EconS 503 - Microeconomic Theory II
Homework #6 - Answer key

1. **Exercises from textbooks:**

   (a) MWG, Chapter 9: Exercise 9.C.7.

   (b) Bolton and Dewatripont, Exercise 6 (see page 650 since all exercises are at the end of the book).

   - See answer key at the end of this handout.

2. **Entry deterrence with a sequence of potential entrants.** The following entry model is inspired on the original paper of Kreps and Wilson (*JET*, 1982). Consider an incumbent monopolist building a reputation as a tough competitor who does not allow entry without a fight. The entrant first decides whether to enter the market, and, if he does, the monopolist chooses whether to fight or acquiesce. If the entrant stays out, the monopolist obtains a profit of $a > 1$, and the entrant gets 0. If the entrant enters, the monopolist gets 0 from fighting and -1 from acquiescing if he is a "tough" monopolist, and -1 from fighting and 0 from acquiescing if he is a "normal" monopolist. The entrant obtains a profit of b if the monopolist acquiesces and $b-1$ if he fights, where $0 < b < 1$. Suppose the entrant believes the monopolist to be tough (normal) with probability p ($1-p$, respectively), while the monopolist observes his own type.

   (a) Depict a game tree representing this incomplete information game.

   (b) Solve for the PBE of this game.

   (c) Suppose the monopolist faces two entrants in sequence, and the second entrant observes the outcome of the first game (there is no discounting). Depict the game tree, and solve for the PBE. [*Hint:* you can use backward induction to reduce the game tree as much as possible before checking for the existence of separating or pooling PBEs. For simplicity, focus on the case in which prior beliefs satisfy $p \leq b$.]

   - See answer key at the end of this handout.

3. **Signaling when the expert receives imprecise signals.** Consider the following signaling model between an expert (E), such a special interest group, and a decision maker (DM), such as a politician. For simplicity, assume that the state of the world is discrete, either $\theta = 1$ or $\theta = 0$ with prior probability $p \in (0,1)$ and $1-p$, respectively. The expert privately observes an informative but noisy signal s, which also takes two discrete values $s \in \{0,1\}$. The precision of the signal is given by the conditional probability

$$\text{prob}(s = k|\theta = k) = q,$$

   where $k = \{0,1\}$, and $q > \frac{1}{2}$. In words, the probability that the signal s coincides with the true state of the world θ is q (precise signal), while the probability of an imprecise signal where $s \neq \theta$ is $1-q$. The time structure of the game is as follows:

1) Nature chooses θ according to the prior p.
2) Expert observes signal s and reports a message $m \in \{0, 1\}$
3) Decision maker observes m and responds with $x \in \{0, 1\}$
4) θ is observed and payoffs are realized

The payoff function for the decision maker is

$$u(x, \theta) = \left(\theta - \frac{1}{2}\right) x$$

while that of the expert is

$$v(m, \theta) = \begin{cases} 1, & \theta = m \\ 0, & \theta \neq m \end{cases}$$

which, in words, indicates that the expert's payoff is 1 when the message she sends coincides with the true realization of the state of the world, but becomes zero otherwise. Importantly, her payoff is unaffected by the signal, which she only uses to infer the actual realization of parameter θ. Intuitively, $v(m, \theta)$ is often understood as a "reputation function" since it provides the expert with a payoff of 1 only when his message was an accurate representation of the true state of the world (which in this model he does not precisely observe).

(a) Is there a Perfect Bayesian equilibrium (PBE) in which the expert reports his signal truthfully?

- Is there a Perfect Bayesian equilibrium (PBE) in which the expert reports his signal truthfully?
- *Updated beliefs.* In a strategy profile where the expert sends a message that coincides with the signal she receives (that is, sending message $m = 1$ after receiving signal $s = 1$, but sending message $m = 0$ after receiving signal $s = 0$)[1], the decision maker and the expert sustain the same beliefs about θ since $m = s$. Specifically, after receiving a signal of $s = 1$ (a message of $m = 1$), both expert and decision maker use Bayes' rule to update their beliefs yielding

$$\mu_1 = \frac{pq}{pq + (1-p)(1-q)}$$

while after receiving a signal of $s = 0$ (a message of $m = 0$), both expert and receiver updated their beliefs as follows

$$\mu_0 = \frac{p(1-q)}{p(1-q) + (1-p)q}$$

- *Decision maker's response.* Given the above beliefs, after receiving a message $m = 1$ from the expert, the decision maker responds with $x = 1$ if

$$\mu_1\left(1 - \frac{1}{2}\right)1 + (1-\mu_1)\left(0 - \frac{1}{2}\right)1 \geq \mu_1\left(1 - \frac{1}{2}\right)0 + (1-\mu_1)\left(0 - \frac{1}{2}\right)0$$

[1] This truthful reporting of signals can be described more compactly by saying that the expert's strategy is a message $m(s) = s$ for every signal $s \in \{0, 1\}$.

or

$$\mu_1 \frac{1}{2} + (1-\mu_1)\left(-\frac{1}{2}\right) \geq 0$$

or, simplifying, $\mu_1 \geq \frac{1}{2}$. From the above expression of posterior belief μ_1, this condition holds if

$$\frac{pq}{pq + (1-p)(1-q)} \geq \frac{1}{2}$$

or, after rearranging, $p \geq 1-q$, which holds by assumption. That is, the decision maker responds with $x = 1$ after receiving message $m = 1$ for all admissible parameter values. Similarly, after receiving a message $m = 0$, the decision maker responds with $x = 1$ if

$$\mu_0 \left(1 - \frac{1}{2}\right) 1 + (1-\mu_0)\left(0 - \frac{1}{2}\right) 1 \geq \mu_0 \left(1 - \frac{1}{2}\right) 0 + (1-\mu_0)\left(0 - \frac{1}{2}\right) 0$$

or, after simplying, $\mu_0 \geq \frac{1}{2}$. From the above expression of posterior belief μ_0, this condition holds if

$$\frac{p(1-q)}{p(1-q) + (1-p)q} \geq \frac{1}{2}$$

or $p \geq q$. In words, the decision maker responds with $x = 1$ after observing message $m = 0$ when the probability of $\theta = 1$, p, is higher than the probability of the expert receiving precise signals, q. Otherwise (when $p < q$), the decision maker responds with $x = 0$ after observing message $m = 0$. Therefore, when $p < q$ we can say that the decision maker responds with an action $x(m) = m$ to every message $m \in \{0, 1\}$ he receives from the sender.

- *Expert's messages - After receiving signal $s = 1$.* If the expert reports her signal truthfully (sending message $m = 1$), her expected payoff is

$$\mu_1 v(1,1) + (1-\mu_1) v(1,0) = \mu_1$$

Intuitively, the above expression says that the expert sends a message $m = 1$ but does not know if the state of the world is $\theta = 1$, which yields a payoff of 1 since $\theta = m$; or if the state of the world is $\theta = 0$, which yields a payoff of zero for her since $\theta \neq m$. If, instead, she misreports her signal (sending message $m = 0$), her expected payoff becomes

$$\mu_1 v(0,1) + (1-\mu_1) v(0,0) = 1 - \mu_1$$

Therefore, the expert truthfully reports her signal if $\mu_1 \geq 1 - \mu_1$, or $\mu_1 \geq \frac{1}{2}$. Using the expression of posterior belief μ_1, we obtain that

$$\frac{pq}{pq + (1-p)(1-q)} \geq \frac{1}{2}$$

which collapses to $p \geq 1-q$. In words, after receiving a signal of $s = 1$, the expert truthfully conveys her signal if the probability of receiving such a signal is higher than the probability of an imprecise signal, $1-q$.

