EconS 503 - Microeconomic Theory II
Homework #5 - Answer key

1. **Cournot competition with uncertain costs.** Consider an industry with two firms competing à la Cournot and facing inverse demand function $p(Q) = 1 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. Every firm i is privately informed about its marginal cost, high or low, denoted as c_H and c_L, respectively, where $1 > c_H > c_L = 0$. Finally, consider that, while firm j cannot observe the realization of firm i's marginal cost (c_H or c_L), firm j knows that that both types are equally likely. Firms then interact in a simultaneous-move game of incomplete information, and in this exercise we seek to find the Bayesian Nash equilibrium of the game.

   (a) Find the best response function for every firm i when its marginal costs are low, $q_i^L(q_j^H, q_j^L)$.

   - When firm i has low costs, it chooses q_i^L to solve the following expected profit maximization problem:

$$\max_{q_i^L \geq 0} \ \pi_i^L\left(q_i^L\right) = \overbrace{\frac{1}{2}\left(1 - q_i^L - q_j^L\right) q_i^L}^{\text{Profits if } j \text{ is low cost}} + \overbrace{\frac{1}{2}\left(1 - q_i^L - q_j^H\right) q_i^L}^{\text{Profits if } j \text{ is high cost}}$$
$$= \left(1 - q_i^L - \frac{q_j^L + q_j^H}{2}\right) q_i^L$$

     which does not include the production cost of firm i because $c_L = 0$. Differentiating with respect to q_i^L and assuming an interior solution, yields

$$\frac{\partial \pi_i^L\left(q_i^L\right)}{\partial q_i^L} = 1 - 2q_i^L - \frac{q_j^L + q_j^H}{2} = 0.$$

     Solving for q_i^L, we find the best response function of firm i when its costs are low, as follows

$$q_i^L\left(q_j^L, q_j^H\right) = \frac{1}{2} - \frac{q_j^L + q_j^H}{4}$$

     which originates at $1/2$, and decreases in its rival's output at a rate of $1/4$ (both when its rival has low and high costs).

   (b) Find the best response function for every firm i when its marginal costs are high, $q_i^H(q_j^H, q_j^L)$.

   - When firm i has high costs, it chooses q_i^H to solve the following expected profit maximization problem:

$$\max_{q_i^H \geq 0} \ \pi_i^H\left(q_i^H\right) = \overbrace{\frac{1}{2}\left(1 - q_i^H - q_j^L\right) q_i^H}^{\text{Profits if } j \text{ is low cost}} + \overbrace{\frac{1}{2}\left(1 - q_i^H - q_j^H\right) q_i^H}^{\text{Profits if } j \text{ is high cost}} - c_H q_i^H$$
$$= \left(1 - c_H - q_i^H - \frac{q_j^L + q_j^H}{2}\right) q_i^H$$

Assuming interior solutions, that is, $q_i^H > 0$, the first order condition satisfies

$$\frac{\partial \pi_i^H \left(q_i^H\right)}{\partial q_i^H} = 1 - c_H - 2q_i^H - \frac{q_j^L + q_j^H}{2} = 0$$

such that the best response function of firm i when its costs are high becomes

$$q_i^H \left(q_j^L, q_j^H\right) = \frac{1 - c_H}{2} - \frac{q_j^L + q_j^H}{4}$$

which originates at $\frac{1-c_H}{2}$, but decreases in its rival's output at a rate of 1/4 (both when its rival has low and high costs).

- Comparing it with firm i's best response function when its costs are low, $q_i^L \left(q_j^L, q_j^H\right)$, we can see that, for a given profile of firm j's output, $\left(q_j^L, q_j^H\right)$, firm i responds producing a larger output when its own costs are low than when they are high if

$$\frac{1}{2} > \frac{1 - c_H}{2}$$

which holds since $c_H > 0$ by definition.

(c) Use your results from parts (a) and (b) to find the Bayesian Nash Equilibrium (BNE) of the game.

