EconS 503 - Microeconomic Theory II
Homework #4 - Answer key

1. **Bargaining with infinite periods and $N \geq 2$ players.** Consider the infinite-period alternating-offer bargaining game presented in class, but let us allow for $N \geq 2$ players. Player 1 is the proposer in period 1, period $N+1$, period $2N+1$, and so on. Similarly, player 2 is the proposer in period 2, period $N+2$, period $2N+2$, and so on. A similar argument applies to any player who becomes the proposer for the first time in period k, becoming again the proposer in period $N+k$, period $2N+k$, etc.

   A division of the surplus at period t is a vector describing the share that each player receives $(d_t^1, d_t^2, ..., d_t^N)$ satisfying $d_t^i \in [0,1]$ for every player i and $\sum_{i=1}^N d_t^i = 1$. Assume that a division must be approved by all other $N-1$ players for it to be accepted (i.e., it requires unanimity). Focus on stationary equilibrium offers, implying that the equilibrium payoff that every player earns only depends on who is the player making offers at that period (himself, the player making offers in the next period, the player making offers in two periods from now, etc.)

   (a) Find the SPE of this game.

   - Let d denote the offer that the player proposing in that period makes himself, d^1 the offer to the player who makes offers in the next period, and, generally, d^k the offer to the player who becomes the proposer in k periods from today.
   - At any period, the player proposing must offer a division d^1 to the player becoming the proposer in the next period that exceeds the discounted value of the equilibrium payoff he anticipates in the next period (when he becomes the proposer, δd. In short, the proposing player's offer, d^1, must satisfy $d^1 \geq \delta d$, which the proposing player reduces as much as possible, making the next player indifferent, that is,

   $$d^1 = \delta d.$$

   - Generally, the proposing player today makes an offer to the player who becomes the proposer k periods from today, d^k, that satisfies

   $$d^k = \delta^{k-1} d,$$

   which gives rise to $N-1$ equations (i.e., $k=2$, $k=3$,..., $k=N$).
   - To simultaneously solve these equations, we start by writing the property that all offers must add up to 1, that is,

   $$d + d^1 + d^2 + ... + d^N = 1$$

   Using now our above result, $d^k = \delta^{k-1} d$, we obtain

   $$d + \delta d + ... + \delta^{N-1} d = 1$$

since $d^1 = \delta d$, $d^2 = \delta^2 d$, ..., $d^{N-1} = \delta^{N-1} d$. Rearranging the above expression, yields

$$d(1 + \delta + ... + \delta^{N-1}) = 1$$

Since the term in parenthesis is a geometric series, we can rewrite this equation as follows

$$d\frac{1-\delta^N}{1-\delta} = 1$$

and, solving for d, we find the equilibrium offer that the proposing player makes himself

$$d = \frac{1-\delta}{1-\delta^N}.$$

Using this result, we obtain that the equilibrium offer that the proposing player makes to the player who will become the proposer in the next period $(k = 2)$ is

$$d^1 = \delta d = \delta\frac{1-\delta}{1-\delta^N}.$$

And, generally, the equilibrium offer that the proposing player makes to the player who will become the proposer k periods from today is

$$d^k = \delta^{k-1} d = \delta^{k-1}\frac{1-\delta}{1-\delta^N}.$$

Summarizing, the equilibrium offers

$$\left(d, d^1, ..., d^{N-1}\right) = \left(\frac{1-\delta}{1-\delta^N}, \delta\frac{1-\delta}{1-\delta^N}, ..., \delta^{N-1}\frac{1-\delta}{1-\delta^N}\right).$$

As in the alternating-offer bargaining game, player 1 makes an offer at the beginning of the game that is immediately accepted by all $N-1$ players, and the game ends. Like in the game with two players, an increase in the common discount factor, δ, decreases player 1's offer to himself, d, while increasing the offers to each of the players who would become the proposers after him (player $2, 3, ..., N-1$).