- *Expert's messages - After receiving signal $s = 0$.* If the expert reports his signal truthfully (that is, sending message $m = 0$), her expected payoff is

$$\mu_0 v(0,1) + (1-\mu_0)v(0,0) = \mu_0 0 + (1-\mu_0)1 = 1-\mu_0$$

  Intuitively, the above expression says that the expert sends a message $m = 0$ but does not know if the state of the world is $\theta = 1$, which yields a payoff of zero for her since $\theta \neq m$; or if the state of the world is $\theta = 0$, which yields a payoff of 1 since $\theta = m$. If, instead, the expert sends message $m = 1$ (lying about her message), her expected payoff becomes

$$\mu_0 v(1,1) + (1-\mu_0)v(1,0) = \mu_0 v(1,1) + (1-\mu_0)v(1,0) = \mu_0$$

  Therefore, the expert truthfully reports her signal if $1-\mu_0 \geq \mu_0$, or $\frac{1}{2} \geq \mu_0$. Examining the expression of posterior belief μ_0, we find that

$$\frac{p(1-q)}{p(1-q)+(1-p)q} \leq \frac{1}{2}$$

  simplifies to $p \leq q$. In words, after receiving a signal of $s = 0$, the expert truthfully conveys her signal if the probability of an accurate signal, q, is higher than the probability of receiving a signal of $s = 1$. Combining the above conditions $p \geq 1-q$ and $p \leq q$, we obtain $1-q \leq p \leq q$.

- *Summary*:
  - When $p \geq q$, a PBE where the expert truthfully reports her signal can be sustained if $1-q \leq p \leq q$ (from the expert) and $p \geq q$ (from the decision maker), which are only compatible when $p = q$. In words, the prior probability of the state of the world being $\theta = 1$, p, must coincide with the probability with which the expert receiving precise signals, q. While the expert truthfully reports her signals to the decision maker, the decision maker does not follow the expert's advise when observing a message of $m = 0$.
  - When $p < q$, a PBE where the expert truthfully reportsher signal can be sustained if $1-q \leq p \leq q$ (from the expert) and $p < q$ (from the decision maker), where the expert sends a message $m(s) = s$ for every signal $s \in \{0,1\}$ she received, while the decision maker responds with an action $x(m) = m$ for every message $m \in \{0,1\}$ he receives. In this PBE, the expert truthfully reports her signals to the decision maker, and the decision maker follows the expert's advise after every message.

EconS 503 – Homework #6

Answer key

Exercise #1 – MWG 9C7

(See handout of Review Session #6)

Exercise #2

a)

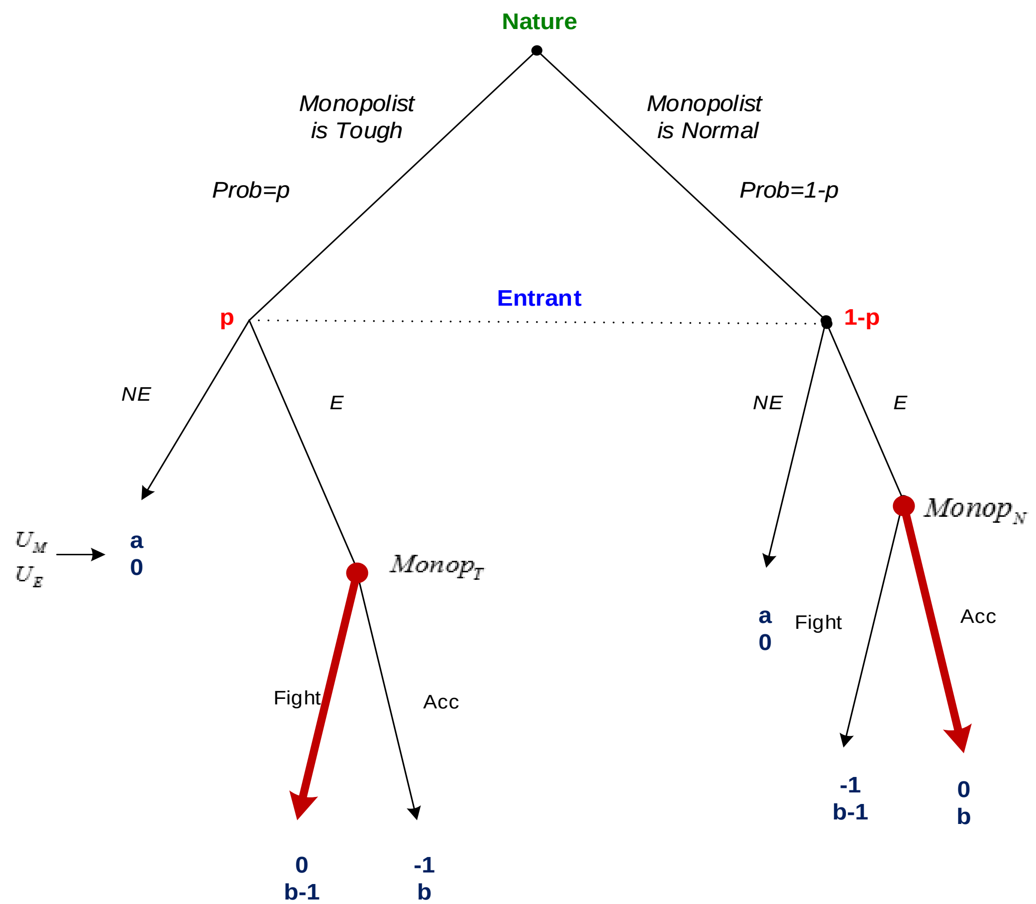

Figure 10.19. Entry deterrence with only one entrant

b) The tough monopolist fights with probability 1, since fight is a dominant strategy for him; while the normal monopolist accommodates with probability 1, since accommodation constitutes a dominant strategy for this type of incumbent. Indeed, at the node labeled with *Monop*$_T$ on the left-hand side of the game tree, the monopolist's payoff from fighting, 0, is larger than from accommodating, -1. In contrast, at the node labeled with $Monop_N$ for the normal monopolist, the monopolist's payoff from fighting, -1, is strictly lower than from accommodating, 0. Hence, the entrant's decision on whether or not to enter will be based on:

$$\text{EU}_E(Enter|p) = p(b-1) + (1-p)b = b - p, \text{ and } \text{EU}_E(Not\ Enter|p) = 0$$

Therefore, the entrant will decide to enter if and only if $b > p > 0$, or alternately, $b > p$. We can, hence, summarize the equilibrium as follows:

The entrant enters if $b > p$, but doesn't enter if $b \leq p$

The incumbent fights if tough, but accommodates if normal.

c) The following game tree depicts an entry game in which the incumbent faces two entrants in sequence

