EconS 503 - Microeconomic Theory II
Homework #2 - Answer key

1. **A strategy that is never a best response, yet is not a dominated strategy.** Consider the following simultaneous-move game between player 1 (in rows) and player 2 (in columns).

		Player 2	
		L	R
Player 1	U	$4,2$	$0,1$
	C	$0,2$	$4,1$
	D	$1,0$	$1,3$

   (a) Show that strategy D is never a best response for player 1.

   - When player 2 chooses strategy L, $u_1(D,L) < u_1(U,L)$. Similarly, when player 2 chooses strategy R, $u_1(D,R) < u_1(C,R)$. Therefore, for all $s_2 \in S_2 = \{L,R\}$, $u_1(D,s_2) \ngeq \max\{u_1(U,s_2), u_1(C,s_2)\}$. In other words, there is no belief that player 1 can sustain about player 2's behavior that would lead player 1 to choose strategy D as his best response.

   (b) Show that strategy D is not strictly dominated by U or C.

   - In the comparison between player 1's strategies U and D, when player 2 chooses strategy L, $u_1(D,L) < u_1(U,L)$ but when player 2 chooses strategy R, $u_1(D,R) > u_1(U,R)$. Similarly, in the comparison between player 1's strategies C and D, when player 2 chooses strategy L, $u_1(D,L) > u_1(C,L)$ but when player 2 chooses strategy R, $u_1(D,R) < u_1(C,R)$. Therefore, strategy D is neither strictly dominated by U nor C.

2. **Mixed strategy NE in a patent rate.** Consider two firms simultaneously choosing an investment level in R&D, $x_i \in \{0,1,...,k\}$ for every firm $i = \{1,2\}$ where $k > 2$. Assume that every firm i's profit function is

$$\pi_i(x_i,x_j) = \begin{cases} R_i - x_i & \text{if } x_i > x_j, \text{ and} \\ -x_i & \text{otherwise} \end{cases}$$

That is, firm i obtains a revenue of R_i if it invests more than firm j ($x_i > x_j$, which means that this firm wins the patent race. Revenue R_i satisfies $R_i > k$, entailing a positive profit for all admissible values of x_i. In contrast, if firm i invests the same or less than firm j, $x_i \leq x_j$, firm i loses the patent race and obtains no revenue from its R&D investment.

   (a) Show that the game has no pure strategy NE.

   - We will rule out the existence of a Nash Equilibrium in pure strategies by reviewing the various instances of pure strategy profile (c_i,c_j). We will divide them into five possible cases and show that no profile in any of these cases is a Nash equilibrium, since at least one of the firms can improve its position if it chooses a different level of investment than the one prescribed for it in the profile.

- *Case 1*: $c_i = c_j = 0$. In such a case, neither firm either gains or loses. Since $c_j = 0$, firm i can improve its position by making a positive investment $c_j > 0$, winning the patent and guaranteeing itself a positive profit.
- *Case 2*: $c_i = c_j > 0$. In such a case, both firms lose their investment. Given that $c_j > 0$, firm i can improve its position by not investing at all, and avoiding any loss.
- *Case 3*: $c_i < c_j < k$. In such a case, firm i does not win the patent and has no positive profit. Given that $c_j < k$, firm i can guarantee itself a positive profit by investing k and winning the patent.
- *Case 4*: $0 < c_i < c_j = k$. In this case, firm i does not win the patent and loses its investment. Given that $c_j = k$, firm i can avoid its loss by investing 0.
- *Case 5*: $0 = c_i < c_j = k$. In such a case, firm j wins the patent, but with the maximum possible investment. Since $c_i = 0$, firm j can increase its profits if it makes a positive investment smaller than k, that will guarantee its winning the patent.

(b) Find a mixed strategy NE for every firm i.

- Consider that firm 1 adopts the strategy

$$\overline{p}_1 = (p_1^0, p_1^1, p_1^2, ..., p_1^k) = \left(\frac{1}{r_2}, \frac{1}{r_2}, \frac{1}{r_2}, ..., 1 - \frac{k}{r_2}\right)$$

which, in words, says that firm 1 mixes among all its pure strategies. In particular, it chooses, with positive probability $\frac{1}{r_2}$, not to make any investment at all and to guarantee itself a payoff of 0.

- In order to show that the above mixed strategy $\overline{p}_1$ is a best response of firm 1 to the strategy:

$$\overline{p}_2 = (p_1^0, p_1^1, p_1^2, ..., p_1^k) = (\frac{1}{r_2}, \frac{1}{r_2}, \frac{1}{r_2}, ..., 1, 1 - \frac{k}{r_2})$$

of firm 2, we mush ascertain that, given $\overline{p}_2$, firm 1 is indifferent among its pure strategies and that they all yield an expected payoff of 0 (like the strategy of refraining from investment, which also guarantees a payoff of 0). This is indeed the case: if firm 1 chooses to invest an amount $c_2 > 0$, it wins the patent and makes a profit of $r_1 - c_1$ as long as firm 2 invests a smaller amount. But firm 1 loses its investment if firm 2 invests an identical or a larger amount.