- Since firms i and j are symmetric, we impose symmetry on the equilibrium output that

$$q^L = q_i^L = q_j^L$$
$$q^H = q_i^H = q_j^H$$

Substituting the above results into the best response functions we found in parts (a) and (b), yields

$$q^L = \frac{1}{2} - \frac{q^L + q^H}{4} \quad \text{from part (a), and}$$
$$q^H = \frac{1 - c_H}{2} - \frac{q^L + q^H}{4} \quad \text{from part (b).}$$

Simultaneously solving for q^L and q^H, we find that the equilibrium output levels are

$$q^{L*} = \frac{4 + c_H}{12} \quad \text{and} \quad q^{H*} = \frac{4 - 5c_H}{12}$$

where, as expected, every firm i produces more output when its marginal cost is low than when it is high, $q^{L*} > q^{H*}$.

- Therefore, the BNE is

$$\left(q_i^{L*}, q_i^{H*}\right) = \left(\frac{4 + c_H}{12}, \frac{4 - 5c_H}{12}\right)$$

for every firm i. For instance, if $c_H = \frac{1}{2}$, we obtain equilibrium output pair of

$$\left(q_i^{L*}, q_i^{H*}\right) = \left(\frac{3}{8}, \frac{1}{8}\right).$$

2. **Price competition with heterogeneous goods and uncertain costs.** Consider two firms competing in prices à la Bertrand and selling heterogeneous goods. The demand function of every firm i is

$$q_i(p_i, p_j) = 1 - \gamma p_i + p_j$$

where $\gamma \geq 1$ denotes the degree of product differentiation (i.e., homogeneous goods when $\gamma = 1$ but differentiated when $\gamma > 1$). Every firm i faces a constant marginal cost of c_H with probability β and a marginal cost c_L with the remaining probability $1 - \beta$, where $1 > c_H > c_L \geq 0$. Every firm i privately observes its own marginal cost, but does not observe the marginal cost of its rival. The probability distribution over costs c_H and c_L is common knowledge among firms.

(a) Find every firm i's best response function when its marginal cost is high, c_H, and its best response function when its marginal cost is low, c_L.

- Firm i, after observing its marginal cost realization c_k, where $k = \{H, L\}$, chooses its price p_i to solve the following profit maximization problem,

$$\max_{p_i>0} \pi_i(p_i) = (p_i - c_k)\left(1 - \gamma p_i + \underbrace{\beta p_j^H + (1-\beta)p_j^L}_{\text{Expected } p_j}\right)$$

where $\overline{p}_j \equiv \beta p_j^H + (1-\beta)p_j^L$ denotes the expected price of firm i's rival. Differentiating with respect to p_i, and assuming interior solutions, we obtain

$$\frac{\partial \pi_i(p_i)}{\partial p_i} = 1 - 2\gamma p_i + \overline{p}_j + \gamma c_k = 0$$

Solving for p_i, we find firm i's best response function

$$p_i^k(\overline{p}_j) = \frac{1+\gamma c_k}{2\gamma} + \frac{1}{2\gamma}\overline{p}_j \quad \text{for every cost } k$$

which originates at $\frac{1+\gamma c_k}{2\gamma}$ and increases in its rival's price, p_j, at a rate of $\frac{1}{2\gamma}$.
- Graphically, as firm i observes a lower realization of its marginal cost (c_L rather than c_H), the vertical intercept of its best response function, $\frac{1+\gamma c_k}{2\gamma}$, decreases, shifting the best response function downward, thus indicating that firm i sets a lower price on its product, for a given expected price of its rival.
- We can now separately evaluate this best response function at each of its possible cost realizations, c_H and c_L, and use the definition of the expected price $\overline{p}_j \equiv \beta p_j^H + (1-\beta)p_j^L$, to obtain

$$p_i^H(p_j^H, p_j^L) = \frac{1+\gamma c_H}{2\gamma} + \frac{1}{2\gamma}\left(\beta p_j^H + (1-\beta)p_j^L\right) \quad \text{when firm } i\text{'s costs are high}$$

$$p_i^L(p_j^H, p_j^L) = \frac{1+\gamma c_L}{2\gamma} + \frac{1}{2\gamma}\left(\beta p_j^H + (1-\beta)p_j^L\right) \quad \text{when firm } i\text{'s costs are low}$$

(b) What are the equilibrium prices?