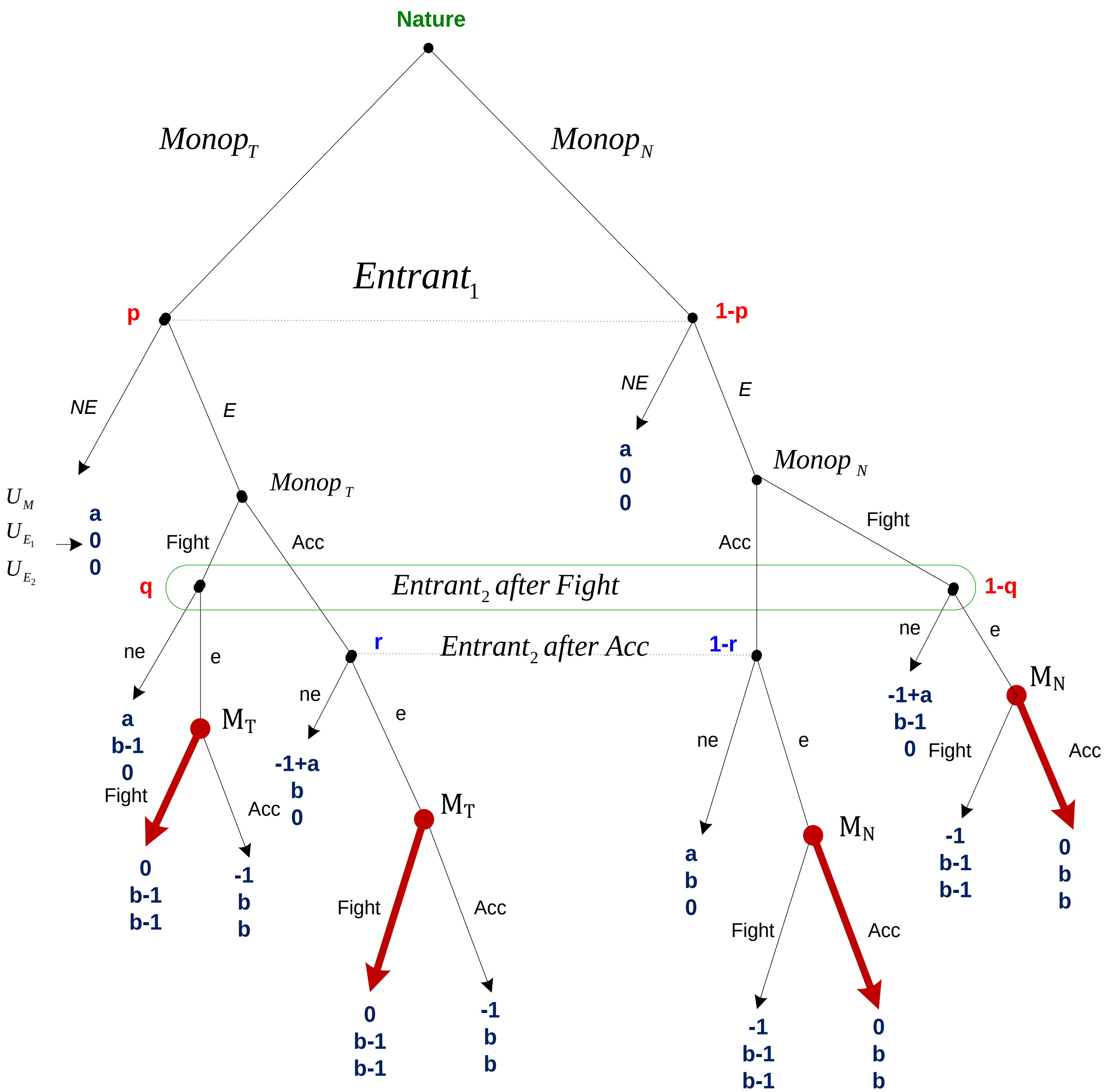

Figure 10.20. Entry deterrence with two entrants in sequence

Hence, applying backward induction on the proper subgames (those labeled with M_T and M_N in the last stages of the game tree) we can reduce the previous game tree to that in figure 10.21. For instance, after the second entrant chooses to enter despite observing a fight with the first entrant (left side of game tree), the tough monopolist chooses between fighting and accommodating in the node labeled M_T. In this setting, the tough monopolist prefers to fight, yielding a payoff of zero, rather than accommodate, which entails a payoff of -1. A similar analysis applies to the other node labeled with M_T (where the second entrant has entered after observing that the incumbent accommodated the first entrant). However, an opposite argument applies for the nodes marked with M_N in the right-hand side of the tree, where the normal monopolist prefers to accommodate the second entrant, regardless of whether he fought or accommodated the first entrant, since his payoff from accommodating (zero) is larger than from fighting (-1).

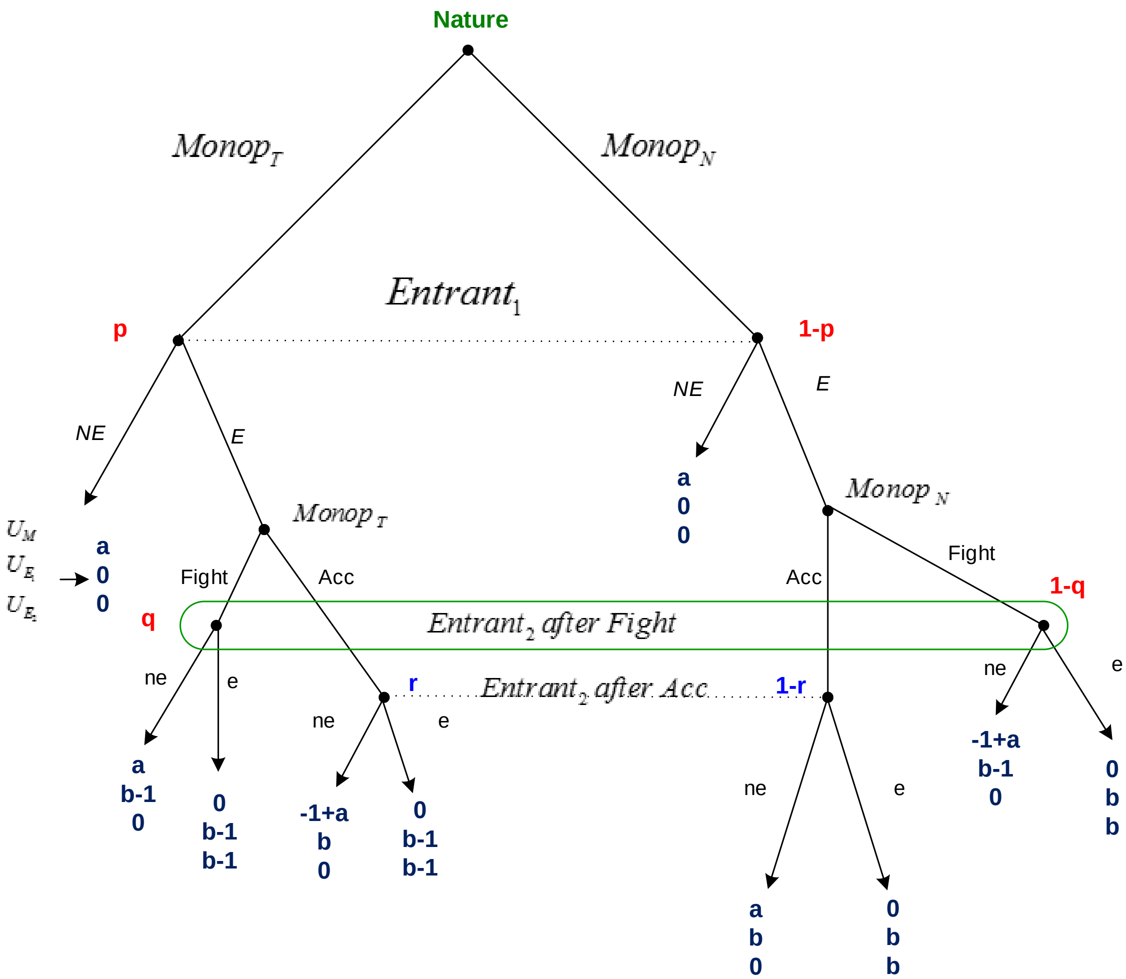

Figure 10.21. Reduced-form game

In addition, note that the first entrant behaves in exactly the same way as in exercise a): entering if and only if $b > p$. Hence, when $p \leq b$, the first entrant enters, as shown in exercise (a). Figure 10.22 shades this choice of the first entrant (green shaded branches).

