Advanced Microeconomic Theory

Chapter 5: Choices under Uncertainty

Outline

- Simple, Compound, and Reduced Lotteries
- Independence Axiom
- Expected Utility Theory
- Money Lotteries
- Risk Aversion
- Prospect Theory and Reference-Dependent Utility
- Comparison of Payoff Distributions

Simple, Compound, and Reduced Lotteries

Simple Lotteries

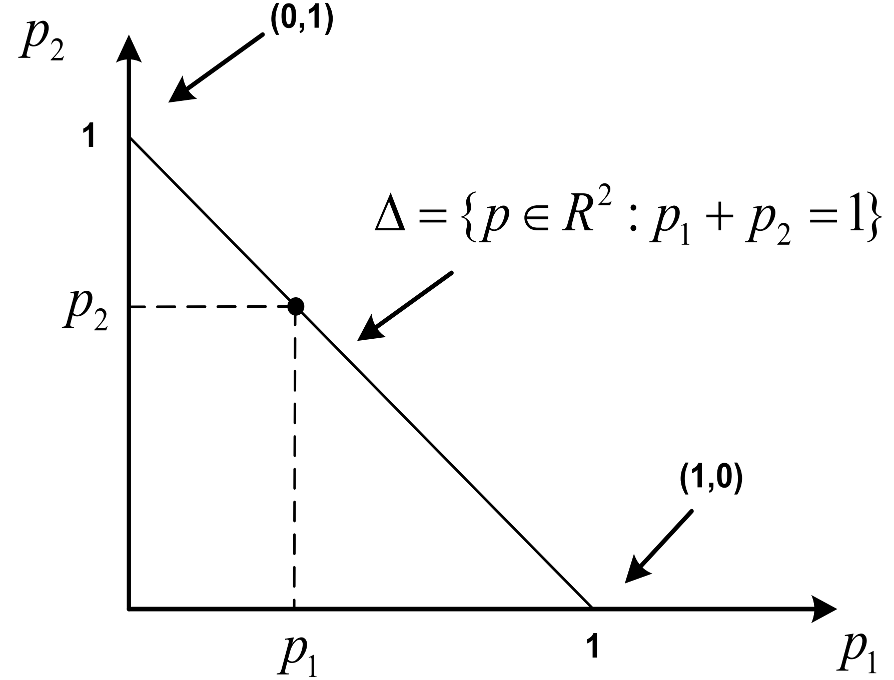
- Consider a set of possible outcomes (or consequences) C.
- The set C can include
  - simple payoffs $C \in \mathbb{R}$ (positive or negative)
  - consumption bundles $C \in \mathbb{R}^L$
- Outcomes are finite (N elements in C, $n = 1,2, \dots, N$)
- Probabilities of every outcome are objectively known
  - p_1 for outcome 1, p_2 for outcome 2, etc.

Simple Lotteries

- ***Simple lottery*** is a list
$$L = (p_1, p_2, \dots, p_N)$$
with $p_n \geq 0$ for all n and $\sum_{n=1}^{N} p_n = 1$, where p_n is interpreted as the probability of outcome n occuring.
- In some books, lotteries are described including the outcomes too.