- When $\delta \to 0$, the above equilibrium offers become

$$\left(d, d^1, ..., d^{N-1}\right) = (1, 0, ..., 0)$$

implying that the proposer players keeps all the surplus when players are extremely impatient. In contrast, when $\delta \to 1$, the equilibrium payoffs become

$$\left(d, d^1, ..., d^{N-1}\right) = \left(\frac{1}{N}, \frac{1}{N}, ..., \frac{1}{N}\right)$$

meaning that all players receive the same offer in equilibrium. This offer, however, is decreasing in the number of players.

(b) Evaluate your results at $N = 2$ and show that they coincide with those in the infinite-period alternating-offer bargaining game presented in class.

- Evaluating our above equilibrium offers in the special case of $N = 2$ players, we obtain

$$(d, d^1) = \left(\frac{1-\delta}{1-\delta^2}, \delta \frac{1-\delta}{1-\delta^2} \right).$$

Using the property that $1 - \delta^2 = (1 + \delta)(1 - \delta)$, we can simplify the above equilibrium offers as follows

$$(d, d^1) = \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta} \right)$$

which coincides with our equilibrium results with two players in class.

(c) Evaluate your results at $N = 3$. Compare them with those in part (b).

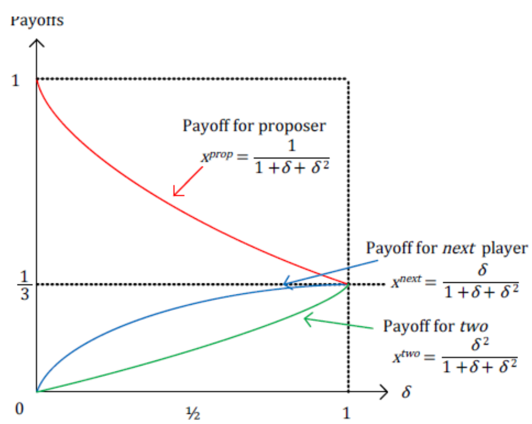
- valuating our above equilibrium offers in the special case of $N = 2$ players, we obtain

$$(d, d^1, d^2) = \left(\frac{1-\delta}{1-\delta^3}, \delta \frac{1-\delta}{1-\delta^3}, \delta^2 \frac{1-\delta}{1-\delta^3} \right).$$

Since $1 - \delta^3 = (1 - \delta)(1 + \delta + \delta^2)$, we can rearrange the equilibrium offers as follows

$$(d, d^1, d^2) = \left(\frac{1}{1+\delta+\delta^2}, \frac{\delta}{1+\delta+\delta^2}, \frac{\delta^2}{1+\delta+\delta^2} \right).$$

which we depict in the next figure.



Intuitively, when players are relatively impatient, the player who makes the first proposal fares better than do the others (he captures most of the surplus). When players are relatively patient, however, all players get relatively similar equilibrium payoffs (approaching $\frac{1}{3}$ when $\delta \to 1$).

2. **Exercises from Tadelis:**

(a) Exercises from Chapter 10: 10.6, 10.9, and 10.11.

- See scanned pages at the end of this handout.

3. Exercise 8.30 (Chapter 8) from the *Advanced Microeconomic Theory* textbook (MIT Press) [**Collusion with Imperfect Monitoring**] Consider the setting in Example 8.15. For simplicity, assume that firms can only choose among three possible output levels,

$$q_i^t = \left\{ \frac{a-c}{3b}, \frac{a-c}{4b}, \frac{3(a-c)}{8b} \right\},$$

which indicate, respectively, the equilibrium output in the Cournot unrepeated game, the collusive output (half of monopoly output), and every firm's optimal deviation to its rival choosing the collusive output. In addition, assume that monitoring is imperfect, but the probability of detecting a deviation increases with your rival's output:

i. If its rival selects the collusive output $\frac{a-c}{4b}$, firm i observes a deviation from such output with probability zero

ii. If its rival produces $\frac{a-c}{3b}$, where $\frac{a-c}{3b} > \frac{a-c}{4b}$ firm i detects a deviation from collusion with probability 15%

iii. If its rival produces $\frac{3(a-c)}{8b}$, where $\frac{3(a-c)}{8b} > \frac{a-c}{3b}$, firm i detects a deviation from collusion with probability 60%

a. Find the minimal discount factor sustaining cooperation in the infinitely repeated game, and compare it with that when firms operate under perfect monitoring (i.e. $\delta > \frac{9}{17}$, as found in Example 8.15).

