EconS 503 - Microeconomic Theory II
Midterm Exam #1 - Answer key

1. **Rationalizable strategies and IDSDS.** Consider the following two-player game, where player 1 (in rows) chooses U, M, or D, while player 2 (in columns) chooses L or R.

		Player 2	
		L	R
Player 1	U	$4,2$	$0,1$
	M	$0,2$	$4,1$
	D	$1,0$	$1,3$

   (a) Assuming that players are restricted to use pure strategies, show that IDSDS has no bite while rationalizability has some bite.

   - *Applying IDSDS.* Starting with player 1, we see that he has no strictly dominated (pure) strategies. Similarly, player 2 has no strictly dominated (pure) strategies, implying that IDSDS has no bite. As a consequence, we could not delete any row or column from the original matrix in our application of IDSDS, implying that all six strategy profiles are our equilibrium prediction according to IDSDS.
   - *Applying rationalizability.* Starting with player 1, we see that strategy D is never a best response (NBR): when player 2 chooses column L, player 1's best response is U; whereas when player 2 chooses column R, player 1's best response is M. In summary, $BR_1(L) = U$ and $BR_1(R) = M$, meaning that player 1 never uses D as his best response against player 2's strategies. We can then delete row D from the original matrix, in our first step of applying rationalizability (deleting strategies that are NBR), obtaining the following 2×2 matrix.

		Player 2	
		L	R
Player 1	U	$4,2$	$0,1$
	M	$0,2$	$4,1$

     We can now turn to player 2, finding his best responses: when player 1 chooses U, player 2's best response is L; and, similarly, when player 1 selects M, player 2's best response is L. More compactly, $BR_2(s_1) = L$ for every $s_1 = \{U, M\}$, implying that R is NBR. Deleting column R from the above 2×2 matrix, yields

		Player 2
		L
Player 1	U	$4,2$
	M	$0,2$

     We can finally turn again to player 1, showing that his best response at this point is $BR_1(L) = U$, implying that M is NBR. Deleting row M, we obtain the following matrix

		Player 2
		L
Player 1	U	$4,2$

Therefore, rationalizability produces a unique equilibrium prediction in this game

$$\text{Rationalizable}=\{(U,L)\}.$$

- Comparing the application of IDSDS and rationalizability, we see that the latter has more bite than the former. This is, however, due to the fact that we restrict players to use pure strategies. In the next part of this exercise, we show that both solution concepts yield the same equilibrium predictions when we allow players to use mixed strategies.

(b) Allowing now for players to use mixed strategies, show that IDSDS and rationalizability produce the same equilibrium results.

- *Applying IDSDS.* Starting with player 1, we can build a randomization between U and M with a expected payoff that strictly dominates D. In particular, when player 2 chooses L, we have that

$$EU_1 = p4 + (1-p)0 > 1$$

which holds if $p > \frac{1}{4}$. Similarly, when player 2 chooses R, we have that

$$EU_1 = p0 + (1-p)4 > 1$$

which holds if $p < \frac{3}{4}$. Therefore, for all satisfying both conditions, $\frac{3}{4} > p > \frac{1}{4}$, we have that player 1's randomization between U and M strictly dominates D, helping us delete row D from the original matrix. This leaves us with the following 2×2 matrix.

		Player 2	
		L	R
Player 1	U	$4,2$	$0,1$
	M	$0,2$	$4,1$

Moving now to player 2, we see that R is strictly dominated by L (when player 1 chooses U, player 2's payoff from L is strictly higher than that from R; and a similar argument applies when player 1 chooses M). After deleting column R from the above matrix, we obtain

		Player 2
		L
Player 1	U	$4,2$
	M	$0,2$

Finally, we can turn to player 1 one more time, noticing that U strictly dominates M since $4 > 0$. After deleting row M, we obtain

		Player 2
		L
Player 1	U	$4, 2$

Therefore, IDSDS yields a unique equilibrium prediction in this game

$$\text{IDSDS}=\{(U, L)\}$$

which coincides with the equilibrium prediction according to rationalizability. This coincidence confirms Pearce's (1984) result about IDSDS and rationalizability yielding the same equilibrium predictions in two-player games each player facing a finite number of pure strategies.