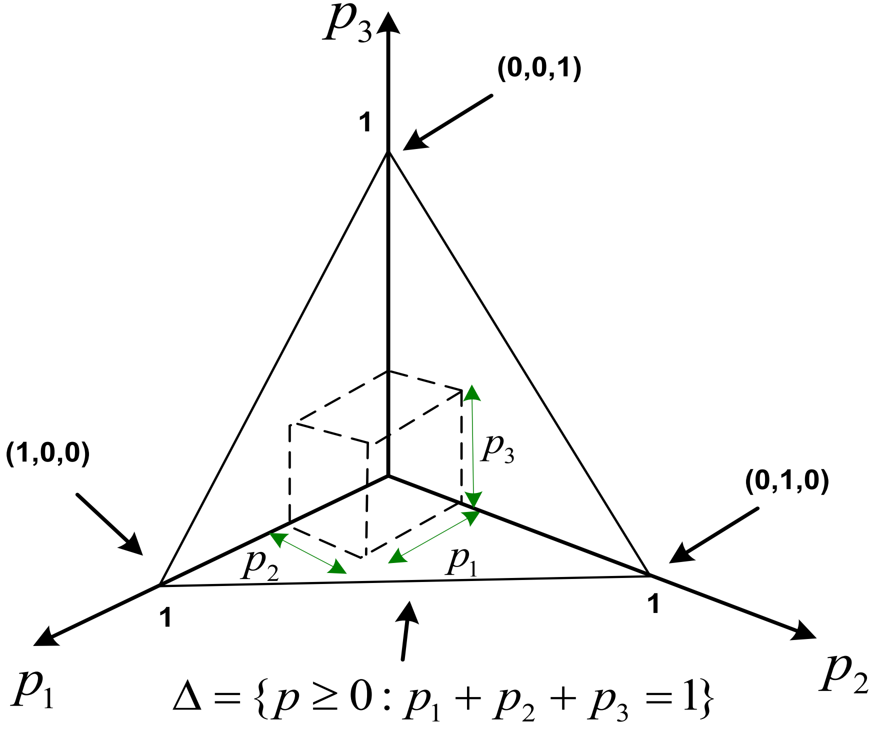
Simple Lotteries

- A simple lottery with 2 possible outcomes
- "Degenerated" probability pairs
  - at (0,1), outcome 2 happens with certainty.
  - at (1,0), outcome 1 happens with certainty.
- Strictly positive probability pairs
  - Individual faces some uncertainty, i.e., $p_1 + p_2 = 1$



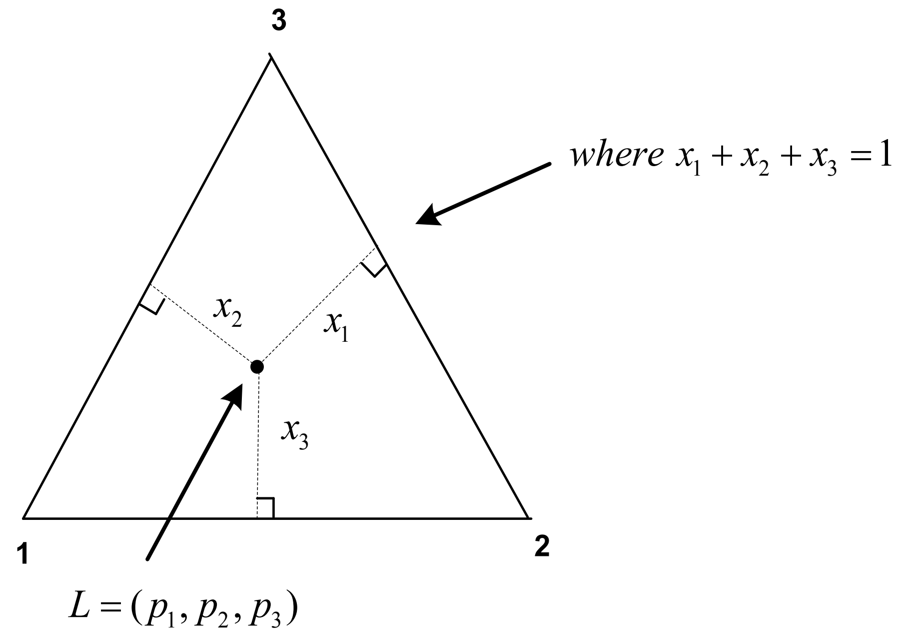
Simple Lotteries

- A simple lottery with 3 possible outcomes (i.e., 3-dim. simplex).
- Intercepts represent degenerated probabilities where one outcome is certain.
- Points strictly inside the hyperplane connecting the three intercepts denote a lottery where the individual faces uncertainty.



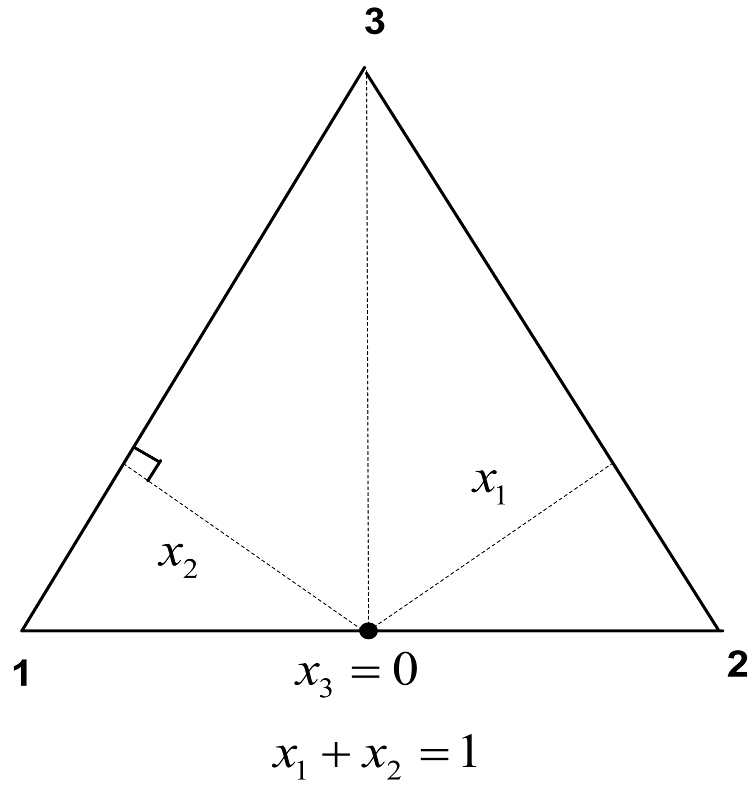
Simple Lotteries

- 2-dim. projection of the 3-dim. simplex
- Vertices represent the intercepts
- The distance from a given point to the side of the triangle measures the probability that the outcome represented at the opposite vertex occurs.