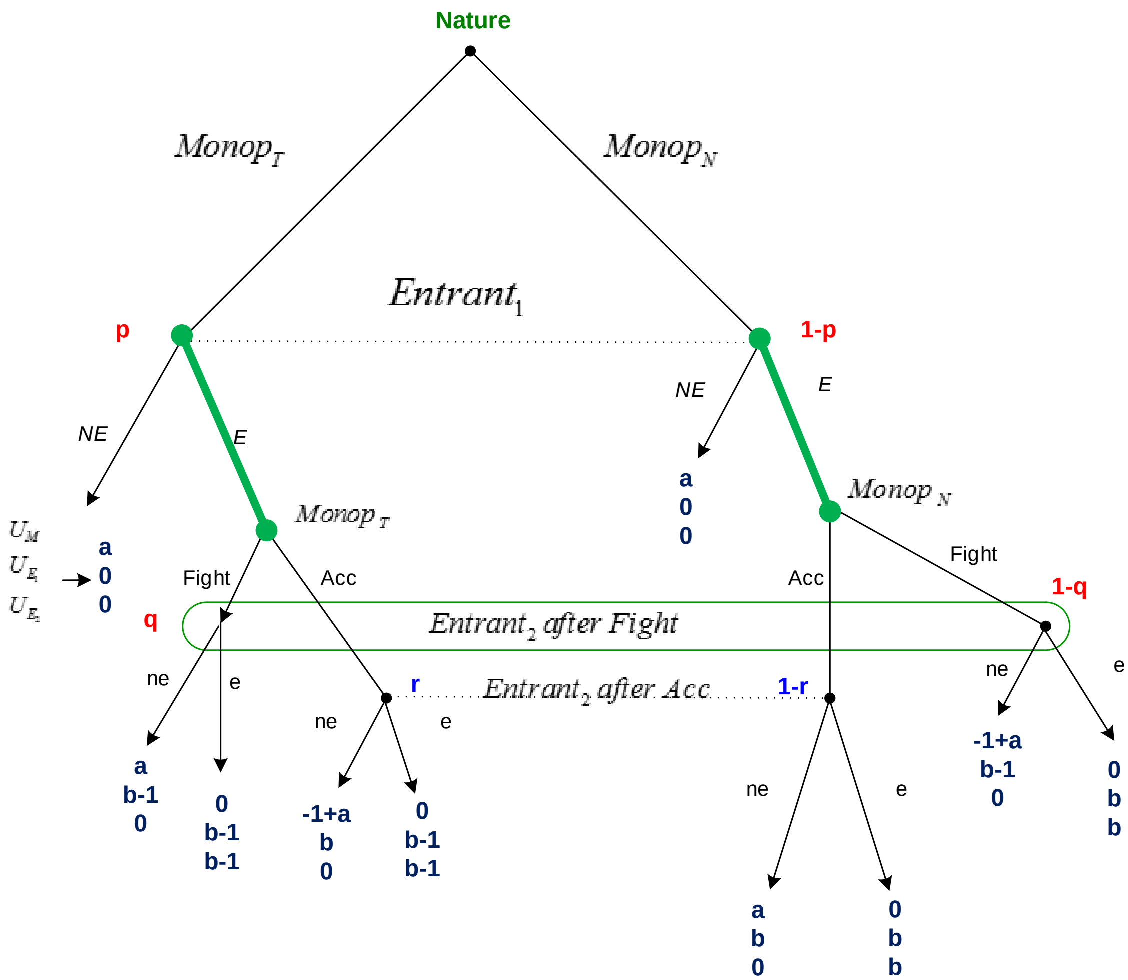

Figure 10.22. Entry of the first potential entrant

Therefore, upon entry, the first entrant gives rise to a beer-quiche type of signaling game, which can be more compactly represented as the game tree in figure 10.23. Intuitively, all elements before $Monop_T$ and $Monop_N$ can be predicted (i.e., the first entrant enters as long as $p \leq b$), while the subsequent stages characterize a signaling game between the monopolist (privately informed about its type) and the second entrant. In this context, the monopolist uses his decision to fight or accommodate the first entrant as a message to the second entrant, in order to convey or conceal his type.

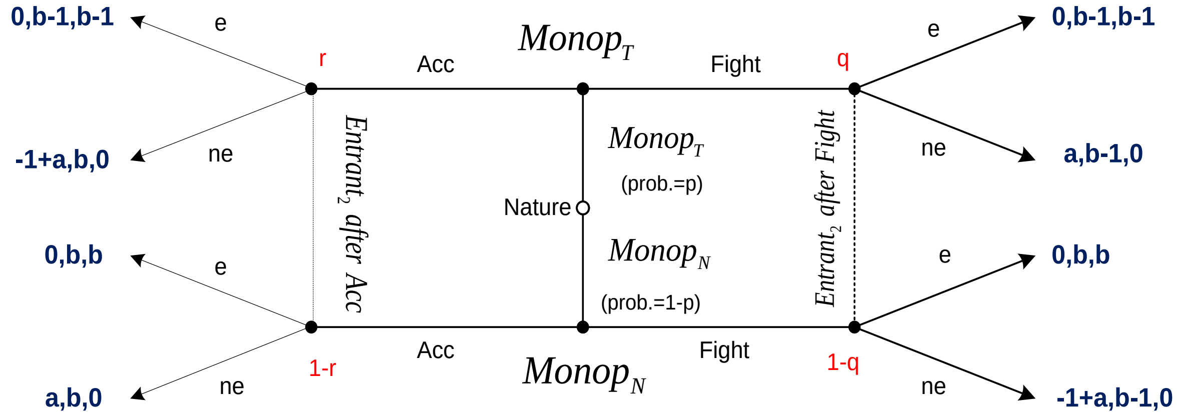

Figure 10.23. A further reduction of the game

(For every triplet of payoffs, the first corresponds to the monopolist, the second to the first entrant, and the third to the second entrant.) Let us now check if a pooling strategy profile in which both types of monopolists accommodate can be sustained as a PBE.

Pooling PBE with Acc. Figure 10.24 shades the braches corresponding to such a pooling strategy.

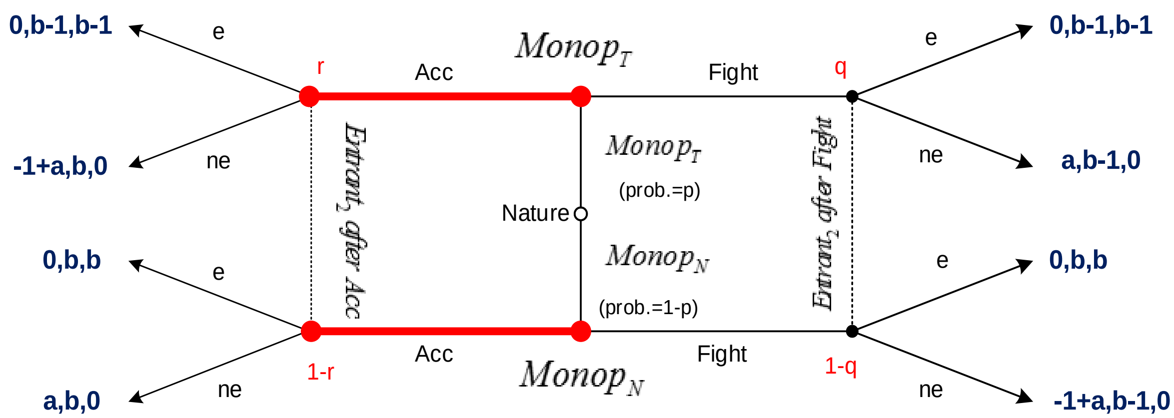

Figure 10.24. Pooling strategy profile - Acc

In this setting, posterior beliefs cannot be updated using Bayes' rule, which entails $r = p$. As in similar exercises, the observation of accommodation by the uninformed entrant does not allow him to further refine his beliefs about the monopolist's type. Hence, the second entrant responds entering (e) after observing that the monopolist accommodates (in equilibrium), since

$$p(b-1) + (1-p)b > p \cdot 0 + (1-p) \cdot 0 \leftrightarrow b > p$$

which holds in this case.