- First, note from Exercise 8.15 that the payoffs for each possible output level are $\left\{ \frac{(a-c)^2}{9b}, \frac{(a-c)^2}{8b}, \frac{9(a-c)^2}{64b} \right\}$, and it is never optimal for a firm to deviate to the Cournot level of output, since it will unambiguously give a lower payoff for any value of δ. For collusion to be sustained, then, it must be true that

$$\frac{1}{1-\delta}\pi^{Coop} > \pi^{Dev} + \delta \left(\underbrace{0.6 \times \frac{1}{1-\delta}\pi^{Cournot}}_{\text{Firm gets caught}} + \underbrace{0.4 \times \frac{1}{1-\delta}\pi^{Coop}}_{\text{Firm doesn't get caught}} \right)$$

that is

$$\frac{1}{1-\delta}\frac{(a-c)^2}{8b} > \frac{9(a-c)^2}{64b} + \delta \left(0.6 \times \frac{1}{1-\delta}\frac{(a-c)^2}{9b} + 0.4 \times \frac{1}{1-\delta}\frac{(a-c)^2}{8b} \right)$$

Rearranging,

$$\frac{1}{8} > \frac{9}{64}(1-\delta) + 0.6\delta \times \frac{1}{9} + 0.4\delta \times \frac{1}{8}$$

and solving for δ yields

$$\delta \left(\frac{23}{960} \right) > \frac{1}{64} \implies \delta > \frac{15}{23}$$

- It is clear to see that the minimal discount factor to sustain collusion is higher when monitoring is imperfect than it is when perfect monitoring is availbale ($\frac{15}{23} > \frac{9}{17}$)

b. What about the general case, where the probability of getting caught is p?

- Just as in part (a), for collusion to be sustained, then, it must be true that

$$\frac{1}{1-\delta}\pi^{Coop} > \pi^{Dev} + \delta\left(\underbrace{p \times \frac{1}{1-\delta}\pi^{Cournot}}_{\text{Firm gets caught}} + \underbrace{(1-p) \times \frac{1}{1-\delta}\pi^{Coop}}_{\text{Firm doesn't get caught}}\right)$$

that is,

$$\frac{1}{1-\delta}\frac{(a-c)^2}{8b} > \frac{9(a-c)^2}{64b} + \delta\left(p \times \frac{1}{1-\delta}\frac{(a-c)^2}{9b} + (1-p) \times \frac{1}{1-\delta}\frac{(a-c)^2}{8b}\right)$$

Rearranging,

$$\frac{1}{8} > \frac{9}{64}(1-\delta) + p\delta \times \frac{1}{9} + (1-p)\delta \times \frac{1}{8}$$

and solving for δ yields

$$\delta\left(\frac{9+8p}{576}\right) > \frac{1}{64} \implies \delta(p) > \frac{9}{9+8p}$$

Figure 8.28 plots $\delta(p)$. Note that when the monitoring is perfect, i.e., $p = 1$, $\delta > \frac{9}{17}$, which is the same as that in Cournot equilibrium.

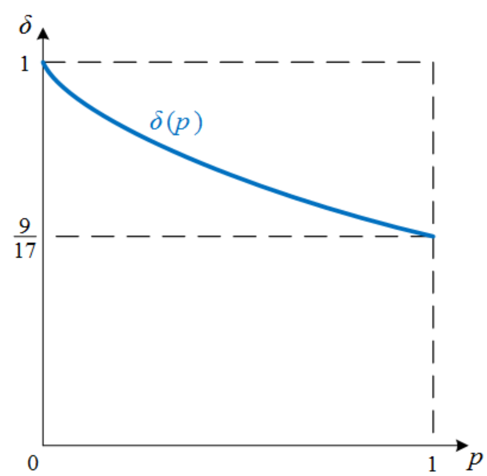


Figure 8.28. Minimal discount factor supporting cooperation.