Simple Lotteries

- A lottery lies on one of the boundaries of the triangle:
  - We can only construct segments connecting the lottery to two of the outcomes.
  - The probability associated with the third outcome is zero.



Compound Lotteries

- Given simple lotteries
$$L_k = (p_1^k, p_2^k, \dots, p_N^k) \text{ for } k = 1,2,\dots,K$$
and probabilities $\alpha_k \geq 0$ with $\sum_{n=1}^{K} \alpha_k = 1$, then the ***compound lottery*** $(L_1, L_2, \dots, L_K; \alpha_1, \alpha_2, \dots, \alpha_K)$ is the risky alternative that yields the simple lottery L_k with probability α_k for $k = 1,2,\dots,K$.
  - Think about a compound lottery as a "lottery of lotteries": first, I have probability α_1 of playing lottery 1, and if that happens, I have probability p_1^1 of outcome 1 occurring.
  - Then, the joint probability of outcome 1 is
$$p_1 = \alpha_1 \cdot p_1^1 + \alpha_2 \cdot p_1^2 + \cdots + \alpha_K \cdot p_1^K$$

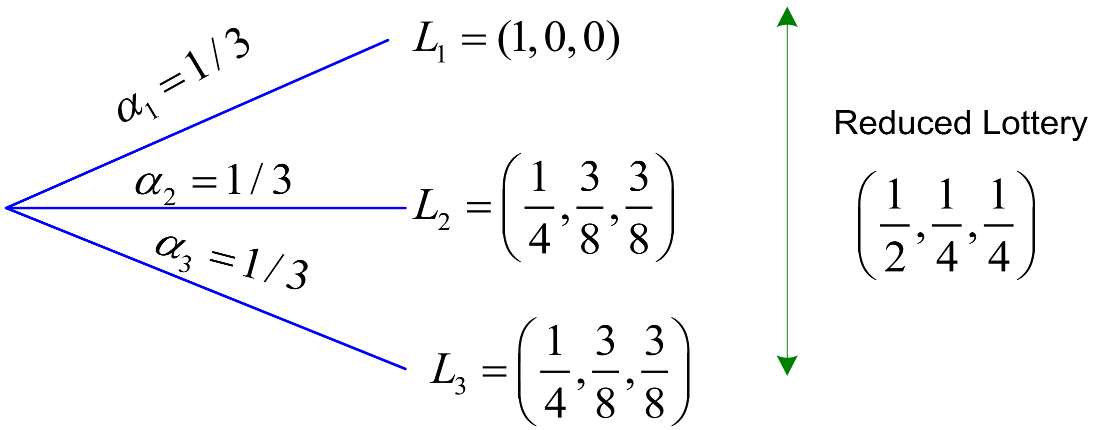
Compound and Reduced Lotteries

- Given that interpretation, the following result should come at no surprise:
  - For any compound lottery $(L_1, L_2, \dots, L_K; \alpha_1, \alpha_2, \dots, \alpha_K)$, we can calculate its corresponding ***reduced lottery*** as the simple lottery $L = (p_1, p_2, \dots, p_N)$ that generates the same ultimate probability distribution of outcomes.
- The reduced lottery L of any compound lottery can be obtained by