If, in contrast, the second entrant observes the off-the-equilibrium message of fight, then this player also enters if

$$q(b-1)+(1-q)b > q \cdot 0 + (1-q) \cdot 0 \leftrightarrow b > q$$

Hence, the second entrant enters regardless of the incumbent's action if off-the-equilibrium beliefs, q, satisfy $q < b$, as depicted in figure 10.25 (see blue shaded branches). Otherwise, the entrant only enters after observing the equilibrium message of Acc.

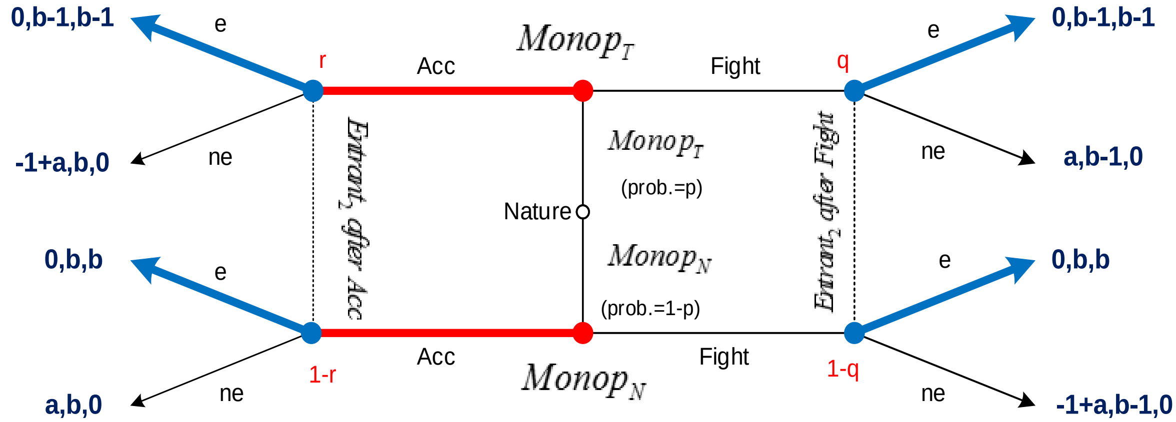

Figure 10.25. Pooling strategy profile – Acc (with responses)

The tough monopolist, M_T, is hence indifferent between Acc (as prescribed) which yields a payoff of 0, and deviate to Fight, which also yields a payoff of zero. A similar argument applies to the normal monopolist, M_N, in the lower part of the game tree. Hence, the pooling strategy profile where both types of incumbents accommodate can be sustained as a PBE.

A remark on the Intuitive Criterion. Let us next show that the above pooling equilibrium, despite constituting a PBE, violates the Cho and Kreps' (1987) Intuitive Criterion. In particular, the tough monopolist has incentives to deviate towards Fight if, by doing so, he is identified as a tough player, $q = 1$, which induces the entrant to respond not entering. In this case, the tough monopolist obtains a payoff of a, which exceeds his equilibrium payoff of 0. In contrast, the normal monopolist doesn't have incentives to deviate since, even if his deviation to Fight deters entry, his payoff from doing so, -1+a, would still be lower than his equilibrium payoff of 0, given that -1+a<0 or a<1. Hence, only the tough monopolist has incentives to deviate, and the entrant's off-the-equilibrium beliefs can thus be restricted to q=1 upon observing that the monopolist fights. Intuitively, the entrant infers that the observation of Fight can only originate from the tough monopolist. In this case, the tough incumbent indeed prefers to select Fight, thus implying that the above pooling PBE violates Cho Kreps' (1983) Intuitive Criterion. □

Let us next check if this pooling strategy profile can be sustained when off-the-equilibrium beliefs satisfy, instead, $q \geq b$, thus inducing the entrant to respond not entering upon observing the off-the-equilibrium message of Fight, as illustrated in the game tree of figure 10.26 (see blue shaded branches in the right-hand side of the tree).

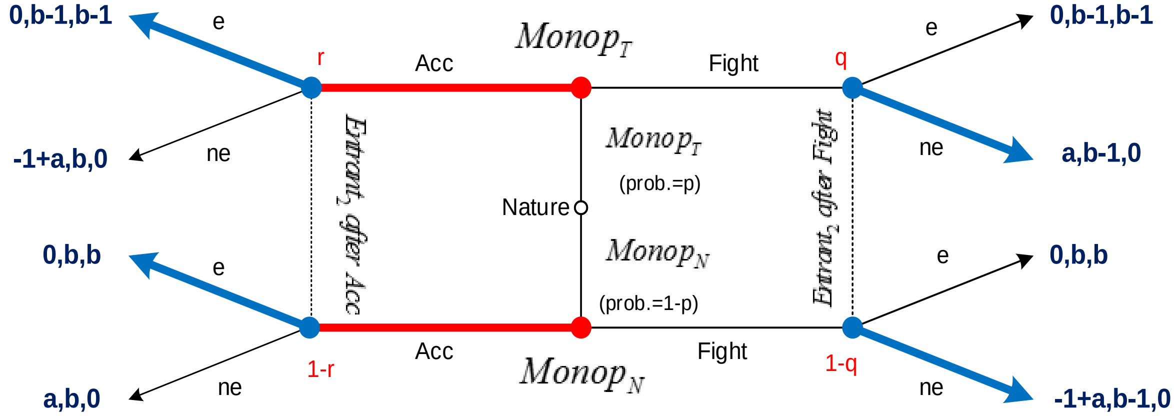

Figure 10.26. Pooling strategy profile – Acc (with responses)

In this case, the M_T has incentives to deviate from Acc, and thus the pooling strategy profile where both M_T and M_N select to Acc cannot be sustained as a PBE.

Pooling PBE with Fight. Let us now examine the opposite pooling strategy profile (Fight, Fight), in which both types of monopolists fight, as depicted in figure 10.27.

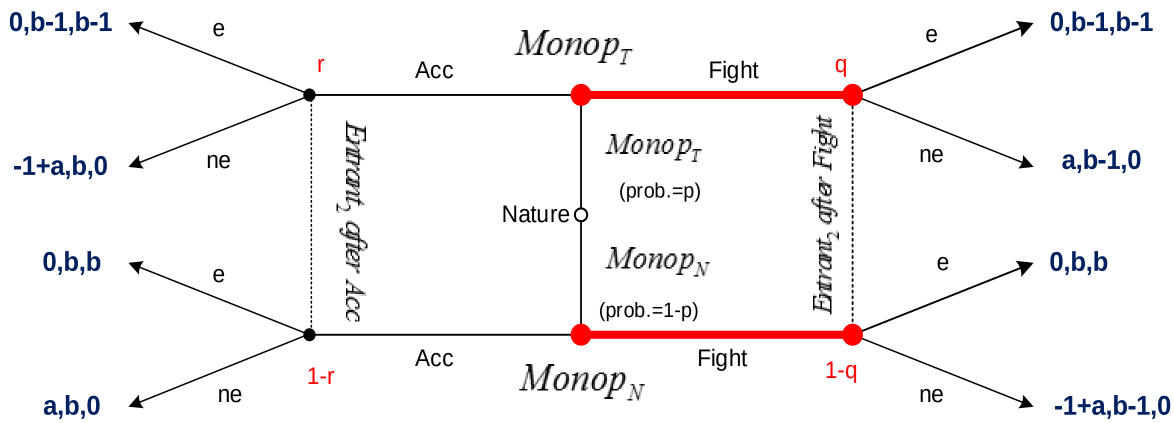

Figure 10.27. Pooling strategy profile - Fight

Hence, equilibrium beliefs after observing Fight cannot be updated, and thus satisfy q=p; while off-the-equilibrium beliefs are arbitrary $r \in [0,1]$ after observing the off-the-equilibrium message of accomodation. Given these beliefs, upon observing the equilibrium message of Fight, the entrant responds entering since