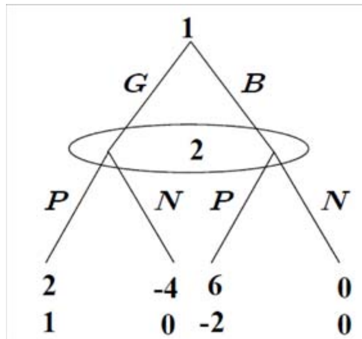


FIGURE 10.1.

the consumer. The "bad" (B) manufacturing procedure costs 0 to the firm, and yields a value of 4 to the consumer. The consumer can choose whether to buy or not at the price p, and this decision must be made before the actual manufacturing procedure is revealed. However, after consumption, the true quality is revealed to the consumer. The choice of manufacturing procedure, and the cost of production, is made before the firm knows whether the consumer will buy or not.

(a) Draw the game tree and the matrix of this game, and find all the Nash equilibria of this game.

Answer: Let player 1 be the firm who can choose G (good) or B (bad), and player 2 is the consumer who can choose P (purchase) or N (not purchase). If, for example, the players choose (G, P) then the firm gets $6-4=2$ and the consumer gets $7-6=1$. In a similar way the complete matrix of this one shot game can be represented as follows:

		Player 2	
		P	N
Player 1	G	$2, 1$	$-4, 0$
	B	$6, -2$	$0, 0$

The extensive form game tree is,■

(b) Now assume that the game described above is repeated twice. (The consumer learns the quality of the product in each period only if he consumes.) Assume that each player tries to maximize the (non-discounted) sum of his stage payoffs. Find *all* the subgame-perfect equilibria of this game.

Answer: It is easy to see that player 1 has a dominant strategy in the stage game: choose B, and player 2's best response is to choose N. This unique Nash equilibrium must be played in the second stage, and by backward induction must also be played in the first stage. hence, it is the unique subgame perfect equilibrium.

(c) Now assume that the game as repeated infinitely many times. Assume that each player tries to maximize the discounted sum of his or her stage payoffs, where the discount rate is $\delta \in (0,1)$. What is the range of discount factors for which the good manufacturing procedure will be used as part of a subgame perfect equilibrium?

Answer: Consider the grim trigger strategies: player 1 chooses G and continues to choose G as long as he chose G in the past and as long as player 2 purchased. Otherwise he chooses B forever after. Player 2 chooses P and continues to choose P as long as he chose P and player 1 chose G. Otherwise he plays N forever after. Player 2 has no incentive to deviate at any stage, but player 1 can gain 4 from switching to B in any period (get 6 instead of 2). He will not have an incentive to deviate if $4 \leq \frac{2}{1-\delta}$, which holds for $\delta \in [\frac{1}{2}, 1)$. ■

(d) Consumer advocates are pushing for a lower price of the drug, say 5. The firm wants to approach the Federal trade Commission and argue that if the regulated price is decreased to 5 then this may have dire consequences for both consumers and the firm. Can you make a formal argument using the parameters above to support the firm? What about the consumers?

Answer: If the price of the drug is lowered to 5 then player 1 has a stronger *relative* temptation to deviate from the grim trigger strategies described in part **c.** above. His gain from deviation is still 4, but the gain from continuing to choose G is only 1 per period and not 2. Hence, he will not have an incentive to deviate if $4 \leq \frac{1}{1-\delta}$, which holds for $\delta \in [\frac{3}{4}, 1)$. Hence, if the firm can argue that $\delta \in [\frac{1}{2}, \frac{3}{4})$ then increasing the price from 4 to 5 will cause the good equilibrium to collapse and no trade will occur. The argument in favor of raising the price can be made if $\delta \in [\frac{3}{4}, 1)$ because then the consumers benefit at the expense of the firm but there is enough surplus to support the good outcome. ■