Given $\overline{p}_2$, the expected payoff of firm 1 is:

$$\begin{aligned}
U_1(c_1, \overline{p}_2) &= (r_1 - c_1) \sum_{n=0}^{c_1-1} p_2^n + (-c_1) \sum_{n=c_1}^{k} p_2^n \\
&= (r_1 - c_1) \sum_{n=0}^{c_1-1} \frac{1}{r_1} + (-c_1) \sum_{n=c_1}^{k-1} \frac{1}{r_1} + (-c_1)\left(1 - \frac{k}{r_1}\right) \\
&= (r_1 - c_1) c_1 \frac{1}{r_1} + (-c_1)(k - c_1)\frac{1}{r_1} + (-c_1)\left(1 - \frac{k}{r_1}\right) \\
&= c_1 \frac{1}{r_1}[(r_1 - c_1) - (k - c_1) + k] - c_1 \\
&= c_1 - c_1 = 0
\end{aligned}$$

as we sought to demonstrate. We have therefore shown that $\overline{p}_1$ is a best reply to $\overline{p}_2$. The same argument, with a reversal of roles between firm 1 and 2, proves that $\overline{p}_2$ is a best reply to $\overline{p}_1$. As a result, strategy profile $(\overline{p}_1, \overline{p}_2)$ is a mixed strategy Nash equilibrium.

3. **Exercises from Tadelis:**

   (a) Chapter 5: Exercises 5.5, 5.11, 5.16, and 5.17.

   (b) Chapter 6: Exercises 6.3, 6.5, 6.7, and 6.11.

   - See answer keys at the end of this handout.

5.5

lamp is 3 for each player and the value of not having one is 0. The Mayor asks each player to either contribute 1 or nothing. If at least two players contribute then the lamp will be erected. If one or less people contribute then the lamp will not be erected, in which case any person who contributed will not get their money back.

(a) Write out or graph each player's best-response correspondence.

Answer: Consider player i with beliefs about the choices of players j and k. If neither j nor k contribute then player i does not want to contribute because the lamp would not be erected and he would lose his contribution. Similarly, if both j and k contribute then player i does not want to contribute because the lamp would be erected without his contribution so he can "free ride" on their contributions. The remaining cases is where only one of the players j and k contribute, in which case by contributing 1 player i receives 3, while by not contributing he receives 0, and hence contributing is a best response. In summary,

$$BR_i(s_j, s_k) = \begin{cases} 0 & \text{if } s_j = s_k \\ 1 & \text{if } s_j \neq s_k \end{cases}.$$

(b) What outcomes can be supported as pure-strategy Nash equilibria?

Answer: The best response correspondence described in (a) above implies that there are two kinds of Nash equilibria: one kind (which is unique) is where no player contributes, and the other kind has two of the three players contributing and the third free riding. Hence, either the lamp being erected with two players contributing or the lamp not being erected with no player contributing can be supported as Nash equilibria. ■

6. **Hawk-Dove:** The following game has been widely used in evolutionary biology to understand how "fighting" and "display" strategies by animals could coexist in a population. For a typical Hawk-Dove game there are resources to

(b) In what way are the best response correspondences different from those in the Cournot game? Why?

Answer: Here the best response function of player i is *increasing* in the choice of player j whereas in the Cournot model it is *decreasing* in the choice of player j. This is because in this game the choices of the two players are strategic complements while in the Cournot game they are strategic substitutes. ■

(c) Find the Nash equilibrium of this game and argue that it is unique.

Answer: We solve two equations with two unknowns,

$$e_1 = \frac{a+e_2}{2} \text{ and } e_2 = \frac{a+e_1}{2},$$

which yield the solution $e_1 = e_2 = a$. It is easy to see that it is unique because it is the only point at which these two best response functions cross. ■

11. **Wasteful Shipping Costs.** Consider two countries, A and B, each with a monopolist that owns the only coal mine in the country, and it produces coal. Let firm 1 be the one located in country A, and firm 2 the one in country B. Let q_i^j, $i \in \{1,2\}$ and $j \in \{A,B\}$ denote the quantity that firm i sells in country j. Consequently, let $q_i = q_i^A + q_i^B$ be the total quantity produced by firm $i \in \{1,2\}$, and let $q^j = q_1^j + q_2^j$ be the total quantity sold in country $j \in \{A,B\}$. The demand for coal in countries A and B is given respectively by,

$$p^j = 90 - q^j,\ j \in \{A,B\},$$

and the costs of production for each firm is given by,

$$c_i(q_i) = 10q_i\,,\ i \in \{1,2\}.$$

(a) Assume that the countries do not have a trade agreement and, in fact, imports in both countries are prohibited. This implies that $q_2^A = q_1^B = 0$ is set as a political constraint. What quantities q_1^A and q_2^B will both firms produce?