$$L = \alpha_1 L_1 + \alpha_2 L_2 + \cdots + \alpha_K L_K \in \Delta$$

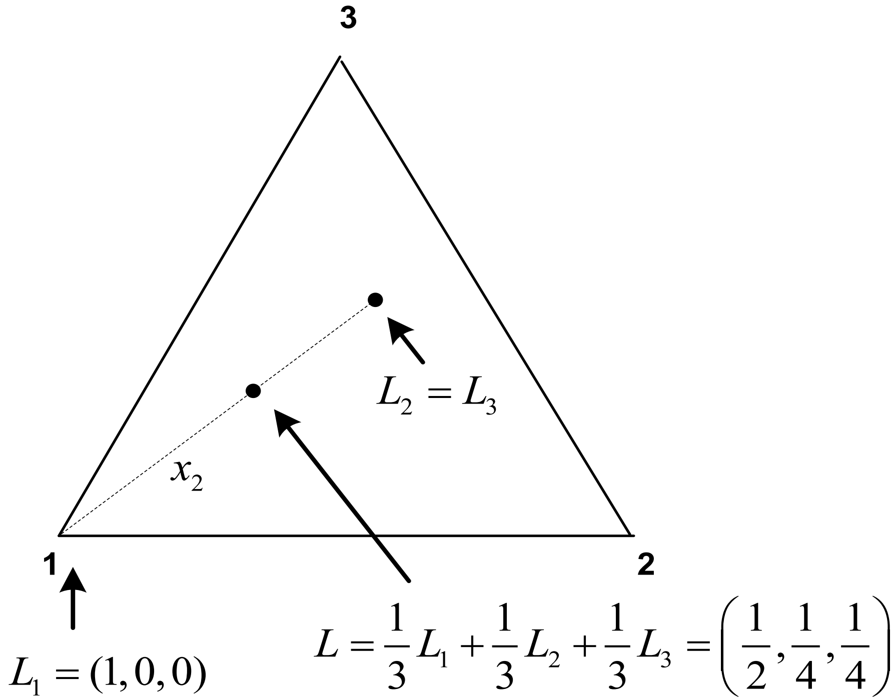
Compound and Reduced Lotteries

- ***Example 1***:
  - All three lotteries are equally likely
  - $P(\text{outcome } 1) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{2}$
  - $P(\text{outcome } 2) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{3}{8} + \frac{1}{3} \cdot \frac{3}{8} = \frac{1}{4}$
  - $P(\text{outcome } 3) = \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot \frac{3}{8} + \frac{1}{3} \cdot \frac{3}{8} = \frac{1}{4}$



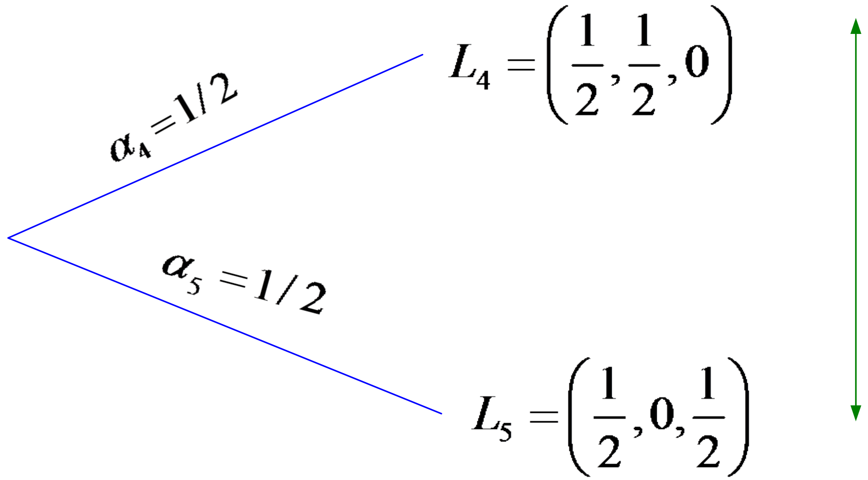
Compound and Reduced Lotteries

- ***Example 1*** (continued):
  - Probability simplex of the reduced lottery of a compound lottery
  - Reduced lottery L assigns the same probability weight to each simple lottery.



Compound and Reduced Lotteries

- ***Example 2***:
  - Both lotteries are equally likely



Reduced Lottery

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$$

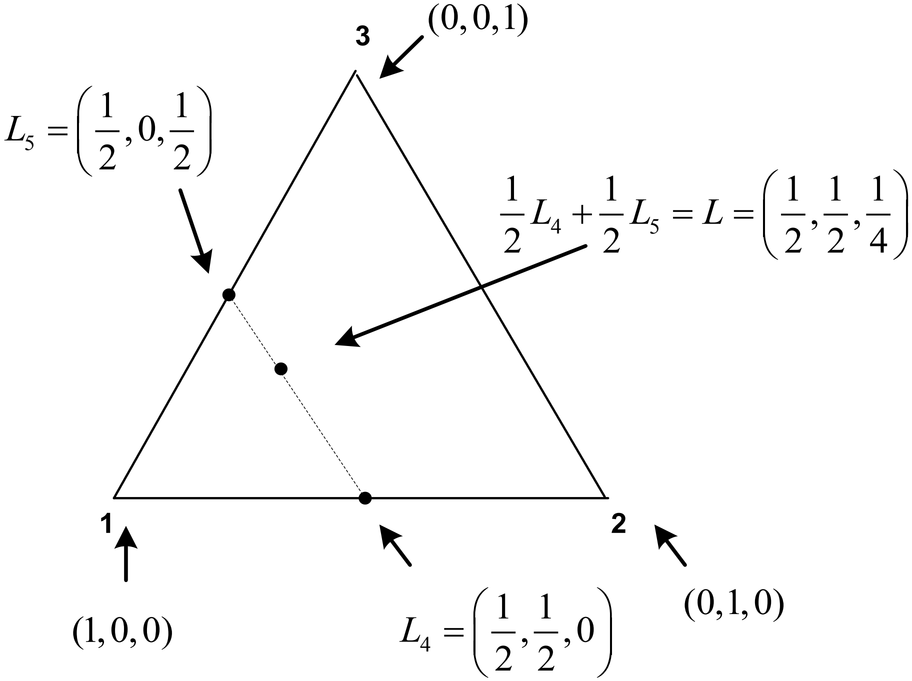
Outcome 1 $\longrightarrow$ $\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{2}$