- In a symmetric equilibrium, both firms set the same price when their marginal costs coincide, that is, $p^k = p_1^k = p_2^k$ for every $k = \{H, L\}$. Using this property in the above system of four equations (two for each firm), we obtain only two equations

$$\begin{aligned} p^H &= \frac{1+\gamma c_H}{2\gamma} + \frac{1}{2\gamma}\left(\beta p^H + (1-\beta)p^L\right) \quad \text{and} \\ p^L &= \frac{1+\gamma c_L}{2\gamma} + \frac{1}{2\gamma}\left(\beta p^H + (1-\beta)p^L\right) \end{aligned}$$

which simplify into

$$\begin{aligned} 2\gamma p^H &= 1+\gamma c_H + \beta p^H + (1-\beta)p^L \quad \text{and} \\ 2\gamma p^L &= 1+\gamma c_L + \beta p^H + (1-\beta)p^L. \end{aligned}$$

Simultaneously solving for p^H and p^L, we find

$$p^H = \frac{2 + c_L(1-\beta) + c_H(\beta + 2\gamma - 1)}{2(2\gamma - 1)}$$

and

$$p^L = \frac{2(1+\gamma c_L) + \beta(c_H - c_L)}{2(2\gamma - 1)}$$

(c) How are equilibrium prices affected by changes in parameter γ and β?

- Differentiating the equilibrium prices with respect to γ, we obtain

$$\frac{\partial p^H}{\partial \gamma} = \frac{\partial p^L}{\partial \gamma} = -\frac{2 + c_L(1-\beta) + \beta c_H}{(2\gamma - 1)^2}$$

which is unambiguously negative. Therefore, as goods become more differentiated (higher γ), equilibrium prices decreases, both when firm i observes a high marginal cost and when it observes a low marginal cost.

- Differentiating the equilibrium prices with respect to β, we find

$$\frac{\partial p^H}{\partial \beta} = \frac{\partial p^L}{\partial \beta} = \frac{c_H - c_L}{2(2\gamma - 1)}$$

which is clearly positive. Intuitively, as the high marginal cost becomes relatively more likely (higher β), equilibrium price increases.

(d) *Numerical example.* Assume that $\gamma = 3/2$, $c_L = 1/4$, and $c_H = 1/2$. Find the equilibrium prices p^H and p^L, and confirm that they increase in β. Then, evaluate the equilibrium prices at $\beta = 0$ and at $\beta = 1$. Interpret.

- Substituting $\gamma = 3/2$, $c_L = 1/4$, and $c_H = 1/2$ into the equilibrium prices, we obtain

$$\begin{aligned} p^L(\beta) &= \frac{2\left(1 + \frac{3}{2} \times \frac{1}{4}\right) + \beta\left(\frac{1}{2} - \frac{1}{4}\right)}{2\left(2 \times \frac{3}{2} - 1\right)} = \frac{11+\beta}{16} \\ p^H(\beta) &= \frac{2 + \frac{1}{4}(1-\beta) + \frac{1}{2}\left(\beta + 2 \times \frac{3}{2} - 1\right)}{2\left(2 \times \frac{3}{2} - 1\right)} = \frac{13+\beta}{16} \end{aligned}$$

and both are monotonically increasing in β. In addition,

$$p^L(0) = \frac{11}{16}$$
$$p^H(0) = \frac{13}{16}$$
$$p^L(1) = \frac{3}{4}$$
$$p^H(1) = \frac{7}{8}$$

so that firms set higher equilibrium prices when they have higher probability to realize a high marginal cost.