Answer: Each firm is a monopolist in its own country. Let and maximizes,

$$\max_{q_i^j \geq 0} (90 - q_i^j)q_i^j - 10q_i^j$$

where either $i = 1$ and $j = A$, or $i = 2$ and $j = B$ (so that $q_2^A = q_1^B = 0$ is set by assumption.) The first order maximization condition is $90 - 2q_i^j - 10 = 0$, which yields $q_1^A = q_2^B = 40$. The payoff for each firm is $1,600$. ■

Now assume that the two countries sign a free-trade agreement that allows foreign firms to sell in their countries without any tariffs. There are, however shipping costs. If firm i sells quantity q_i^j in the foreign country (i.e., firm 1 selling in B or firm 2 selling in A) then shipping costs are equal to $10q_i^j$. Assume further that *each firm* chooses a pair of quantities q_i^A, q_i^B simultaneously, $i \in \{1, 2\}$, so that a profile of actions consists of four quantity choices.

(b) Model this as a normal form game and find a Nash equilibrium of the game you described. Is it unique?

Answer: This game has two players, $i \in \{1, 2\}$, each choosing a strategy that consists of two non-negative quantities, $(q_i^A, q_i^B) \in \mathbb{R}_+^2$, and the payoff of the two players are given by,

$$\begin{aligned} v_1(q_1^A, q_1^B, q_2^A, q_2^B) &= (90 - q_1^A - q_2^A)q_1^A + (90 - q_1^B - q_2^B)q_1^B - 10(q_1^A + q_1^B) - 10q_1^B, \\ v_2(q_1^A, q_1^B, q_2^A, q_2^B) &= (90 - q_1^A - q_2^A)q_2^A + (90 - q_1^B - q_2^B)q_2^B - 10(q_2^A + q_2^B) - 10q_2^A, \end{aligned}$$

where the first term is the firm's revenue in market A, the second is the revenue in market B, the third is the total production cost and the last is the shipping cost. Given beliefs (q_2^A, q_2^B) about what firm 2 chooses to produce, firm 1's optimization requires two partial derivatives with respect to q_1^A and q_1^B as follows,

$$\begin{aligned}\frac{\partial v_1(q_1^A, q_1^B, q_2^A, q_2^B)}{\partial q_1^A} &= 90 - q_2^A - 2q_1^A - 10 = 0, \\ \frac{\partial v_1(q_1^A, q_1^B, q_2^A, q_2^B)}{\partial q_1^B} &= 90 - q_2^B - 2q_1^B - 20 = 0,\end{aligned}$$

which in turn lead to the two parts of firm 1's best response function,[3]

$$q_1^A = \frac{80 - q_2^A}{2}, \tag{5.5}$$

$$q_1^B = \frac{70 - q_2^B}{2}. \tag{5.6}$$

It is easy to see that the objective of firm 2 is symmetric to that of firm 1 and hence we can directly write down firm 2's best responses as,

$$q_2^A = \frac{70 - q_1^A}{2}, \tag{5.7}$$

$$q_2^B = \frac{80 - q_1^B}{2}. \tag{5.8}$$

The Nash equilibrium is solved by finding a profile of strategies $(q_1^A, q_1^B, q_2^A, q_2^B)$ for which (5.5), (5.6), (5.7) and (5.8) all hold simultaneously. From (5.5) and (5.7) we obtain $q_1^A = 30$ and , $q_2^A = 20$. Similarly, from (5.6) and (5.8) we obtain $q_1^B = 20$ and , $q_2^B = 30$. The payoff of each firms would be equal to $1,300$.

Now assume that before the game you described in (b) is played, the research department of firm 1 discovered that shipping coal with the current ships causes the release of pollutants. If the firm would disclose this report to the World-Trade-Organization (WTO) then the WTO would prohibit the use of the current ships. Instead, a new shipping

[3] Because the payoff function has no interactions between the markets (i.e., it is *separable* in the two markets so that there are no interactions through the cost function) then q_1^A depends only on q_2^A and q_1^B depends only on q_2^B (and vice versa for firm 2). If costs were not linear then this would not be the case and the solution would involve solving four equations wit four unknowns simultaneously.

technology would be offered that would increase shipping costs to $40q_i^j$ (instead of $10q_i^j$ as above).

(c) Would firm 1 be willing to release the information to the WTO? Justify your answer with an equilibrium analysis.

Answer: To answer this we need to solve the Nash equilibrium with the more expensive shipping technology and compare the profits to that of the current cheaper technology. We know that a monopolist (or competitive firm) would never prefer a more expensive technology to a cheaper one, but here there may be interesting strategic effects: the more expensive shipping technology will dampen competition. The new payoff functions are

$$\begin{aligned} v_1(q_1^A, q_1^B, q_2^A, q_2^B) &= (90 - q_1^A - q_2^A)q_1^A + (90 - q_1^B - q_2^B)q_1^B - 10(q_1^A + q_1^B) - 40q_1^B, \\ v_2(q_1^A, q_1^B, q_2^A, q_2^B) &= (90 - q_1^A - q_2^A)q_2^A + (90 - q_1^B - q_2^B)q_2^B - 10(q_2^A + q_2^B) - 40q_2^A, \end{aligned}$$

