EconS 503 - Microeconomic Theory II
Homework #5 - Due date: March 15th, 2021

1. **Cournot competition with uncertain costs.** Consider an industry with two firms competing à la Cournot and facing inverse demand function $p(Q) = 1 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. Every firm i is privately informed about its marginal cost, high or low, denoted as c_H and c_L, respectively, where $1 > c_H > c_L = 0$. Finally, consider that, while firm j cannot observe the realization of firm i's marginal cost (c_H or c_L), firm j knows that that both types are equally likely. Firms then interact in a simultaneous-move game of incomplete information, and in this exercise we seek to find the Bayesian Nash equilibrium of the game.

   (a) Find the best response function for every firm i when its marginal costs are low, $q_i^L(q_j^H, q_j^L)$.

   (b) Find the best response function for every firm i when its marginal costs are high, $q_i^H(q_j^H, q_j^L)$.

   (c) Use your results from parts (a) and (b) to find the Bayesian Nash Equilibrium (BNE) of the game.

2. **Price competition with heterogeneous goods and uncertain costs.** Consider two firms competing in prices à la Bertrand and selling heterogeneous goods. The demand function of every firm i is

$$q_i\left(p_i, p_j\right) = 1 - \gamma p_i + p_j$$

   where $\gamma \geq 1$ denotes the degree of product differentiation (i.e., homogeneous goods when $\gamma = 1$ but differentiated when $\gamma > 1$). Every firm i faces a constant marginal cost of c_H with probability β and a marginal cost c_L with the remaining probability $1 - \beta$, where $1 > c_H > c_L \geq 0$. Every firm i privately observes its own marginal cost, but does not observe the marginal cost of its rival. The probability distribution over costs c_H and c_L is common knowledge among firms.

   (a) Find every firm i's best response function when its marginal cost is high, c_H, and its best response function when its marginal cost is low, c_L.

   (b) What are the equilibrium prices?

   (c) How are equilibrium prices affected by changes in parameter γ and β?

   (d) *Numerical example.* Assume that $\gamma = 3/2$, $c_L = 1/4$, and $c_H = 1/2$. Find the equilibrium prices p^H and p^L, and confirm that they increase in β. Then, evaluate the equilibrium prices at $\beta = 0$ and at $\beta = 1$. Interpret.

3. **Expected revenue in the first-price auction.** Consider the first-price auction with $N \geq 2$ bidders, where every bidder i independently draws his value for the object, v_i.

(a) Assuming that every bidder's valuation is distributed according to a generic cumulative distribution function $F(v_i)$, find the seller's expected revenue from the auction.

(b) *Uniformly distributed valuations.* If every bidder's valuation is uniformly distributed, $F(v_i) = v_i$, where $v_i \in [0, 1]$, what is the seller's expected revenue from this auction?

(c) Does the seller's expected revenue found in part (b) increase or decrease in the number of bidders? What is the seller's expected revenue when $N \to +\infty$?

(d) *Exponentially distributed valuations.* Consider now that individual valuations are drawn from an exponential distribution,

$$F(v_i) = 1 - \exp(-\lambda v_i)$$

where $v_i \in [0, +\infty)$, and there are $N = 2$ bidders. Find the seller's expected revenue in this context. How does expected revenue change with the parameter λ? Interpret your results.

(e) *Other distribution forms.* Consider the following distribution function,

$$F(v_i) = (1 + \alpha) v_i - \alpha v_i^2$$

where $v_i \in [0, 1]$, and parameter α satisfies $\alpha \in [-1, 1]$. When $\alpha = 0$, this function collapses to the uniform distribution, $F(v_i) = v_i$; when $\alpha > 0$, it becomes concave, thus putting more probability weight on low valuations; and when $\alpha < 0$, it is convex, assigning more probability weight on high valuations. Find the seller's expected revenue in the setting of $N = 2$ bidders, and compare to the seller's revenue under second-price auction. How is this revenue affected by parameter α? Interpret your results.

4. **Second-price auctions with budget constrained bidders, based on Che and Gale (1998).**[1] Consider a second-price auction with $N \geq 2$ bidders, but assume that every bidder privately observes his valuation for the object, v_i, and his budget, w_i. Bidder i's type in this context is, then, a pair (v_i, w_i), where both v_i and w_i are independently drawn from the $[0, 1]$ interval, that is, $(v_i, w_i) \in [0, 1]^2$. For simplicity, assume that if a bidder wins the auction and the winning price is above his budget, w_i, he cannot afford to pay this price, and the seller imposes a fine on the buyer for having to renege.

   (a) Show that every bidder i finds it dominated to bid above his budget, $b_i > w_i$.

   (b) If bidder i's valuation, v_i, satisfies $v_i \leq w_i$ (i.e., his budget constraint does not bind), show that bidding according to his valuation, $b_i = v_i$ (as in Exercise 1.2) is still a weakly dominant strategy in the second-price auction.

   (c) If bidder i's valuation, v_i, satisfies $v_i > w_i$ (i.e., his budget constraint binds), show that submitting a bid equal to his budget, $b_i = w_i$, is a weakly dominant strategy.

[1] Che, Yeon-Koo and Ian Gale (1998) "Standard Auctions with Financially Constrained Bidders," The Review of Economic Studies, 65(1), pp. 1-21.

(d) Combine your results from parts (b) and (c) to describe the equilibrium bidding function in the second-price auction with budget constraints, $b_i(v_i, w_i)$. Depict it as a function of v_i.