3. **Expected revenue in the first-price auction.** Consider the first-price auction with $N \geq 2$ bidders, where every bidder i independently draws his value for the object, v_i.

   (a) Assuming that every bidder's valuation is distributed according to a generic cumulative distribution function $F(v_i)$, find the seller's expected revenue from the auction.

   - For compactness, let us define $G(x) \equiv (F[x])^{N-1}$ to be the joint cumulative distribution function for $N-1$ bidders, where valuation x satisfies $x \in [0,1]$. Then the above optimal bidding function in the first-price auction can be rewritten as

$$\begin{aligned} b_i(v_i) &= v_i - \underbrace{\frac{\int_0^{v_i} F(x)^{N-1} dx}{F(v_i)^{N-1}}}_{\text{bid shading}} \\ &= v_i - \frac{1}{G(v_i)} \int_0^{v_i} G(x)\, dx \\ &= \frac{1}{G(v_i)} \int_0^{v_i} x g(x)\, dx \end{aligned}$$

     From an *ex-ante* point of view (before observing his own valuation for the object), bidder i's expected payment to the seller is given by the probability of winning the auction times the bid he pays for the object upon winning, that is,

$$\begin{aligned} m(v_i) &= \Pr(win) \times b_i(v_i) \\ &= (F[v_i])^{N-1} \times b_i(v_i) \\ &= G(v_i) \times b_i(v_i) \\ &= G(v_i) \times \frac{1}{G(v_i)} \int_0^{v_i} x g(x)\, dx \\ &= \int_0^{v_i} x g(x)\, dx \end{aligned}$$

     where the second line indicates that bidder i wins the auction if his valuation v_i is higher than everyone else's, that is, $v_i \geq v_j$ for every bidder $j \neq i$.

The probability of his valuation exceeding that of every other bidder is given by $(F[v_i])^{N-1}$, which we represented more compactly in the third line as $G(v_i) \equiv (F[v_i])^{N-1}$. In the fourth line, we just insert the equilibrium bidding function found above, $b_i(v_i)$, and in the last line we simplify the expression.

- Since the seller cannot observe bidders' values, he finds the expected payment from each bidder i, $E[\pi_i(v_i)]$, and then sums up for all N bidders, $\sum_{i=1}^{N} E[\pi_i(v_i)]$, which gives us the seller's revenue from the auction (this is, of course, understood from an *ex-ante* perspective since the seller does not observe bidders' valuations). We find the seller's revenue as follows

$$R^1 = \sum_{i=1}^{N} E[\pi_i(v_i)] = N \int_0^1 \underbrace{\left[\int_0^{v_i} xg(x)\,dx\right]}_{\text{Expected payment, } \pi_i(v_i)} f(v_i)\,dv_i$$

(b) *Uniformly distributed valuations.* If every bidder's valuation is uniformly distributed, $F(v_i) = v_i$, where $v_i \in [0,1]$, what is the seller's expected revenue from this auction?

- When valuations are uniformly distributed, we obtain that $G(v_i) = v_i^{N-1}$, which implies that

$$g(v_i) = G'(v_i) = (N-1)\,v_i^{N-2},$$

and that $f(v_i) = F'(v_i) = 1$. Therefore, the seller's expected revenue is

$$\begin{aligned}
R^1 &= N \int_0^1 \left[\int_0^{v_i} xg(x)\,dx\right] f(v_i)\,dv_i \\
&= N \int_0^1 \left[\int_0^{v_i} \underbrace{(N-1)\,x^{N-1}}_{xg(x)} dx\right] \underbrace{1}_{f(v_i)} dv_i \\
&= N(N-1) \int_0^1 \left[\frac{x^N}{N}\right]_0^{v_i} dv_i \\
&= (N-1) \int_0^1 v_i^N\,dv_i \\
&= (N-1) \left[\frac{v_i^{N+1}}{N+1}\right]_0^1 \\
&= (N-1)\left(\frac{1}{N+1} - 0\right) \\
&= \frac{N-1}{N+1}.
\end{aligned}$$

- This revenue, $R^1 = \frac{N-1}{N+1}$, coincides with the expected revenue in the second-price auction when valuations are uniformly distributed, $R^2 = \frac{N-1}{N+1}$, found in class.