and following the same arguments in part (b) above, the four equations that will define the best responses of both firms are,

$$q_1^A = \frac{80 - q_2^A}{2}, \tag{5.9}$$

$$q_1^B = \frac{40 - q_2^B}{2}. \tag{5.10}$$

and,

$$q_2^A = \frac{40 - q_1^A}{2}, \tag{5.11}$$

$$q_2^B = \frac{80 - q_1^B}{2}. \tag{5.12}$$

From (5.9) and (5.11) we obtain $q_1^A = 40$ and , $q_2^A = 0$. Similarly, from (5.10) and (5.12) we obtain $q_1^B = 0$ and , $q_2^B = 40$. The payoff of each firms would be equal to $1,600$, as we calculated in part (a) above. Hence, the firm would like to disclose the information and let the WTO impose a ban that would effectively kill cross-border competition. ■

5.16

"mass" or quantity of buyers in the interval $[a, b]$ is equal to $b - a$.) Imagine two firms, players 1 and 2 who are located at each end of the interval (player 1 at the 0 point and player 2 at the 1 point.) Each player i can choose its price p_i, and each customer goes to the vendor who offers them the highest value. However, price alone does not determine the value, but distance is important as well. In particular, each buyer who buys the product from player i has a net value of $v - p_i - d_i$ where d_i is the distance between the buyer and vendor i, and represents the transportation costs of buying from vendor i. Thus, buyer $a \in [0, 1]$ buys from 1 and not 2 if $v - p_1 - d_1 > v - p_2 - d_2$, and if buying is better than getting zero. (Here $d_1 = a$ and $d_2 = 1 - a$. The buying choice would be reversed if the inequality is reversed.) Finally, assume that the cost of production is zero.

(a) Assume that v is very large so that all the customers will be served by at least one firm, and that some customer $x^* \in [0, 1]$ is indifferent between the two firms. What is the best response function of each player?

Answer: Because customer x^*'s distance from firm 1 is x^* and his distance from firm 2 is $1 - x^*$, his indifference implies that

$$v - p_1 - x^* = v - p_2 - (1 - x^*)$$

which gives the equation for x^*,

$$x^* = \frac{1 + p_2 - p_1}{2} .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$\begin{aligned} q_1(p_1, p_2) &= x^* = \frac{1 + p_2 - p_1}{2} , \\ q_1(p_1, p_2) &= 1 - x^* = \frac{1 + p_1 - p_2}{2} . \end{aligned}$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1 + p_2 - p_1}{2} \right) p_1$$

which yields the first order condition

$$1 + p_2 - 2p_1 = 0 \ ,$$

implying the best response function

$$p_1 = \frac{1}{2} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{1}{2} + \frac{p_1}{2}.$$

■

(b) Assume that $v = 1$. What is the Nash equilibrium? Is it unique?

Answer: If we use the best response functions calculated in part (a) above then we obtain a unique Nash equilibrium $p_1 = p_2 = 1$, and this implies that $x^* = \frac{1}{2}$ so that each firm gets half the market. However, when $v = 1$ then the utility of customer $x^* = \frac{1}{2}$ is $v - p_1 - \frac{1}{2} = 1 - 1 - \frac{1}{2} = -\frac{1}{2}$, implying that he would prefer not to buy, and by continuity, an interval of customers around x^* would also prefer not to buy. his violated the assumptions we used to calculate the best response functions.[5] So, the analysis in part (a) is invalid when $v = 1$. It is therefore useful to start with the monopoly case when $v = 1$ and see how each firm would have priced if the other is absent. Firm 1 maximizes

$$\max_{p_1} \ (1 - p_1)p_1$$

which yields the solution $p_1 = \frac{1}{2}$ so that everyone in the interval $x \in [0, \frac{1}{2}]$ wished to buy from firm 1 and no other customer would buy. By symmetry, if firm 2 were a monopoly then the solution would be $p_2 = \frac{1}{2}$ so that everyone in the interval $x \in [\frac{1}{2}, 1]$ would buy from firm 2 and no other customer would buy. But this implies that if both firms set their

[5] We need $v \geq 1.5$ for customer $x^* = \frac{1}{2}$ to be just indifferent between buying and not buying when $p_1 = p_2 = 1$. All the other customers will strictly prefer buying.

monopoly prices $p_1 = p_2 = \frac{1}{2}$ then each would maximize profits ignoring the other firm, and hence this is the (trivially) unique Nash equilibrium. ■

(c) Now assume that $v = 1$ and that the transportation costs are $\frac{1}{2}d_i$, so that a buyer buys from 1 if and only if $v - p_1 - \frac{1}{2}d_1 > v - p_2 - \frac{1}{2}d_2$. Write the best response function of each player and solve for the Nash Equilibrium.