$$p(b-1) + p \cdot b > p \cdot 0 + (1-p) \cdot 0 \leftrightarrow b > p$$

which holds by definition. If, in contrast, the entrant observes the off-the-equilibrium message of Acc, then it responds entering if

$$r(b-1) + r \cdot b > p \cdot 0 + (1-r) \cdot 0 \leftrightarrow b > r$$

Hence, if b>r, the entrant enters both after observing Fight (in equilibrium) and Acc (off-the-equilibrium path). If, instead, $b \leq r$, then the entrant only responds entering after observing Fight, but is deterred otherwise. We next separately analyze each case. Figure 10.28 illustrates the entrant's responses when b>r.

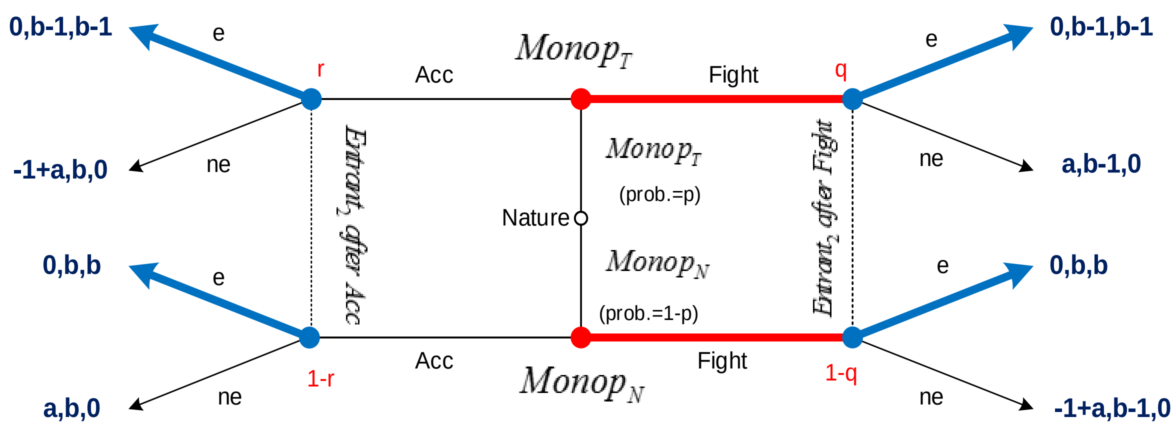

Figure 10.28. Pooling strategy profile – Fight (with responses)

In this context, the tough monopolist is indifferent between sticking to the equilibrium strategy of Fight, obtaining zero profits, or accommodating, which also yields zero profits. A similar argument applies to the normal monopolist in the lower part of the game tree. Hence, in this case the pooling strategy profile (Fight, Fight) can be supported as a PBE if off-the-equilibrium beliefs, r, satisfy r<b.

If, instead, $r \geq b$, then the entrant is deterred upon observing the off-the-equilibrium message of Acc, as figure 10.29 depicts.

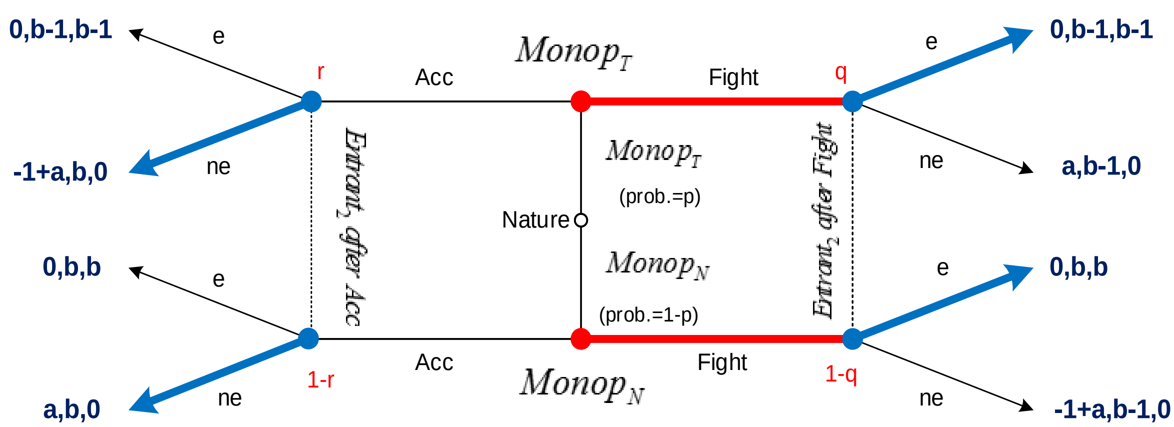

Figure 10.29 Pooling strategy profile – Fight (with responses)

The pooling strategy profile cannot be sustained in this context, since M_N has incentives to deviate towards Acc, obtaining a payoff of a, which exceeds its payoff of zero when he Fights. Hence, the pooling strategy profile (Fight, Fight) cannot be supported as a PBE when off-the-equilibrium beliefs satisfy $r \geq b$.

Separating PBE (Fight, Acc). Let us next examine if the separating strategy profile in which only the tough monopolist fights can be sustained as a PBE. Figure 10.30 depicts this strategy profile.

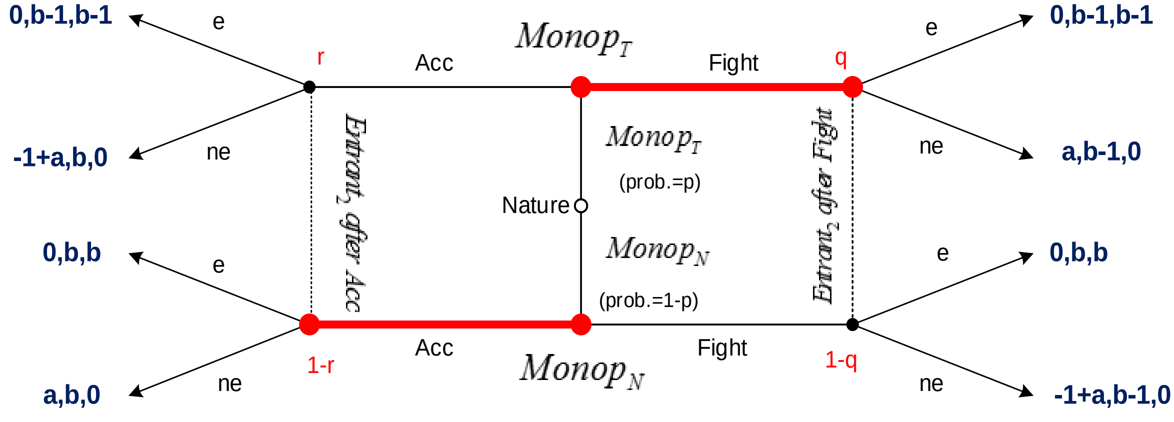

Figure 10.30. Separating strategy profile (Fight, Acc)

In this case, entrant's beliefs are updated to q=1 and r=0 using Bayes' rule, implying that, upon observing Fight, the entrant is deterred from the market since b-1<0, given that b<1 by definition. Upon observing Acc, the entrant is instead attracted to the market since b>0. Figure 10.31 illustrates the entrant's responses (see blue shaded arrows).