2. **Strict Nash equilibrium.** Consider the following definition: A strategy profile $s^* \equiv (s_1^*, ..., s_N^*)$ is a *strict Nash equilibrium* (SNE) if it satisfies

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*) \text{ for every player } i, \text{ and every } s_i \in S_i.$$

You probably noticed that this definition is almost identical to the definition of Nash equilibrium (NE), except for using a strict, rather than weak, inequality sign. In this exercise we connect both solution concepts, but first examine the relationship between a strict Nash equilibrium and IDSDS.

(a) Show that if strategy profile s^* is a NE, it doesn't need to be a SNE. An example suffices.

- Consider the following payoff matrix

		Player 2		
		L	M	R
	U	$\underline{6},\underline{5}$	$7,3$	$\underline{8},4$
Player 1	C	$4,3$	$10,6$	$5,\underline{8}$
	D	$5,2$	$\underline{11},8$	$4,\underline{10}$

- *Nash equilibrium.* First, we underline best response payoffs of each player to find the Nash equilibria of the game. We identify only one Nash equilibrium (U, L), where every player chooses best responses to his opponent's equilibrium strategies.
- *Strict Nash equilibrium.* To see that this game has no SNE, note that there is no strictly dominant strategies, neither for player 1 nor for player 2. From our discussion in parts (a) and (b), this implies that there is no SNE.

(b) Show that if strategy profile s^* is a SNE, the game has no mixed strategy NE.

- If s^* is a SNE, then, it is the only strategy profile surviving IDSDS and, in turn, s^* is the unique pure strategy NE of the game. Both players play their strictly dominant strategies in this NE, and there is no belief that other player will play different strategies. Thus, we cannot sustain a mixed strategy NE.

3. **Contests with symmetric valuations and two players.** Consider a contest where two players compete to earn a prize and the probability of winning the prize is a function of a player's own investment relative to the aggregate investment of all the players. Contests are often used to model promotions within a firm (where every worker invests time and effort into being selected for a promotion), political campaigns (where candidates invest money and resources to capture a larger share of votes), and R&D races (where firms invest resources into discovering a new product, such as drug). We consider two players, players i and j, who assign the same value to the prize, $V \geq 1$. The probability of winning the prize is given by

$$p_i = \frac{x_i^r}{x_i^r + x_j^r}$$

where x_i denotes player i's investment and the parameter $2 \geq r \geq 1$ represents the effectiveness of player i's investment, which is assumed to be the same for every player. For simplicity, we normalize the cost of every unit of investment to one dollar.

(a) Setup and solve the utility maximization program for the players.

- We begin with player i, and the case for player j is analyzed analogously. Player i's solves

$$\begin{aligned} \max_{x_i \geq 0} EU_i \left[x_i | V\right] &= p_i V + (1 - p_i)\, 0 - x_i \\ &= \frac{x_i^r}{x_i^r + x_j^r} V - x_i \end{aligned}$$

since he incurs x_i regardless of whether he wins the prize or not. Taking first order conditions with respect to x_i, we obtain

$$\frac{r x_i^{r-1} \left(x_i^r + x_j^r\right) - r x_i^r x_i^{r-1}}{(x_i^r + x_j^r)^2} V - 1 = 0$$

Rearranging, yields

$$\frac{r x_i^{r-1} x_j^r}{(x_i^r + x_j^r)^2} V = 1$$

Invoking symmetry, $x_i^* = x_j^* = x^*$, this expression becomes

$$\frac{r x^{r-1} x^r}{(x^r + x^r)^2} V = 1,$$

which simplifies to

$$\frac{r x^{2r-1}}{4 x^{2r}} V = 1,$$

or $\frac{r}{4x} V = 1$. Solving for x, we find the equilibrium investment

$$x^* = \frac{rV}{4}.$$

(b) How does the equilibrium investment of player i change with V and r?

- Equilibrium investment, $x^* = \frac{rV}{4}$, increases in both V and r. Intuitively, bidders invest more if they assign a higher valuation to the prize (higher V), and their investments are more effective (higher r).
- The intuition behind the first effect is clear but that behind the second effect is a bit more intricate. An increase in r produces an increase in the numerator of the probability of winning, $\frac{x_i^r}{x_i^r + x_j^r}$, by $x_i^r \ln(x_i)$, while an increase in r produces an increase in the denominator by $x_i^r \ln(x_i) + x_j^r \ln(x_j)$. Comparing these two effects, it's clear that that effect in the denominator is larger,

$$x_i^r \ln(x_i) + x_j^r \ln(x_j) > x_i^r \ln(x_i)$$

thus reducing the probability that player i wins the prize. As a result, every player increases its investment to compensate for the lower probability of winning the prize.