(c) Does the seller's expected revenue found in part (b) increase or decrease in the number of bidders? What is the seller's expected revenue when $N \to +\infty$?

- The expected revenue is increasing in the number of bidders, N, but at a decreasing rate, since

$$\frac{\partial R^1}{\partial N} = \frac{2}{(N+1)^2} > 0, \text{ and}$$
$$\frac{\partial^2 R^1}{\partial N^2} = -\frac{4}{(N+1)^3} < 0.$$

Intuitively, as more bidders participate in the auction, they submit more aggressive bids. As a result, the expected winning bid that the seller collects is higher. For example, when there are $N = 2$ bidders, the seller generates a revenue of $R^1 = \frac{2-1}{2+1} = \frac{1}{3}$, and increases to $R^1 = \frac{3-1}{3+1} = \frac{1}{2}$ when there are $N = 3$ bidders competing for the object. Figure 3.1 depicts the expected revenue in the first-price auction, R^1, as a function of the number of bidders, N.

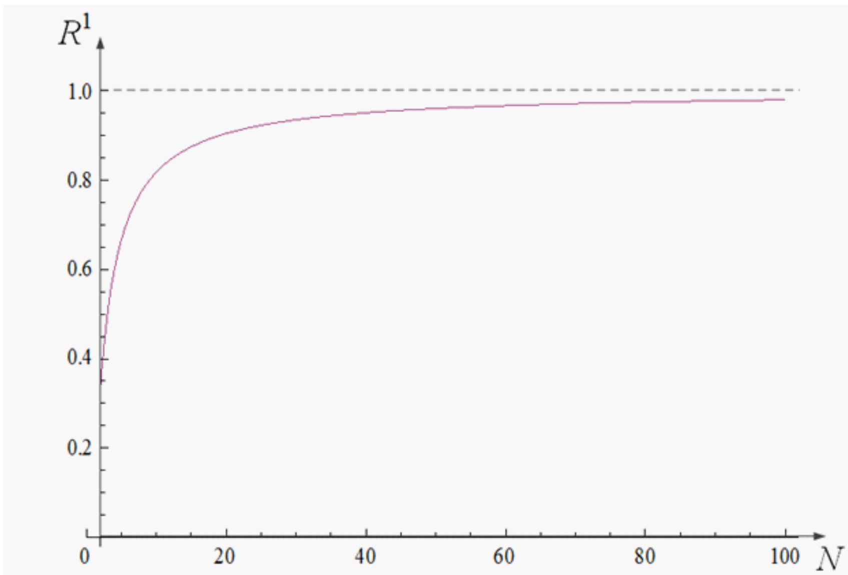

Figure 3.1. Seller's expected revenue in the first-price auction, R^1, as a function of N.

- Furthermore, when the number of bidders grows to infinity, note that

$$\lim_{N\to\infty} R^1 = \lim_{N\to\infty} \left(1 - \frac{2}{N+1}\right) = 1$$

so that the seller obtains a revenue of \$1 from the bidder with the highest valuation.

(d) *Exponentially distributed valuations.* Consider now that individual valuations are drawn from an exponential distribution,

$$F(v_i) = 1 - \exp(-\lambda v_i)$$

where $v_i \in [0, +\infty)$, and there are $N = 2$ bidders. Find the seller's expected revenue in this context. How does expected revenue change with the parameter λ? Interpret your results.