Outcome 2 $\longrightarrow$ $\frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot 0=\frac{1}{4}$

Outcome 3 $\longrightarrow$ $\frac{1}{2}\cdot 0+\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$

Compound and Reduced Lotteries

- ***Example 2*** (continued):
  - Probability simplex of the reduced lottery of a compound lottery



Compound and Reduced Lotteries

- Consumer is indifferent between the two compound lotteries which induce the same reduced lottery
  - This was illustrated in the previous Examples 1 and 2 where, despite facing different compound lotteries, the consumer obtained the same reduced lottery.
- We refer to this assumption as the ***Consequentialist hypothesis***:
  - Only consequences, and the probability associated to every consequence (outcome) matters, but not the route that we follow in order to obtain a given consequence.

Preferences over Lotteries

- For a given set of outcomes C, consider the set of all simple lotteries over C, $\mathcal{L}$.
- We assume that the decision maker has a *complete* and *transitive* preference relation $\succsim$ over lotteries in $\mathcal{L}$, allowing him to compare any pair of simple lotteries L and L'.
  - ***Completeness***: For any two lotteries L and L', either $L \succsim L'$ or $L' \succsim L$, or both.
  - ***Transitivity***: For any three lotteries L, L' and L'', if $L \succsim L'$ and $L' \succsim L''$, then $L \succsim L''$.

Preferences over Lotteries

- ***Extreme preference for certainty***:
  – $L \succsim L'$ if and only if

$$\max_{n \in N} p_n \geq \max_{n \in N} p'_n$$

  – The decision maker is only concerned about the probability associated with the most likely outcome.

Preferences over Lotteries

- ***Smallest size of the support***:
  - $L \succsim L'$ if and only if

$$\text{supp}(L) \leq \text{supp}(L')$$

    where $\text{supp}(L) = \{n \in N: p_n > 0\}$.

  - The decision maker prefers the lottery whose probability distribution is concentrated over the smallest set of possible outcomes.

Preferences over Lotteries

- ***Lexicographic preferences***:
  - First, order outcomes from most preferred (outcome 1) to least preferred (outcome n).
  - Then $L \succsim L'$, if and only if

$$p_1 > p_1', \text{ or}$$
$$\text{If } p_1 = p_1' \text{ and } p_2 > p_2', \text{ or}$$
$$\text{If } p_1 = p_1' \text{ and } p_2 = p_2' \text{ and } p_3 > p_3', \text{ or}$$
$$\dots$$

  - The decision maker weakly prefers lottery L to L' if outcome 1 is more likely to occur in lottery L than in lottery L'.
  - If outcome 1 is as likely to occur in both lotteries, he moves to outcome 2; and so on.

Preferences over Lotteries

- ***The worst case scenario***:
  - First, attach a number $v(z)$ to every outcome $z \in C$, that is, $v(z) \in \mathbb{R}$.
  - Then $L \gtrsim L'$ if and only if

$$\min\{v(z)\colon\ p(z) > 0\} > \min\{v(z)\colon\ p'(z) > 0\}$$

  - The decision maker prefers lottery L if the lowest utility he can get from playing lottery L is higher than the lowest utility he can obtain from playing lottery L'.

Preferences over Lotteries

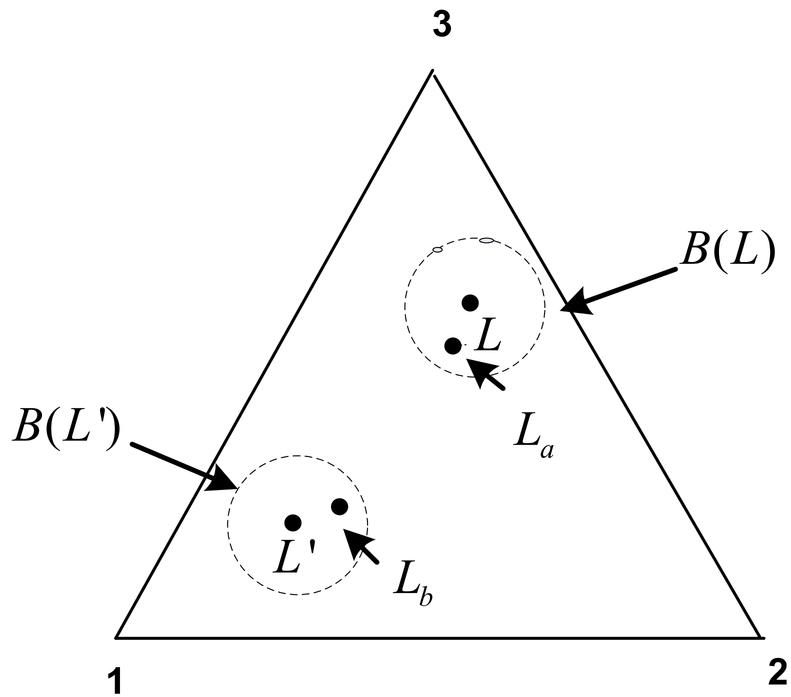
- Continuity of preferences over lotteries:
  - ***Continuity 1***: For any three lotteries L, L', and L'', the sets