4. **Stackelberg competition with two asymmetric firms**. Consider an industry where two firms producing a homogeneous good, and facing a linear demand function $p(Q) = 1 - Q$, where $Q = q_1 + q_2$ denotes aggregate output. Firms interact in the following sequential-move game: In the first stage, firm 1 (the industry leader) chooses its output q_1. In the second stage, firm 2 (the industry follower) observes the leader's output q_1 and responds with its own output level q_2.

The leader, because of its experience in the industry, faces a lower marginal cost than the follower, that is, c_l and c_f satisfy $1 > c_f \geq c_l > 0$.

(a) Find the follower's best response function.

- The follower's profit-maximization problem is

$$\max_{q_2 \geq 0} \ \pi_2 = (1 - q_1 - q_2)q_2 - c_f q_2$$

Differentiating with respect to q_2, we obtain

$$1 - 2q_2 - q_1 - c_f = 0$$

rearranging, and solving for q_2, we find firm 2's best response function

$$q_2(q_1) = \frac{1 - c_f}{2} - \frac{1}{2}q_1.$$

which originates at a vertical intercept of $q_2 = \frac{1-c_f}{2}$ and decreases in the leader's output, q_1, at a rate of $1/2$, crossing the horizontal axis at $q_1 = 1 - c$.

(b) Find the leader's equilibrium output.

- The leader anticipates the follower's best response function in the second stage, $q_2(q_1)$, so its profit-maximization problem is

$$\max_{q_1 \geq 0} \ \pi_1 = [1 - q_1 - q_2(q_1)]\, q_1 - c_l q_1$$

Inserting firm 2's best response function in this problem, we obtain

$$\begin{aligned} \max_{q_1 \geq 0} \ \pi_1 &= \left[1 - q_1 - \overbrace{\left(\frac{1 - c_f}{2} - \frac{1}{2}q_1 \right)}^{q_2(q_1)} \right] q_1 - c_l q_1 \\ &= \frac{1 - q_1 + c_f - 2c_l}{2} q_1. \end{aligned}$$

which is only a function of the leader's choice variable (its output level q_1).

- Differentiating the leader's profit with respect to q_1, we find

$$1 - 2q_1 + c_f - 2c_l = 0$$

Solving for q_1, we obtain the leader's equilibrium output

$$q_1^* = \frac{1 + c_f - 2c_l}{2}$$

which is decreasing in its own cost,c_l, but increasing in the follower's cost, c_f.

(c) Find firm 2's output level in equilibrium.

- The follower observes the leader's equilibrium output, q_1^*, and inserts it into its best response function to obtain its equilibrium output

$$\begin{aligned} q_2^* &= \frac{1-c_f}{2} - \frac{1}{2}\overbrace{\left(\frac{1+c_f-2c_l}{2}\right)}^{q_1^*} \\ &= \frac{1-c_f}{2} - \frac{1+c_f-2c_l}{4} \\ &= \frac{1-3c_f+2c_l}{4} \end{aligned}$$

which is decreasing in its own cost, c_f, but increasing in the leader's cost, c_l.

(d) Under which conditions are both firms active in the industry? Under which conditions is only the leader active? Under which conditions is only the follower active?

- *Both firms are active.* Both firms are active in the industry when both

$$q_1^* = \frac{1+c_f-2c_l}{2} > 0 \text{ and } q_2^* = \frac{1-3c_f+2c_l}{4} > 0.$$

Solving for c_l in $q_1^* > 0$, we find that the leader is active if

$$c_l < \frac{1+c_f}{2}.$$

Similarly, solving for c_l in $q_2^* > 0$, we find that the follower is active if

$$c_l > \frac{3c_f-1}{2}.$$

Therefore, combining these two conditions on c_l, we obtain that both firms are active if the leader's costs are intermediate, that is,

$$\frac{1+c_f}{2} > c_l > \frac{3c_f-1}{2}$$

- *Only the leader is active.* The leader is the only active firm in the industry if condition $c_l < \frac{1+c_f}{2}$ holds but $c_l > \frac{3c_f-1}{2}$ does not. In other words, for the entrant to be inactive, we need $c_l \leq \frac{3c_f-1}{2}$. Comparing the leader and the follower's cutoffs, we find that