Answer: Like in part (a), assume that customer x^*'s distance from firm 1 is x^* and his distance from firm 2 is $1 - x^*$, and he is indifferent between buying from either, so his indifference implies that

$$v - p_1 - \frac{1}{2}x^* = v - p_2 - \frac{1}{2}(1 - x^*)$$

which gives the equation for x^*,

$$x^* = \frac{1}{2} + p_2 - p_1 \ .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$\begin{aligned} q_1(p_1, p_2) &= x^* = \frac{1}{2} + p_2 - p_1 \ , \\ q_1(p_1, p_2) &= 1 - x^* = \frac{1}{2} + p_1 - p_2 \ . \end{aligned}$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1}{2} + p_2 - p_1\right) p_1$$

which yields the first order condition

$$\frac{1}{2} + p_2 - 2p_1 = 0 \ ,$$

implying the best response function

$$p_1 = \frac{1}{4} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{1}{4} + \frac{p_1}{2}.$$

The Nash equilibrium is a pair of prices for which these two best response functions hold simultaneously, which yields $p_1 = p_2 = \frac{1}{2}$, and $x^* = \frac{1}{2}$. To verify that this is a Nash equilibrium notice that for customer x^*, the utility form buying from firm 1 is $v - p_1 - \frac{1}{2} = 1 - \frac{1}{2} - \frac{1}{2} = 0$ implying that he is indeed indifferent between buying or not, which in turn implies that every other customer prefer buying over not buying. ■

(d) Following your analysis in (c) above, imagine that transportation costs are αd_i, with $\alpha \in [0, \frac{1}{2}]$. What happens to the Nash equilibrium as $\alpha \to 0$? What is the intuition for this result?

Answer: Using the assumed indifferent customer x^*, his indifference implies that

$$v - p_1 - \alpha x^* = v - p_2 - \alpha(1 - x^*)$$

$$v - p_1 - \alpha x = v - p_2 - \alpha(1 - x)$$

which gives the equation for x^*,

$$x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1) .$$

It follows that under the assumptions above, given prices p_1 and p_2, the demands for firms 1 and 2 are given by

$$\begin{aligned} q_1(p_1, p_2) &= x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1) , \\ q_1(p_1, p_2) &= 1 - x^* = \frac{1}{2} + \frac{1}{2\alpha}(p_1 - p_2) . \end{aligned}$$

Firm 1's maximization problem is

$$\max_{p_1} \left(\frac{1}{2} + \frac{1}{2\alpha}(p_2 - p_1)\right) p_1$$

which yields the first order condition

$$\frac{1}{2} + \frac{p_2}{2\alpha} - \frac{p_1}{\alpha} = 0 \ ,$$

implying the best response function

$$p_1 = \frac{\alpha}{2} + \frac{p_2}{2}.$$

A symmetric analysis yields the best response of firm 2,

$$p_2 = \frac{\alpha}{2} + \frac{p_1}{2}.$$

$$p_2 = \frac{\alpha}{2} + \frac{\frac{\alpha}{2} + \frac{p_2}{2}}{2}.$$

The Nash equilibrium is a pair of prices for which these two best response functions hold simultaneously, which yields $p_1 = p_2 = \alpha$, and $x^* = \frac{1}{2}$. From the analysis in (c) above we know that for any $\alpha \in [0, \frac{1}{2})$ customer x^* will strictly prefer to buy over not buying and so will every other customer. We see that as α decreases, so do the equilibrium prices, so that at the limit of $\alpha = 0$ the prices will be zero. The intuition is that the transportation costs d cause firms 1 and 2 to be differentiated, and this "softens" the Bertrand competition between the two firms. When the transportation costs are higher this implies that competition is less fierce and prices are higher, and the opposite holds for lower transportation costs.■

17. **To vote or not to vote:** Two candidates, D and R, are running for mayoral election in a town with n residents. A total of $0 < d < n$ residents support candidate D while the remainder $r = n - d$ support candidate R. The value for each resident for having their candidate win is 4, for having him tie is 2, and for having him lose is 0. Going to vote costs each resident 1.

    (a) Let $n = 2$ and $d = 1$. Write down this game as a matrix and solve for the Nash equilibrium.

    Answer: The game is between the residents as the candidates seem not to play a role and the question is whether to vote or not to vote. Letting Y denote "yes" vote and N denote "no" vote, the matrix representation of this two player game is

		Player 2	
		Y	N
Player 1	Y	$1,1$	$3,0$
	N	$0,3$	$2,2$

If both vote or both don't vote then there is a tie and the only difference is the cost of voting. If only one votes then his candidate wins and he exerts the voting costs, while the other gains and expends nothing. Voting is a dominant strategy so (Y,Y) is the unique Nash equilibrium. ■

(b) Let $n > 2$ be an even number and let $d = r = \frac{n}{2}$. Find all the Nash equilibria.