    $\{\alpha \in [0,1]: \alpha L + (1-\alpha)L' \succsim L''\} \subset [0,1]$ and
    $\{\alpha \in [0,1]: L'' \succsim \alpha L + (1-\alpha)L'\} \subset [0,1]$

    are closed.
  - ***Continuity 2***: if $L \succ L'$, then there are neighborhoods of L and L', $B(L)$ and $B(L')$, such that for all $L_a \in B(L)$ and $L_b \in B(L')$, we have $L_a \succ L_b$.

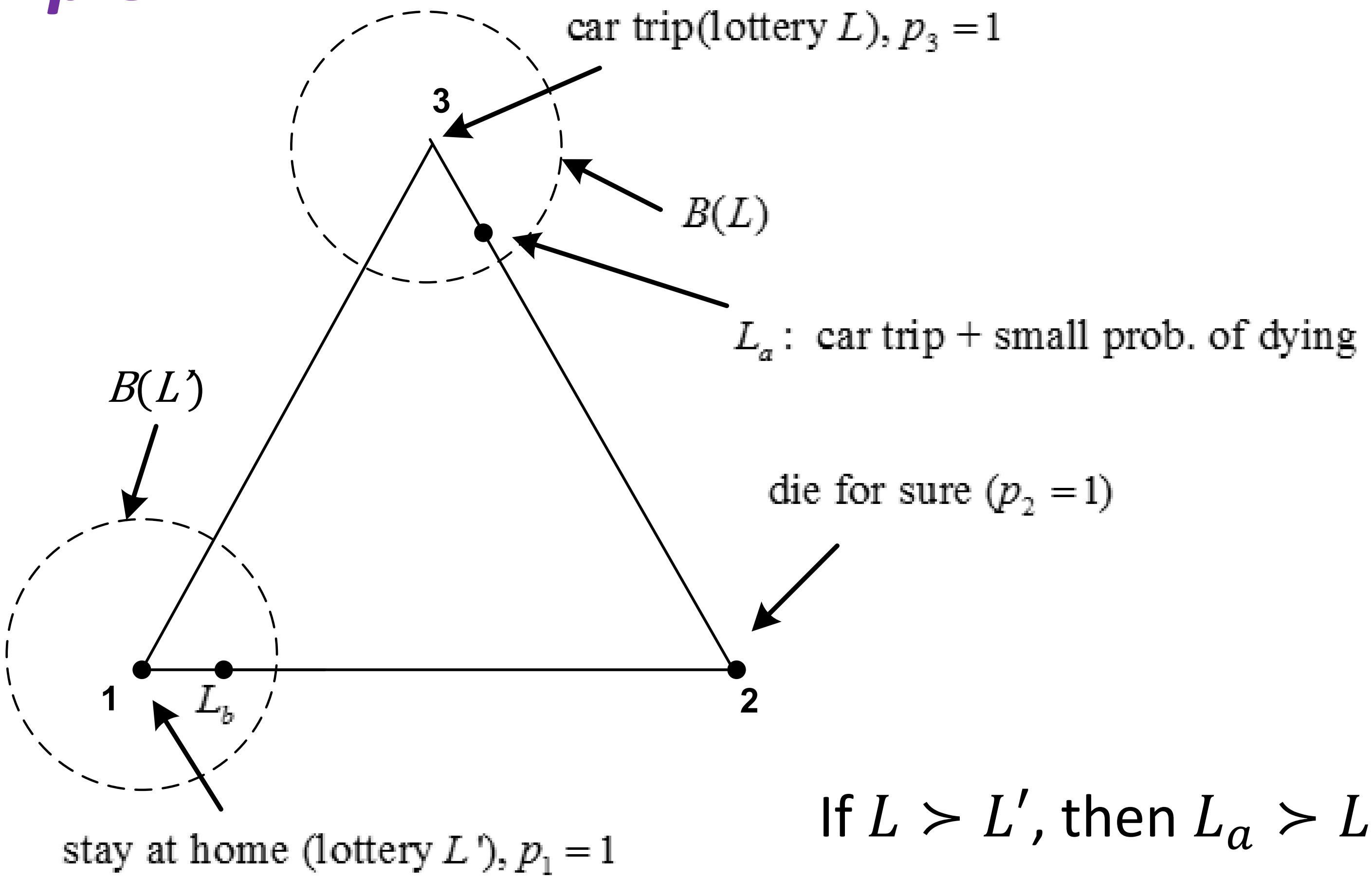
Preferences over Lotteries

- Small changes in the probability distribution of lotteries L and L' do not change the preference over the two lotteries.