$$\frac{1+c_f}{2} - \frac{3c_f-1}{2} = 1 - c_f$$

which is positive since $c_f < 1$ by assumption. Graphically, this result means that cutoff $\frac{1+c_f}{2}$ lies to the right-hand side of cutoff $\frac{3c_f-1}{2}$. Therefore, condition $c_l \leq \frac{3c_f-1}{2}$ implies $c_l < \frac{1+c_f}{2}$ or, formally, $c_l \leq \frac{3c_f-1}{2}$ is a sufficient condition for $c_l < \frac{1+c_f}{2}$. Then, for the leader to be the only active firm, we only need condition $c_l \leq \frac{3c_f-1}{2}$. Intuitively, the leader's cost advantage is so strong that the follower decides to be inactive.

- *Only the follower is active.* The follower is the only active firm in the industry if $c_l > \frac{3c_f-1}{2}$ and $c_l \geq \frac{1+c_f}{2}$. From our above discussion, we know that $\frac{1+c_f}{2} > \frac{3c_f-1}{2}$, implying that condition $c_l \geq \frac{1+c_f}{2}$ is sufficient for $c_l > \frac{3c_f-1}{2}$. In other words, the follower is the only active firm if $c_l \geq \frac{1+c_f}{2}$ holds.
- In summary, our above discussion identifies three regions of c_l. These regions are depicted in figure 1 with c_l in the vertical axis and c_f in the horizontal axis, along with cutoff $c_l = \frac{3c_f-1}{2}$, originating at $c_l = -\frac{1}{2}$ and reaching a maximum height of 1; and cutoff $c_l = \frac{1+c_f}{2}$, which originates at $c_l = \frac{1}{2}$ that also reaches a maximum height of 1. The three regions are the following:
  (i) When the leader is relatively efficient, $c_l \leq \frac{3c_f-1}{2}$, it is the only active firm in the industry;
  (ii) If the leader's cost advantage is moderate, $\frac{3c_f-1}{2} < c_l < \frac{1+c_f}{2}$, both firms are active; and
  (iii) If the leader's cost advantage is low, $c_l \geq \frac{1+c_f}{2}$, the leader is inactive and only the follower is active.

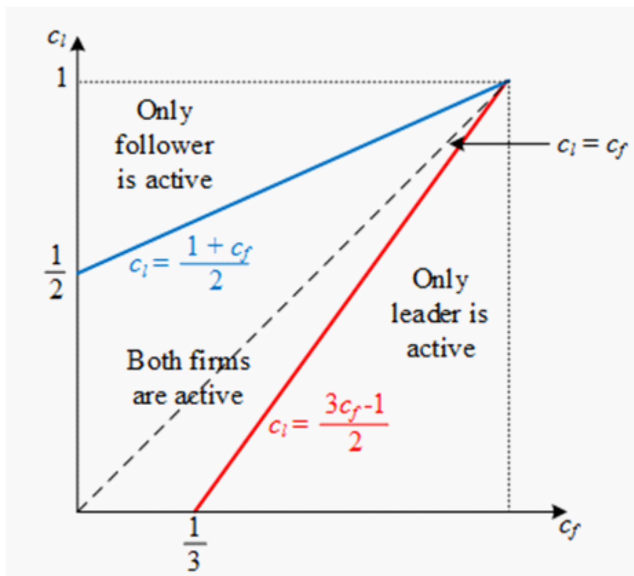

Figure 1. Output profiles in Stackelberg.

- **Remark:** Note that condition $c_l \geq \frac{1+c_f}{2}$ entails that the leader suffers from a cost disadvantage relative to the follower since $\frac{1+c_f}{2} > c_f$ simplifies to $1 > c_f$, which holds by assumption. Intuitively, in all cost pairs (c_l, c_f) in Region (iii) the leader suffers from a cost disadvantage, leading it to shut down, but it also suffers from a cost disadvantage (although smaller) in Region (ii), where the leader produces a positive output level.
  Similarly, it is easy to show that condition $c_l \leq \frac{3c_f-1}{2}$ entails that the leader benefits from a cost advantage relative to the follower since $\frac{3c_f-1}{2} < c_f$ simplifies to $c_f < 1$, which holds by definition. Therefore, in all cost pairs of Region (i) and in a portion of Region (ii), the leader has a cost advantage.