Answer: Observe that everyone voting is a Nash equilibrium. Like in part (a) there will be a tie and every player's payoff is 1, while if he chose not to vote then his candidate will lose and his payoff will be 0, hence it is a Nash equilibrium. We now show that no other profile of strategies is a Nash equilibrium in three steps. Let d' and r' denote the number of member of each group that plan on voting. (*i*) Assume that an identical number of voters from each side votes so that there is a tie but some voters are not voting, that is, $d' = r' < \frac{n}{2}$. In this case any one of the voters who is not voting would prefer to deviate, expend a voting cost of 1 and increase his payoff from 2 to 3 because he would tip the election. Hence, this cannot be a Nash equilibrium. (*ii*) Now assume that the number of supporters of candidate D is is at least 2 more than that of candidate R, that is, $d' \geq r' + 2$. (A symmetric argument will apply to the case of $r' \geq d' + 2$.) In this case any one of the D supporters who plans to vote knows that his vote is redundant, and hence he would prefer not to vote and save the voting costs. Hence, this cannot be a Nash equilibrium. (*iii*) Now assume that the number of supporters of candidate D is is exactly 1 more than that of candidate R, that is, $d' = r' + 1$. (A symmetric argument will apply to the case of $r' = d' + 1$.) In this case any one of the R supporters who does not plan to vote knows that his vote can turn a loss into a tie, and hence he would prefer to vote and change the election giving him a payoff of 1 instead of 0. Hence, this too cannot be a Nash equilibrium. This covers all the possible scenarios and shows that everyone voting is the unique Nash equilibrium. ■

(c) Assume now that the costs of voting are equal to 3. How does your answer to (a) and (b) change?

Answer: The two player game is now

		Player 2	
		Y	N
Player 1	Y	$-1,-1$	$1,0$
	N	$0,1$	$2,2$

and the dominated strategy is voting, implying that the unique Nash equilibrium is for the players not to vote, (N, N). A similar argument to part (b) above shows that all players not voting is the unique Nash equilibrium. ■

18. **Political Campaigning:** Two candidates are competing in a political race. Each candidate i can spend $s_i \geq 0$ on adds that reach out to voters, which in turn increases the probability that candidate i wins the race. Given a pair of spending choices (s_1, s_2), the probability that candidate i wins is given by $\frac{s_i}{s_1+s_2}$. If neither spends any resources then each wins with probability $\frac{1}{2}$. Each candidate values winning at a payoff of $v > 0$, and the cost of spending s_i is just s_i.

 (a) Given two spend levels (s_1, s_2), write the expected payoff of a candidate i.

6.3

(a) Find a strategy different from $(\sigma_2(L), \sigma_2(C), \sigma_2(R)) = (0, \frac{1}{2}, \frac{1}{2})$ that strictly dominates the pure strategy L for player 2. Argue that you can find an infinite number of such strategies.

Answer: The expected payoff of any player in a matrix game is continuous in the probabilities of his mixed strategy (because it is a linear function of the probability weights), and hence if we "tweak" the strategy $(\sigma_2(L), \sigma_2(C), \sigma_2(R)) = (0, \frac{1}{2}, \frac{1}{2})$ just a little bit then the payoffs will be the same for any choice of player 1. For example, take $\sigma_2' = (\sigma_2'(L), \sigma_2'(C), \sigma_2'(R)) = (0, \frac{4}{10}, \frac{6}{10})$. The expected payoff of player 2 from σ_2' against any one of the three strategies of player 1 are,

$$\begin{aligned} v_2(U, \sigma_2') &= 0.4 \times 4 + 0.6 \times 0 = 1.6 > 1 = v_2(U, L) \\ v_2(U, \sigma_2') &= 0.4 \times 0 + 0.6 \times 5 = 3 > 2 = v_2(U, L) \\ v_2(U, \sigma_2') &= 0.4 \times 4 + 0.6 \times 3 = 3.4 > 3 = v_2(U, L) \end{aligned}$$

which shows that σ_2' strictly dominates L. It is therefore follows by the continuity of the expected payoff function that any one of the infinitely many mixed strategies that puts weights close to 0.5 on C and the remaining probability on R will dominate L.[1] ■

(b) Find a strategy different from $(\sigma_1(U), \sigma_1(M), \sigma_1(D)) = (0, \frac{1}{2}, \frac{1}{2})$ that strictly dominates the pure strategy U for player 1 in the game remaining after one stage of elimination. Argue that you can find an infinite number of such strategies.

Answer: This is an identical procedure as for part (a).

4. **Monitoring:** An employee (player 1) who works for a boss (player 2) can either work (W) or shirk (S), while his boss can either monitor the employee (M) or ignore him (I). Like most employee-boss relationships, if the employee is working then the boss prefers not to monitor, but if the boss is not

[1] A more elegant way Of writing this would be to choose a mixed strategy $\sigma_2' = (0, \frac{1}{2} + \varepsilon, \frac{1}{2} - \varepsilon)$ and show that for small enough values of ε it follows that σ_2' strictly dominates L, and it follows that there are infinitely many such values of ε.

6.5

depends on whether he catches a robber, who is player 2. If the robber prowls the streets then the police officer will catch him and obtain a payoff of 20. If the robber stays in his hideaway then the officer's payoff is 0. The robber must choose between staying hidden or prowling the street. If he stays hidden then his payoff is 0, while if he walks the street his payoff is (−10) if the officer is patrolling the streets, and it is 10 if the officer is at the coffee shop.