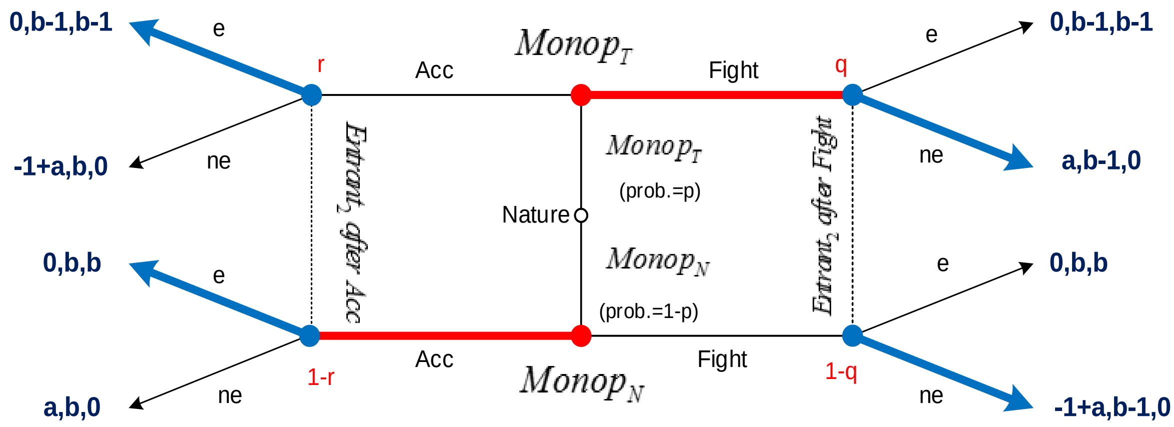

Figure 10.31. Separating strategy profile (Fight, Acc), with responses

In this setting, no type of monopolist has incentives to deviate: (1) the tough monopolist obtains a payoff of a by fighting (as prescribed) but only 0 from deviating towards Acc; and similarly (2) the normal monopolist obtains 0 by accommodating (as prescribed) but a negative payoff, -1+a, by deviating towards Fight, given that a<1 by definition. Hence, this separating strategy profile can be sustained as a PBE.

Separating PBE (Acc, Fight). Let us now check if the alternative separating strategy profile, in which only the normal monopolist fights, can be supported as a PBE (We know, this strategy sounds crazy, but we want to formally show that it cannot be sustained as a PBE.) Figure 10.32 illustrates this strategy profile.

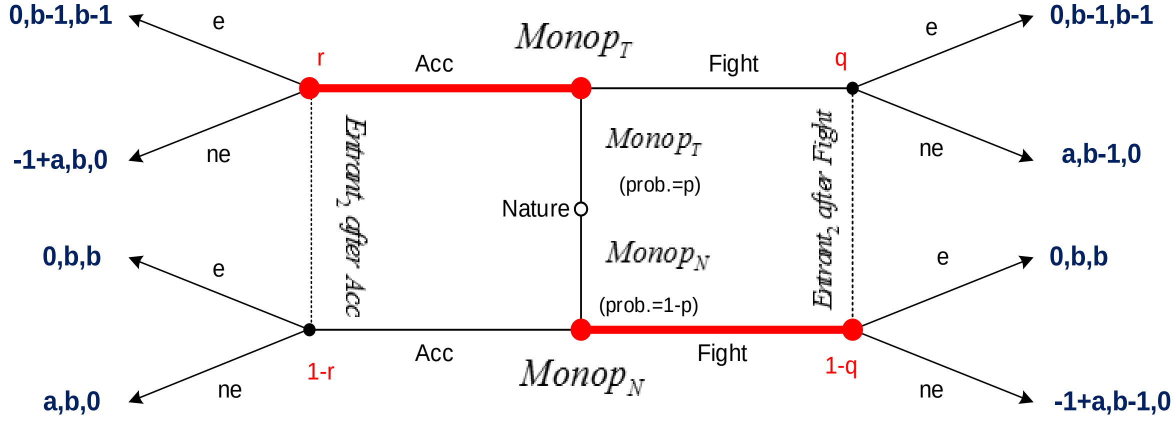

Figure 10.32. Separating strategy profile (Acc, Fight)

In this setting, the entrant's beliefs can be updated to r=1 and q=0, inducing the entrant to respond not entering after observing Acc, but entering after observing Fight, as depicted in figure 10.33 (see blue shaded branches).

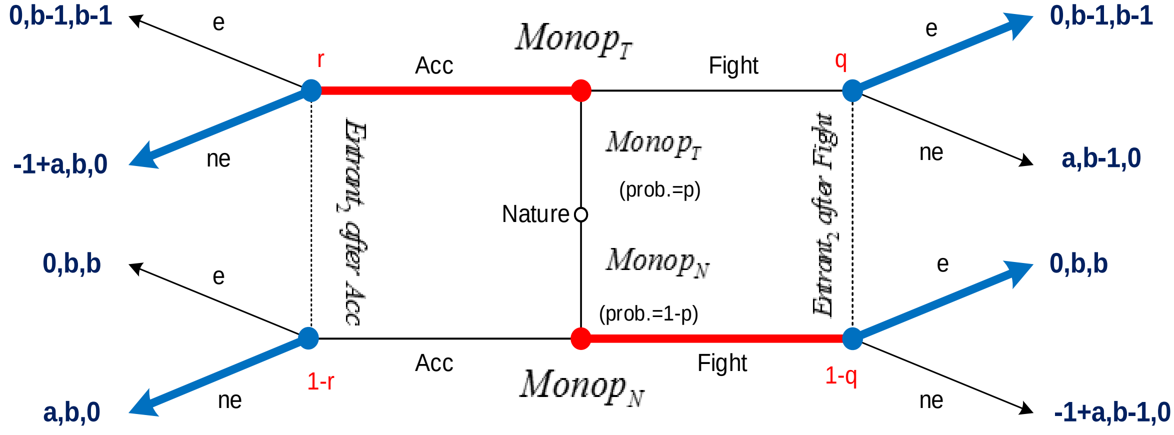

Figure 10.33. Separating strategy profile (Acc, Fight), with responses

Given these responses by the entrant, the tough monopolist has incentives to deviate from Acc, which yields a negative payoff of -1+a, to Fight, which yields a higher payoff of zero. Therefore, this separating strategy cannot be supported as a PBE.

Exercise #3

Chapter 3

Hidden Information, Signaling

3.1 Question 6

Consider a firm that can invest an amount I in a project generating high observable cash flow $C > 0$ with probability θ and 0 otherwise: $\theta \in \{\theta_L, \theta_H\}$ with $\theta_H - \theta_L \equiv \Delta > 0$ and $\Pr[\theta = \theta_L] = \beta$. The firm needs to raise I from external investors who do not observe the value of θ. Assume that $\theta_L C - I > 0$. Everybody is risk neutral and there is no discounting.

1. Suppose that the firms can only promise to repay an amount R chosen by the firm (with $0 \leq R \leq C$) when cash flow is C and 0 otherwise. Can a good firm signal its type?

