $t = w + fq_1 + sP$. Discuss how it depends on σ_P^2 and on its relation with σ_q^2. Compare your solution with that in the Chapter.

4.4.1 CEO Compensation

Note that this question uses notation that is not consistent with the exposition in section 4.6.1. The program for this problem is the following,

$$\max_{a,w,f,s} E\left(q_1 + q_2 - t\right)$$

subject to

$$E\left(-e^{-\eta[t-\psi(a)]}\right) \geq -e^{-\eta\bar{t}} \tag{IR}$$

$$a \in \arg\max_{\hat{a}} E\left(-e^{-\eta[t-\psi(\hat{a})]}\right) \tag{IC}$$

$$t = w + fq_1 + sP$$

$\Leftrightarrow$

$$\max_{a,w,f,s} 2a - (w + fa + 2sa)$$

subject to

$$w + fa + 2sa - \frac{\eta}{2}\left(f^2\sigma_q^2 + s^2\sigma_P^2 + 2sf\sigma_{qP}\right) - \frac{c}{2}a^2 \geq \bar{t}$$

$$a \in \arg\max_{\widehat{a}} w + f\widehat{a} + 2s\widehat{a} - \frac{\eta}{2}\left(f^2\sigma_q^2 + s^2\sigma_P^2 + 2sf\sigma_{qP}\right) - \frac{c}{2}\widehat{a}^2.$$

We can apply the first-order approach and substitute the first order condition

$$a = \frac{f + 2s}{c}$$

for the incentive compatibility constraint. Introducing this efficient level of effort and substituting the outside opportunity level plus the risk premium plus the cost of effort for the wage, we obtain

$$\max_{f,s} 2\frac{f + 2s}{c} - \frac{\eta}{2}\left(f^2\sigma_q^2 + s^2\sigma_P^2 + 2sf\sigma_{qP}\right) - \frac{(f + 2s)^2}{2c} - \bar{t}.$$

The first order conditions with respect to f and s are respectively

$$\frac{2}{c} - \eta\left(f^*\sigma_q^2 + s^*\sigma_{qP}\right) - \frac{f^* + 2s^*}{c} = 0$$

$$\frac{4}{c} - \eta\left(s^*\sigma_P^2 + f^*\sigma_{qP}\right) - 2\frac{f^* + 2s^*}{c} = 0.$$

After some rewriting, the equations become

$$f^* = \frac{2 - s^*(2 + \eta c\sigma_{qP})}{1 + \eta c\sigma_q^2}$$

$$f^* = \frac{2 - s^*(2 + \frac{\eta c\sigma_P^2}{2})}{1 + \frac{\eta c\sigma_{qP}}{2}},$$

and we finally find

$$f^* = \frac{\sigma_P^2 - 2\sigma_{qP}}{2\sigma_q^2 + \frac{\sigma_P^2}{2} - 2\sigma_{qP} + \frac{\eta c}{2}(\sigma_P^2\sigma_q^2 - \sigma_{qP}^2)}$$

$$s^* = \frac{2\sigma_q^2 - \sigma_{qP}}{2\sigma_q^2 + \frac{\sigma_P^2}{2} - 2\sigma_{qP} + \frac{\eta c}{2}(\sigma_P^2\sigma_q^2 - \sigma_{qP}^2)}.$$

4.4.2 Comparison

Compared to the example in Chapter 4, the importance of the agent's effort has doubled for the principal. Among the contractible variables, only the impact of a change in effort on the stock price has doubled. Although the agent's effort

now determines the output in two periods, the output in the second period is not contractible. The stock price therefore becomes a more precise indicator of effort than the contractible output. If we consider the relative size of the incentives at the optimum

$$\frac{f^*}{s^*} = \frac{\sigma_P^2 - 2\sigma_{qP}}{2\sigma_q^2 - \sigma_{qP}}$$

and compare this to the equivalent ratio in Chapter 4[1]

$$\frac{f^*}{s^*} = \frac{\sigma_P^2 - \sigma_{qP}}{\sigma_q^2 - \sigma_{qP}},$$

we notice that the principal puts relatively more weight on the stock prices. This weight is exactly double when both indicators are independent, $\sigma_{qP} = 0$.

4.5 Question 14

Consider the following principal-agent problem. There is a project whose probability of success is a (a is also the effort made by the risk-neutral agent, at cost a^2). In case of success the return is R, and in case of failure the return is 0. The parameter R can take two values, X with probability λ and 1 with probability $1 - \lambda$. To undertake the project, the agent needs to borrow an amount I from the principal. The sequence of events is as follows:

- First, the principal offers the agent a debt contract, with face value D_0. The agent accepts or rejects this contract.

- Second, nature determines the value R that would occur in case of success. This value is observed by both principal and agent. The principal can then choose to lower the debt from D_0 to D_1.

- The agent chooses a level of effort a. This level is not observed by the principal.

- The project succeeds or not. If the project succeeds, the agent pays the minimum of R and the face value of debt D_1.

Answer the following questions:

1. Compute the subgame-perfect equilibrium of this game as a function of I, λ, and X.

2. When do we have $D_1 < D_0$? Discuss.

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[1] In section 4.6.1 f and s and w and t are interchanged.