(a) Write down the matrix form of this game.

Answer: Let P denote patrol and C coffee shop for player 1, and S is the robber's choice of prowling while H is remaining hidden. The game is therefore

player 1	Player 2: S	Player 2: H
P	$20, -10$	$0, 0$
C	$10, 10$	$10, 0$

■

(b) Draw the best response functions of each player.

Answer: Let p be the probability that player 1 chooses P and q the probability that player 2 chooses S. It follows that $v_1(P, q) > v_1(C, q)$ if and only if $20q > 10$, or $q > \frac{1}{2}$, and $v_2(p, S) > v_2(p, H)$ if and only if $-10p + 10(1-p) > 0$, or $p < \frac{1}{2}$. It follows that for player 1,

$$BR_1(q) = \begin{cases} p = 0 & \text{if } q < \frac{1}{2} \\ p \in [0,1] & \text{if } q = \frac{1}{2} \\ p = 1 & \text{if } q > \frac{1}{2} \end{cases}$$

and for player 2,

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p < \frac{1}{2} \\ q \in [0,1] & \text{if } p = \frac{1}{2} \\ q = 0 & \text{if } p > \frac{1}{2} \end{cases} .$$

Notice that these are identical to the best response functions for the matching pennies game (see Figure 6.3). ■

(c) Find the Nash equilibrium of this game. What kind of game does this game remind you of?

Answer: From the two best response correspondences the unique Nash equilibrium is $(p, q) = (\frac{1}{2}, \frac{1}{2})$ and the game's strategic forces are identical to those in the Matching Pennies game. ■

6. **Declining Industry:** Consider two competing firms in a declining industry that cannot support both firms profitably. Each firm has three possible choices as it must decide whether or not to exit the industry immediately, at the end of this quarter, or at the end of the next quarter. If a firm chooses to exit then its payoff is 0 from that point onward. Every quarter that both firms operate yields each a loss equal to -1, and each quarter that a firm operates alone yields a payoff of 2. For example, if firm 1 plans to exit at the end of this quarter while firm 2 plans to exit at the end of the next quarter then the payoffs are $(-1, 1)$ because both firms lose -1 in the first quarter and firm 2 gains 2 in the second. The payoff for each firm is the sum of its quarterly payoffs.

(a) Write down this game in matrix form.

Answer: Let E denote immediate exit, T denote exit this quarter, and N denote exit next quarter.

		Player 2		
		E	T	N
Player 1	E	$0, 0$	$0, 2$	$0, 4$
	T	$2, 0$	$-1, -1$	$-1, 1$
	N	$4, 0$	$1, -1$	$-2, -2$

(b) Are there any strictly dominated strategies? Are there any weakly dominated strategies?

Answer: There are no strictly dominated strategies but there is a weakly dominated one: T. To see this note that choosing both E and

6.7

Answer: Let N denote no time, O denote one week, and T denote two weeks. The matrix game is,

		Player 2		
		N	O	T
Player 1	N	$1.5, 1.5$	$0, 2$	$0, 1$
	O	$2, 0$	$0.5, 0.5$	$-1, 1$
	T	$1, 0$	$1, -1$	$-0.5, -0.5$

The payoffs are derived by the fact that a tie is an equal chance of getting 3 so each player gets 1.5 in expectation. ■

(b) Are there any strictly dominated strategies? Are there any weakly dominated strategies?

Answer: Each one of the three strategies can be a strict best response: N is a best response to T, O is a best response to N, and T is a best response to O. Hence, no strategy is strictly or weakly dominated. ■

(c) Find the unique mixed strategy Nash equilibrium.

Answer: Let $\sigma_i = (\sigma_{iN}, \sigma_{iO}, 1 - \sigma_{iN} - \sigma_{iO})$ denote a mixed strategy for player i. Because the game is symmetric it suffices to solve the indifference conditions for one player. For player i to be indifferent between N and O,

$$1.5\sigma_{iN} = 2\sigma_{iN} + 0.5\sigma_{iO} - (1 - \sigma_{iN} - \sigma_{iO})$$

and for him to be indifferent between N and T,

$$1.5\sigma_{iN} = \sigma_{iN} + \sigma_{iO} - 0.5(1 - \sigma_{iN} - \sigma_{iO})$$

Solving these two equations with two unknowns yields $\sigma_{iN} = \sigma_{iO} = \frac{1}{3}$ implying that the unique mixed strategy Nash equilibrium has the players mixing between all three pure strategies with equal probability. ■

8. **Market entry:** There are 3 firms that are considering entering a new market. The payoff for each firm that enters is $\frac{150}{n}$ where n is the number of firms that enter. The cost of entering is 62.

players deviates from this strategy and choose to bid $\overline{x}+\varepsilon < 1$ then he will win with probability 1 and receive a payoff of $1-(\overline{x}+\varepsilon) > 0$, contradicting that $\overline{x}_1 = \overline{x}_2 = \overline{x} < 1$ is an equilibrium. ■

v. Show that $F_i(x)$ being uniform over $[0, 1]$ is a symmetric Nash equilibrium of this game.