- The seller's expected revenue, with $N = 2$ bidders is

$$R^1 = 2\int_0^\infty \left[\int_0^{v_i} xg(x)\,dx\right] f(v_i)\,dv_i.$$

Since

$$G(v_i) = F(v_i)^{2-1} = F(v_i) = 1 - \exp(-\lambda v_i),$$

we obtain that

$$g(v_i) = G'(v_i) = \lambda \exp(-\lambda v_i).$$

In addition, $f(v_i) = F'(v_i) = \lambda \exp(-\lambda v_i)$, so the above expected revenue becomes

$$\begin{aligned} R^1 &= 2\int_0^\infty \left[\int_0^{v_i} x\underbrace{\lambda \exp(-\lambda x)}_{g(x)}dx\right] f(v_i)\,dv_i \\ &= -2\int_0^\infty \left[\int_0^{v_i} x d\exp(-\lambda x)\right] f(v_i)\,dv_i \end{aligned}$$

Integrating by parts, yields

$$\begin{aligned} R^1 &= -2\int_0^\infty \left\{[x\exp(-\lambda x)]_0^{v_i} - \int_0^{v_i} \exp(-\lambda x)\,dx\right\} f(v_i)\,dv_i \\ &= -2\int_0^\infty \left\{v_i\exp(-\lambda v_i) + \frac{1}{\lambda}[\exp(-\lambda x)]_0^{v_i}\right\} f(v_i)\,dv_i \\ &= -2\int_0^\infty \left\{v_i\exp(-\lambda v_i) + \frac{1}{\lambda}[\exp(-\lambda v_i) - 1]\right\} \underbrace{\lambda\exp(-\lambda v_i)}_{f(v_i)}dv_i \\ &= -2\int_0^\infty \lambda v_i \exp(-2\lambda v_i)\,dv_i - 2\int_0^\infty \exp(-2\lambda v_i)\,dv_i + 2\int_0^\infty \exp(-\lambda v_i)\,dv_i \\ &= \int_0^\infty v_i d\exp(-2\lambda v_i) - 2\int_0^\infty \exp(-2\lambda v_i)\,dv_i - \frac{2}{\lambda}\int_0^\infty d\exp(-\lambda v_i) \end{aligned}$$

Solving each integral and rearranging, we obtain

$$R^1 = [v_i\exp(-2\lambda v_i)]_0^\infty - 3\int_0^\infty \exp(-2\lambda v_i)\,dv_i - \frac{2}{\lambda}[\exp(-\lambda v_i)]_0^\infty.$$

Finally, applying the L'Hôpital rule, we find

$$\begin{aligned} R^1 &= \lim_{v_i\to\infty} \frac{\frac{\partial v_i}{\partial v_i}}{\frac{\partial \exp(2\lambda v_i)}{\partial v_i}} + \frac{3}{2\lambda}[\exp(-2\lambda v_i)]_0^\infty + \frac{2}{\lambda} \\ &= \lim_{v_i\to\infty} \frac{1}{2\lambda\exp(2\lambda v_i)} - \frac{3}{2\lambda} + \frac{2}{\lambda} \\ &= \frac{1}{2\lambda}. \end{aligned}$$

- This revenue, $R^1 = \frac{1}{2\lambda}$, coincides with the expected revenue in the second-price auction when valuations are exponentially distributed, $R^2 = \frac{1}{2\lambda}$, found in part (e) of exercise 1.7. The same comparative statics as in that exercise apply: an increase in parameter λ implies that the exponential distribution assigns a larger probability weight on low valuations producing a decrease in the seller's expected revenue.

(e) *Other distribution forms.* Consider the following distribution function,

$$F(v_i) = (1+\alpha)v_i - \alpha v_i^2$$

where $v_i \in [0,1]$, and parameter α satisfies $\alpha \in [-1,1]$. When $\alpha = 0$, this function collapses to the uniform distribution, $F(v_i) = v_i$; when $\alpha > 0$, it becomes concave, thus putting more probability weight on low valuations; and when $\alpha < 0$, it is convex, assigning more probability weight on high valuations. Find the seller's expected revenue in the setting of $N = 2$ bidders, and compare to the seller's revenue under second-price auction. How is this revenue affected by parameter α? Interpret your results.