Preferences over Lotteries

- ***Example***:



If $L \succ L'$, then $L_a \succ L_b$.

Preferences over Lotteries

- The continuity assumption, as in consumer theory, implies the existence of a utility function $U: \mathcal{L} \to \mathbb{R}$ such that

$$L \succsim L' \text{ if and only if } U(L) \geq U(L')$$

- However, we first impose an additional assumption in order to have a more structured utility function.
  - The following assumption is related with consequentialism: the *Independence axiom*.

Preferences over Lotteries

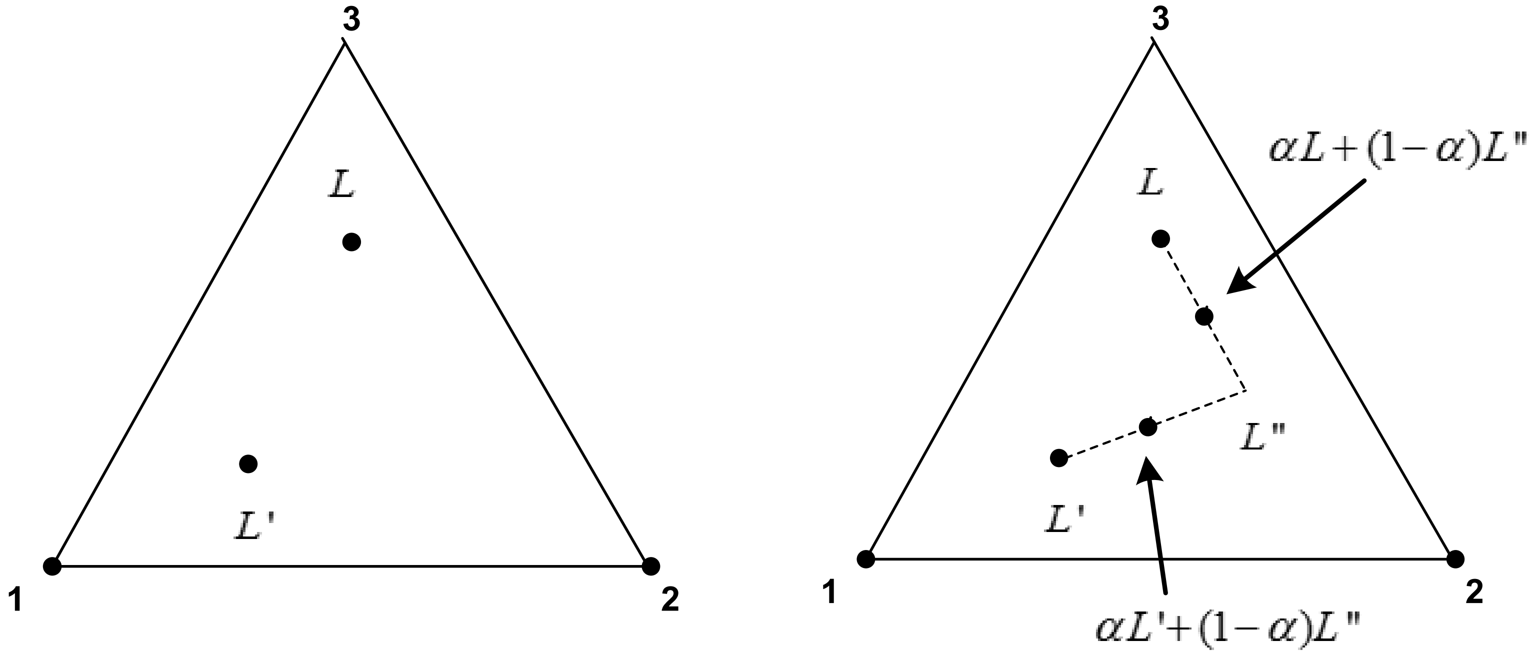
- ***Independence Axiom (IA)***: a preference relation satisfies IA if, for any three lotteries L, L', and L'', and $\alpha \in (0,1)$ we have
$$L \succsim L' \text{ if and only if}$$
$$\alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$$

- *Intuition*: If we mix each of two lotteries, L and L', with a third one (L''), then the preference ordering of the two resulting compound lotteries is independent of the particular third lottery .