Answer: Imagine that player 2 is playing according to the proposed strategy $F_2(x)$ uniform over $[0, 1]$. If player 1 bids some value $s_1 \in [0, 1]$ then his expected payoff is

$$\Pr\{s_1 > s_2\}(1-s_1)+\Pr\{s_1 < s_2\}(-s_1) = s_1(1-s_1)+(1-s_1)(-s_1) = 0$$

implying that player 1 is willing to bid any value in the $[0, 1]$ interval, and in particular, choosing a bid according to $F_1(x)$ uniform over $[0, 1]$. Hence, this is a symmetric Nash equilibrium. ■

11. **Bribes:** Two players find themselves in a legal battle over a patent. The patent is worth 20 for each player, so the winner would receive 20 and the loser 0. Given the norms of the country they are in, it is common to bribe the judge of a case. Each player can offer a bribe secretly, and the one whose bribe is the largest is awarded the patent. If both choose not to bribe, or if the bribes are the same amount, then each has an equal chance of being awarded the patent. If a player does bribe, then bribes can be either a value of 9 or of 20. Any other number is considered to be very unlucky and the judge would surely rule against a party who offers a different number.

(a) Find the unique pure-strategy Nash equilibrium of this game.

Answer: The game is captured in the following two player matrix, where Z represents no payment, N represents a bribe of 9 and T a bribe of 20. For example, if both choose 9 then they have an equal chance of getting 20, so the expected payoff is $\frac{1}{2} \times 20 - 9 = 1$,

Player 2

Player 1	Z	N	T
Z	10, 10	0, 11	0, 0
N	11, 0	1, 1	−9, 0
T	0, 0	0, −9	−10, −10

It is easy to see that T is strictly dominated by N. In the remaining game, Z is strictly dominated by N, and hence (N, N) is the unique Nash equilibrium. ■

(b) If the norm were different so that a bribe of 15 were also acceptable, is there a pure strategy Nash equilibrium?

Answer: Now the game is as follows (where F denotes a bribe of 15),

Player 2

Player 1	Z	N	F	T
Z	10, 10	0, 11	0, 5	0, 0
N	11, 0	1, 1	−9, 5	−9, 0
F	5, 0	5, −9	−5, −5	−15, 0
T	0, 0	0, −9	0, −15	−10, −10

Using the best responses of each player it is easy to see that there is no pure strategy Nash equilibrium. ■

(c) Find the symmetric mixed-strategy Nash equilibrium for the game with possible bribes of 9, 15 and 20.

Answer: Note first that T is weakly dominated by Z, so consider the game without T,

Player 2

Player 1	Z	N	F
Z	10, 10	0, 11	0, 5
N	11, 0	1, 1	−9, 5
F	5, 0	5, −9	−5, −5

Let $\sigma_i = (\sigma_{iZ}, \sigma_{iN}, \sigma_{iF})$ denote a mixed strategy for player i where $\sigma_{iF} = 1 - \sigma_{iZ} - \sigma_{iN}$. The game is symmetric so for player 1 to be indifferent between Z and T it must be that,

$$10\sigma_{2Z} = 11\sigma_{2Z} + \sigma_{2N} - 9(1 - \sigma_{2Z} - \sigma_{2N})$$

which implies that $\sigma_{2N} = \frac{9}{10} - \sigma_{2Z}$. For player 1 to be indifferent between Z and F it must be that,

$$10\sigma_{2Z} = 5\sigma_{2Z} + 5\sigma_{2N} - 5(1 - \sigma_{2Z} - \sigma_{2N})$$

which implies that $\sigma_{2N} = \frac{1}{2}$. Hence, the unique (mixed strategy) Nash equilibrium has each player play $\sigma_i = (\frac{2}{5}, \frac{1}{2}, \frac{1}{10})$. ■

12. **The Tax Man:** A citizen (player 1) must choose whether or not to file taxes honestly or whether to cheat. The tax man (player 2) decides how much effort to invest in auditing and can choose $a \in [0, 1]$, and the cost to the tax man of investing at a level a is $c(a) = 100a^2$. If the citizen is honest then he receives the benchmark payoff of 0, and the tax man pays the auditing costs without any benefit from the audit, yielding him a payoff of $(-100a^2)$. If the citizen cheats then his payoff depends on whether he is caught. If he is caught then his payoff is (-100) and the tax man's payoff is $100 - 100a^2$. If he is not caught then his payoff is 50 while the tax man's payoff is $(-100a^2)$. If the citizen cheats and the tax man chooses to audit at level a then the citizen is caught with probability a and is not caught with probability $(1 - a)$.

    (a) If the tax man believes that the citizen is cheating for sure, what is his best response level of a?

    Answer: The tax man maximizes $a(100 - 100a^2) + (1 - a)(0 - 100a^2) = 100a - 100a^2$. The first-order optimality condition is $100 - 200a = 0$, yielding $a = \frac{1}{2}$. ■

    (b) If the tax man believes that the citizen is honest for sure, what is his best response level of a?