7. **Diluted Happiness:** Consider a relationship between a bartender and a customer. The bartender serves bourbon to the customer, and chooses $x \in [0, 1]$, which is the proportion of bourbon in the drink served, while $1 - x$ is the proportion of water. The cost of supplying such a drink (standard 4 once glass) is cx where $c > 0$. The Customer, without knowing x, decides on whether or not to buy the drink at the market price p. If he buys the drink, his payoff is $vx - p$, and the bartender's payoff is $p - cx$. Assume that $v > c$, and all payoffs are common knowledge. If the customer does not buy the drink, he gets 0, and the bartender gets $-(cx)$. because the customer has some experience, once the drink is bought and he tastes it, he learns the value of x, but this is only after he pays for the drink.

   (a) Find all the Nash equilibria of this game.

   Answer: The customer has to buy the drink without knowing its content, implying that the bartender has a dominant strategy which is to choose $x = 0$ once the customer pays for the drink. But anticipating that, the customer would not buy the drink. Hence, the unique Nash equilibrium is for the customer not to buy and the bartender to choose $x = 0$ if he does buy. ■

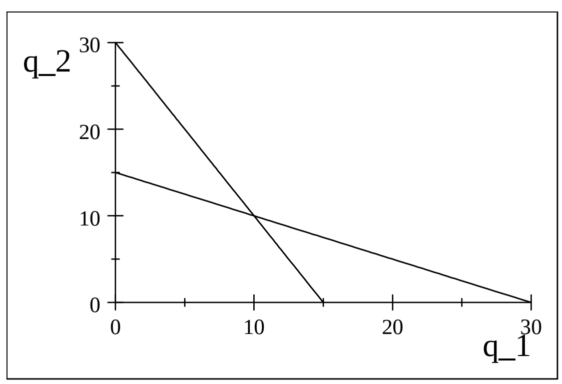
   (b) Now assume that the customer is visiting town for 10 days, and this "bar game" will be played for each of the 10 evenings that the customer is in town. Assume that each player tries to maximize the (non-discounted) sum of his stage payoffs. Find all subgame-perfect equilibria of this game.

   Answer: The game just unravels: in the last period they must play or $\delta^2 + \delta - 1 \geq 0$, which results in $\delta \geq \frac{1}{2}\sqrt{5} - \frac{1}{2} \approx 0.618$. The reason we need a larger discount factor is that the punishment is less severe as it lasts for only two periods and not infinitely many. ■

9. **Negative Externalities:** Two firms are located adjacent to one another and each imposes an external cost on the other: the detergent that Firm 1 uses in it's laundry business makes the fish that firm 2 catches in the lake taste funny, and the smoke that firm 2 uses to smoke its caught fish makes the clothes that firm 1 hands out to dry smell funny. As a consequence, each firms profits are increasing it its own production and decreasing in the production of its neighboring firm. In particular, if q_1 and q_2 are the firms' production levels then their per-period (stage game) profits are given by $v_1(q_1, q_2) = (30 - q_2)q_1 - q_1^2$ and $v_2(q_1, q_2) = (30 - q_1)q_2 - q_2^2$.

   (a) Draw the firms' best response functions and find the Nash equilibrium of the stage game. How does this compare to the Pareto optimal stage-game profit levels?

   Answer: Each firm maximizes $v_1(q_i, q_j) = (30 - q_j)q_i - q_i^2$ and the first order condition is $30 - q_j - 2q_i = 0$, resulting in the best response function $q_i = \frac{30-q_j}{2}$ as drawn in the following figure:



   The unique Nash equilibrium is $q_1 = q_2 = 10$ giving each firm a profit of 100. To solve for the Pareto optimal outcome we can maximize the sum of profits,

$$\max_{q_1, q_2} S(q_1, q_2) = (30 - q_2)q_1 - q_1^2 + (30 - q_1)q_2 - q_2^2$$

and the two first order conditions are

$$\frac{\partial S(q_1,q_2)}{\partial q_1} = 30 - q_2 - 2q_1 - q_2 = 0$$
$$\frac{\partial S(q_1,q_2)}{\partial q_2} = 30 - q_1 - 2q_2 - q_1 = 0$$

and solving them together yields $q_1 = q_2 = 7\frac{1}{2}$ and the profits of each firm are $112\frac{1}{2}$. ■

(b) For which levels of discount factors can the firms support the Pareto optimal level of quantities in an infinitely repeated game?