- Since $F(v_i) = (1+\alpha)v_i - \alpha v_i^2$, its associated density function is

$$f(v_i) = F'(v_i) = 1 + \alpha - 2\alpha v_i.$$

Since we only have $N = 2$ bidders in this context, $g(v_i) = f(v_i)$, so we obtain that

$$\begin{aligned}
R^1 &= 2\int_0^1 \left[\int_0^{v_i} x g(x)\,dx\right] f(v_i)\,dv_i \\
&= 2\int_0^1 \left[\int_0^{v_i} x\underbrace{(1+\alpha-2\alpha x)}_{g(x)}dx\right] f(v_i)\,dv_i \\
&= 2\int_0^1 \left[\frac{(1+\alpha)v_i^2}{2} - \frac{2\alpha v_i^3}{3}\right]\underbrace{(1+\alpha-2\alpha v_i)}_{f(v_i)}dv_i \\
&= \frac{1}{3}\int_0^1 \left[3(1+\alpha)^2 v_i^2 - 10\alpha(1+\alpha)v_i^3 + 8\alpha^2 v_i^4\right] dv_i \\
&= \frac{1}{3}\left[(1+\alpha)^2 - \frac{5\alpha(1+\alpha)}{2} + \frac{8\alpha^2}{5}\right] \\
&= \frac{\alpha^2 - 5\alpha + 10}{30}.
\end{aligned}$$

Differentiating R^1 with respect to α, we find that

$$\frac{\partial R^1}{\partial \alpha} = \frac{2\alpha - 5}{30}$$

which is positive if $2\alpha > 5$, or $\alpha > 2.5$. Since parameter α satisfies $\alpha \in [-1,1]$ by definition, we obtain that R^1 is decreasing in α. Intuitively, as the probability of low valuations increases (higher α), the seller's expected revenue decreases. As a remark, note that when $\alpha = 0$, the expected revenue simplifies to $R^1 = \frac{1}{3}$, as shown in part (d) of this exercise where, for $N = 2$ bidders, we found that $R^1 = \frac{2-1}{2+1} = \frac{1}{3}$.

4. **Second-price auctions with budget constrained bidders, based on Che and Gale (1998).**[1] Consider a second-price auction with $N \geq 2$ bidders, but assume that every bidder privately observes his valuation for the object, v_i, and his budget, w_i. Bidder i's type in this context is, then, a pair (v_i, w_i), where both v_i and w_i are independently drawn from the $[0, 1]$ interval, that is, $(v_i, w_i) \in [0, 1]^2$. For simplicity, assume that if a bidder wins the auction and the winning price is above his budget, w_i, he cannot afford to pay this price, and the seller imposes a fine on the buyer for having to renege.

   (a) Show that every bidder i finds it dominated to bid above his budget, $b_i > w_i$.

   - Let us consider a bidder who submits a bid above his own budget, $b_i > w_i$, and wins the auction, paying the second-highest bid $h_i = \max_{j \neq i}\{b_j\}$, which denotes the highest bid among all losing bidders $j \neq i$. Two cases can arise:
     - If $h_i \leq w_i$, then he also wins submitting a bid $b_i = w_i$, that entails a price h_i too. Therefore, he would have incentives to deviate to a bid equal to his budget, $b_i = w_i$.
     - If $h_i > w_i$, he cannot afford to pay the winning price h_i, does not win the object and, in addition, receives a fine from the seller. In this context, he would have been better off submitting any bid equal or lower than his budget, $b_i \leq w_i$.

     In summary, we found that bidder i is weakly or strictly better off submitting a bid $b_i = w_i$ than bidding strictly above his budget, $b_i > w_i$. In other words, bidding strictly above his budget, $b_i > w_i$, is a dominated strategy.

   (b) If bidder i's valuation, v_i, satisfies $v_i \leq w_i$ (i.e., his budget constraint does not bind), show that bidding according to his valuation, $b_i = v_i$ (as in Exercise 1.2) is still a weakly dominant strategy in the second-price auction.