2. Suppose now that the firm also has the possibility of pledging some assets as collateral for the loan: Should a "default" occur (the firm being unable to repay R), an asset of value K to the firm is transferred to the creditor whose valuation is xK with $0 < x < 1$. The size of the collateral K is a choice variable. Give a necessary and sufficient condition for the "best" Perfect Bayesian Equilibrium to be separating. How does it depend on β and x? Explain.

3.1.1 No Collateral

Both firms would want to undergo the project since $\theta_L C > I$. A good firm cannot signal its type, since for a separating equilibrium to exist we need $R_H \neq R_L$. However, this cannot be an equilibrium. This can be seen clearly from the incentive compatibility condition for a firm of type i

$$\theta_i (C - R_i) \geq \theta_i (C - R_j).$$

Whenever $R_i \neq R_j$ at least one type of firm will want to deviate.

Intuitively speaking, since both firms receive C when the project is successful and 0 when it fails, and we only have one repayment instrument, the bad firm can perfectly mimic the good firm.

3.1.2 Collateral

Separating Equilibrium

The best separating equilibrium is clearly the one with the least amount of K so $K_L = 0$. The loss $(1-x)K$ is higher for a low-type firm since it has a higher probability of being in default.

A separating equilibrium can be supported by the following beliefs:

$$\begin{aligned} \Pr(\theta = \theta_L | K > K^*) &= 0 \\ \Pr(\theta = \theta_L | K \leq K^*) &= 1 \end{aligned}$$

and we have

$$\begin{aligned} K_L &= 0 \\ R_L &= \frac{I}{\theta_L}. \end{aligned}$$

This holds since otherwise the low type could offer a higher payment and still be better off–there is no benefit from a positive K.

Thus we shall have the following incentive compatibility and individual rationality constraints:

$$\begin{aligned} \theta_L R_L &= I && \text{(IRL)} \\ xK^H(1-\theta_H) + \theta_H R_H &= I && \text{(IRH)} \\ \theta_L(C - R_L) &\geq \theta_L(C - R_H) - (1-\theta_L)K_H && \text{(ICL)} \\ \theta_H(C - R_H) - (1-\theta_H)K_H &\geq \theta_H(C - R_L) && \text{(ICH)} \end{aligned}$$

Rewriting (ICL) we obtain

$$R_L - R_H \leq \frac{1-\theta_L}{\theta_L} K_H.$$

Similarly, re-arranging (ICH) gives

$$R_L - R_H \geq \frac{1-\theta_H}{\theta_H} K_H.$$

Putting these expressions together we obtain

$$\frac{1-\theta_H}{\theta_H} K_H \leq R_L - R_H \leq \frac{1-\theta_L}{\theta_L} K_H.$$

This works even if x is very small. The intuition is that the high type benefits from a lower R more often and suffers from the loss of K less often, since

$\theta_H > \theta_L$. Thus the best separating equilibrium minimizes the use of (socially) wasteful collateral, that is K_H is set as low as possible. Hence, in equilibrium, only (ICL) is binding and (ICH) is slack. Thus, in what follows we can ignore (ICH). Solving the following equalities which we obtained from the constraints using the fact that (ICL) is binding in equilibrium and combining (IRL) and (ICL), we find

$$xK^H(1-\theta_H) + \theta_H R_H = I \quad \text{(IRH)}$$
$$K_H(1-\theta_L) + \theta_L R_H = I. \quad \text{(IRL, ICL)}$$

Rewriting these conditions yields

$$\begin{aligned} R_H &= \frac{I}{\theta_H} - \frac{1-\theta_H}{\theta_H} x K_H \\ R_H &= \frac{I}{\theta_L} - \frac{1-\theta_L}{\theta_L} K_H, \end{aligned}$$

and after some algebraic manipulation we obtain

$$\begin{aligned} K_H &= \frac{\Delta I}{\theta_H (1-\theta_L) - x\theta_L (1-\theta_H)} \\ R_H &= \frac{I}{\theta_H} - \frac{1-\theta_H}{\theta_H} \frac{x\Delta I}{\theta_H (1-\theta_L) - x\theta_L (1-\theta_H)}. \end{aligned}$$

From above, we have

$$\begin{aligned} K_L &= 0 \\ R_L &= \frac{I}{\theta_L}, \end{aligned}$$

since this is the best, or the least cost equilibrium. K is costly and hence there is no reason to use it in the low state. Notice that this separating equilibrium always exists.

Pooling Equilibrium

We can compare this to the best pooling equilibrium, where

$$\begin{aligned} K^P &= 0 \\ R^P &= \frac{I}{\theta_H - \beta\Delta}. \end{aligned}$$

However, this pooling equilibrium may not exist. It will exist if and only if

$$\theta_H \left(C - R^P\right) \geq \theta_H (C - R) - (1-\theta_H) K$$

where

$$\begin{aligned} R &= \frac{I - (1-\theta_L) xK}{\theta_L} \\ I &= \theta_L R + (1-\theta_L) xK, \end{aligned}$$

which follows from the assumption that for any deviation from $K^P = 0$ the investors will believe that the firm is of low type. So the pooling equilibrium is sustainable if there are no deviations given these beliefs. This is the worst belief in the sense that if we cannot find a pooling equilibrium supported by these beliefs then no pooling equilibrium exists (there will always be a profitable deviation from it). Combining the equations, we obtain

$$\theta_H \left(C - \frac{I}{\theta_H - \beta\Delta} \right) \geq \theta_H \left(C - \frac{I - (1-\theta_L)\, xK}{\theta_L} \right) - (1-\theta_H)\, K,$$

which is equivalent to

$$x \leq (\theta_H I (\frac{1}{\theta_L} - \frac{1}{\beta\theta_L + (1-\beta)\theta_H}) + (1-\theta_H)K)\frac{\theta_L}{\theta_H(1-\theta_L)K}.$$

Thus, the smaller x or β the more likely is the existence of a pooling equilibria. This is intuitive. A smaller x means that the signal is more costly, and hence a profitable deviation from the least-cost pooling equilibrium that has no costly collateral, is very difficult. Similarly, with a smaller β the less likely it is that the firm is a bad type (so a smaller cross-subsidy is needed).

Comparison

One way to compare the different equilibria would be to compare ex-ante expected profits of the firm for the separating and pooling equilibrium (compare section 3.1.1). The expected profits for the separating equilibrium are given by

$$\begin{aligned}
\pi^S &= (1-\beta)\,\pi_H^S + \beta\pi_L^S \\
&= (1-\beta)\,[\theta_H(C - R_H) - (1-\theta_H)K_H] + \beta\theta_L(C - R_L) \\
&= C\,[\theta_L + (1-\beta)\,\Delta] - I - (1-\beta)\,(1-\theta_H)(1-x)K_H.
\end{aligned}$$

where K_H is as defined above.

In contrast, expected profits for the pooling equilibrium are

$$\begin{aligned}
\pi^P &= (1-\beta)\,\pi_H^P + \beta\pi_L^P \\
&= (1-\beta)\,\theta_H(C - R^P) + \beta\theta_L(C - R^P) \\
&= [\theta_L + (1-\beta)\,\Delta]\,(C - R^P) \\
&= [\theta_L + (1-\beta)\,\Delta]\,C - I.
\end{aligned}$$

Hence, we have

$$\pi^P > \pi^S,$$

whenever the pooling equilibrium exists. The pooling equilibrium leads to higher profits as it avoids the use of (wasteful) collateral. Thus, the best perfect Bayesian equilibrium (when defined in this way) is separating if and only if the pooling equilibrium does not exist. Another way would be look for one equilibria Pareto-dominating the other. Here we would see when both types prefer the pooling equilibrium. This happens when β is very small and thus the tiny gain from signalling (avoiding the infinitesimal cross subsidy) is smaller than the costly signal. See section 3.1.1 for details.