Answer: We consider grim trigger strategies of the form "I will choose $q_i = 7.5$ and continue to do so as long as both chose this value. If anyone ever deviates I will revert to $q_i = 10$ forever." The best deviation from $q_i = 7.5$ given that $q_j = 7.5$ is to choose the best response to 7.5 which is $\frac{30-7.5}{2} = 11.25$, and the profit from deviating is $(30 - 7\frac{1}{2})11\frac{1}{4} - (11\frac{1}{4})^2 = \frac{2025}{16} = 126\frac{9}{16}$. Thus, each player will not want to deviate if

$$126\frac{9}{16} + \delta\frac{100}{1-\delta} \leq \frac{112\frac{1}{2}}{1-\delta}$$

which holds for $\delta \in [\frac{9}{17}, 1)$. ■

10. **Law Merchants (revisited):** Consider the three person game described in section **??**. A subgame perfect equilibrium was constructed with a bond equal to 2, and a wage paid by every player P_2^t to player 3 equal to $w = 0.1$, and it was shown that it is indeed an equilibrium for any discount factor $\delta \geq 0.95$. Show that a similar equilibrium, where players P_1^t trust players P_2^t who post bonds, players P_2^t post bonds and cooperate, and player 3 follows the contract in every period, for any discount factor $0 < \delta < 1$.

Answer: First notice that the bond need not be equal to 2 because player P_2^t only gains 1 from deviating. Hence, any bond of value $1 + \varepsilon > 1$ will deter player P_2^t from choosing to defect instead of cooperate. Second, notice that for any wage to the third party of $1 - \varepsilon < 1$, player P_2^t still get a positive surplus $\varepsilon > 0$ from engaging the services of the third party. Hence, for any value of $\varepsilon \in (0, 1)$, posting a bond of $1 + \varepsilon$ and paying the third party $1 - \varepsilon$ guarantees that player P_2^t will choose to employ the third party and cooperates if trusted, and in turn, P_1^t will choose to trust. We are left to see whether the third party prefers to return the bond as promised or if he would deviate and give up the future stream of all income. By deviating the third party pockets the bon worth $1 + \varepsilon$, and gives up the future series of wages $1 - \varepsilon$ for all future periods. Hence, he will not deviate if

$$1 + \varepsilon \leq \frac{\delta}{1 - \delta}(2 - \varepsilon),$$

which for $\varepsilon \in (0, 1)$ holds for $\delta \in (\frac{1+\varepsilon}{2}, 1)$. Hence, for any $\delta > \frac{1}{2}$ there exists a small enough $\varepsilon > 0$ for which the inequality above holds. ■

11. **Trading Brand Names:** Show that the strategies proposed in Section **??** constitute a subgame perfect equilibrium of the sequence of trust games.

    Answer: Consider any player P_2^t, $t > 1$. Under the proposed strategies, if trust was never abused and the name was bought up till period $t - 1$ then (i) by buying the name and cooperating he is guaranteed a payoff of 1, (ii) by buying the name and defecting he receives 2 but cannot sell the name to the next player 2 and hence he gets $2 - p^* < 1$, and (iii) by not buying the name he gets 0. Hence, for any t the strategy of P_2^t is a best response. Consider player P_2^1. If he (i) by creating the name and cooperating he is guaranteed a payoff of $1 + p^* > 2$, (ii) by not creating the name he gets 0. Hence, the strategy of P_2^1 is a best response. Last, it is easy to see that any player 1 can expect cooperation, and hence trusting is a best response conditional on no one ever defecting and the name being created and transmitted. ■

12. **Folk Theorem (revisited):** Consider the infinitely repeated trust game described in Figure 10.1.

    (a) Draw the convex hull of average payoffs.

    Answer: ■