   - In this case, the budget constraint does not bind. Therefore, we can follow the same argument as in Exercise 1.2 to show that bidding according to his valuation, $b_i = v_i$, is a weakly dominant strategy in the second-price auction.

   (c) If bidder i's valuation, v_i, satisfies $v_i > w_i$ (i.e., his budget constraint binds), show that submitting a bid equal to his budget, $b_i = w_i$, is a weakly dominant strategy.

   - If $v_i > w_i$, it is easy to show that bidder i finds $b_i = w_i$ is a weakly dominant strategy. Two cases can arise depending on how h_i ranks relative to w_i:
     - If $h_i \leq w_i$, he wins the object, paying h_i, which he can afford. If he decreases his bid, he still pays h_i if he wins the object, but he may lose the auction if his bid satisfies $b_i < h_i$, so he has no strict incentives to deviate from $b_i = w_i$.

[1] Che, Yeon-Koo and Ian Gale (1998) "Standard Auctions with Financially Constrained Bidders," The Review of Economic Studies, 65(1), pp. 1-21.

- If $h_i > w_i$, he either loses the auction because the highest bid among his rivals, h_i, exceeds his bid, $b_i = w_i$, or wins the auction which is not affordable for bidder i in this case. Recall that if he bids above his budget and wins, where $b_i > h_i > w_i$, he would not be able to pay the price, losing the auction and receiving a penalty from the seller after reneging. Therefore, deviations from bidding $b_i = w_i$ does not strictly increase bidder i's payoff.

Overall, we showed that bidding $b_i = w_i$ yields a payoff that bidder i cannot strictly increase by submitting a different bid.

(d) Combine your results from parts (b) and (c) to describe the equilibrium bidding function in the second-price auction with budget constraints, $b_i(v_i, w_i)$. Depict it as a function of v_i.

- We found that, every bidder i submits a bid equal to his valuation, v_i, when his budget constraint is not binding (as in part b), and a bid equal to his budget, w_i, otherwise (as in part c). More compactly,

$$b_i(v_i, w_i) = \begin{cases} v_i & \text{if } v_i \leq w_i \\ w_i & \text{otherwise.} \end{cases}$$

As depicted in figure 1.8, this bid coincides with the 45^0-line, $b_i = v_i$, when his valuation satisfies $v_i \leq w_i$, but becomes $b_i = w_i$ otherwise (see the flat segment). Graphically, when the budget constraint does not bind, $v_i \leq w_i$, the bidder behaves as in a standard second-price auction, where $b_i = v_i$, but otherwise submits a bid equal to his budget w_i.

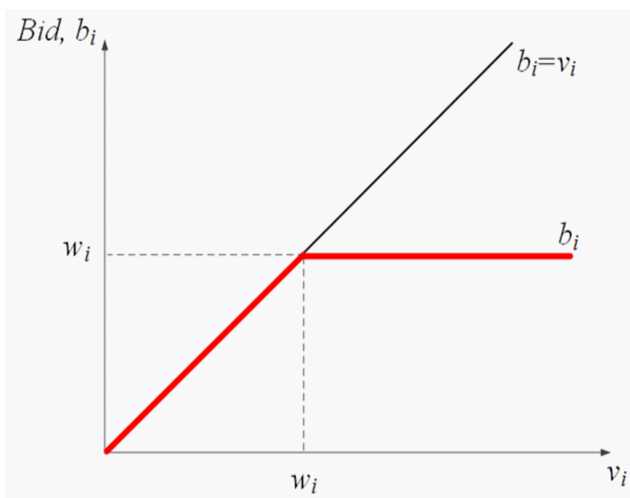

Figure 1.8. Equilibrium bids in a second-price auction with budget constraints.

This bidding function is often presented as

$$b_i(v_i, w_i) = \min\{v_i, w_i\},$$

since bidder i's bid is the lowest of his valuation for the object (when such valuation is lower than his budget, w_i) and his budget (when his valuation exceeds his budget).