Preferences over Lotteries

- $L \gtrsim L'$ if and only if

$$\alpha L + (1-\alpha)L'' \gtrsim \alpha L' + (1-\alpha)L''$$



Preferences over Lotteries

- ***Example 1*** (intuition):
  - The decision maker prefers lottery L to L', $L \succsim L'$
  - Construct a compound lottery by a coin toss:
    - play lottery L if heads comes up
    - play lottery L'' if tails comes up
  - By IA, if $L \succsim L'$, then

$$\frac{1}{2}L + \frac{1}{2}L'' \succsim \frac{1}{2}L' + \frac{1}{2}L''$$

Preferences over Lotteries

- ***Example 2*** (violations of IA):
  - Extreme preference for certainty
  - Consider two simple lotteries L and L' for which $L \sim L'$.
  - Construct two compound lotteries for which

$$\frac{1}{2}L + \frac{1}{2}L \nsim \frac{1}{2}L' + \frac{1}{2}L$$

  - If $L \sim L'$, then it must be that

$$\max\{p_1, p_2, \ldots, p_n\} = \max\{p_1', p_2', \ldots, p_n'\}$$

Preferences over Lotteries

- ***Example 2*** (violations of IA):
  - Compound lottery $\frac{1}{2}L + \frac{1}{2}L$ coincides with simple lottery L.
  - Hence, $\max\{p_1, p_2, \dots, p_n\}$ is used to evaluate lottery L.
  - But compound lottery $\frac{1}{2}L' + \frac{1}{2}L$ is a reduced lottery with associated probabilities
    $$\max\left\{\frac{1}{2}p_1' + \frac{1}{2}p_1, \dots, \frac{1}{2}p_n' + \frac{1}{2}p_n\right\}$$
    which might *differ* from $\max\{p_1', p_2', \dots, p_n'\}$.

Preferences over Lotteries

- ***Example 2*** (violations of IA, a numerical example):
  - Consider two simple lotteries
    $$L = (0.4, 0.5, 0.1),\ L' = (0.5, 0, 0.5)$$
  - Hence,
    $$\max\{0.4, 0.5, 0.1\} = 0.5 = \max\{0.5, 0, 0.5\}$$
    implying that $L \sim L'$.
  - However, the compound lottery $\frac{1}{2}L' + \frac{1}{2}L$ entails probabilities
    $$\left(\frac{0.4 + 0.5}{2}, \frac{0.5 + 0}{2}, \frac{0.1 + 0.5}{2}\right) = (0.45, 0.25, 0.3)$$
    implying that $\max\{0.45, 0.25, 0.3\} = 0.45$.

Preferences over Lotteries

- ***Example 2*** (violations of IA, a numerical example):
  - Therefore,

$$\max\{0.4, 0.5, 0.1\} = 0.5 > 0.45 = \max\{0.45, 0.25, 0.3\}$$

and thus $L = \frac{1}{2}L + \frac{1}{2}L \succ \frac{1}{2}L' + \frac{1}{2}L$.

  - This violates the IA, which requires

$$\frac{1}{2}L + \frac{1}{2}L \sim \frac{1}{2}L' + \frac{1}{2}L$$

Preferences over Lotteries

- ***Example 3*** (violations of IA, "worst case scenario"):
  - Consider $L \succ L'$.
  - Then, the compound lottery $\frac{1}{2}L + \frac{1}{2}L$ does *not* need to be preferred to $\frac{1}{2}L' + \frac{1}{2}L$.
  - *Example*:
    - Consider the simple lotteries $L = (1,3)$ and $L' = (10,0)$, with probabilities (p_1, p_2) and (p_1', p_2'), respectively.
    - This implies

$$\min\{v(z): p(z) > 0\} = 1 \text{ for lottery } L$$

$$\min\{v(z): p'(z) > 0\} = 0 \text{ for lottery } L'$$

    - Hence, $L \succ L'$.

Preferences over Lotteries

- ***Example 3*** (violations of IA, "worst case scenario"):
  - *Example* (continued):
    - However, the compound lottery $\frac{1}{2}L + \frac{1}{2}L'$ is $\left(\frac{11}{2}, \frac{3}{2}\right)$, whose worst possible outcome is $\frac{3}{2}$, which is preferred to that of $\frac{1}{2}L + \frac{1}{2}L$, which is 1.
    - Hence, despite $L \succ L'$ over simple lotteries,

$$L = \frac{1}{2}L + \frac{1}{2}L \prec \frac{1}{2}L + \frac{1}{2}L',$$

which violates the IA.

Expected Utility Theory

Expected Utility Theory

- The utility function $U: \mathcal{L} \rightarrow \mathbb{R}$ has the ***expected utility*** (EU) form if there is an assignment of numbers $(u_1, u_2, \dots, u_N)$ to the N possible outcomes such that, for every simple lottery $L = (p_1, p_2, \dots, p_N) \in \mathcal{L}$ we have

$$U(L) = p_1 u_1 + \cdots + p_N u_N$$

  - A utility function with the EU form is also referred to as a **von-Neumann-Morgenstern** (vNM) expected utility function.
  - Note that this function is *linear* in the probabilities.

Expected Utility Theory

- Hence, a utility function $U: \mathcal{L} \to \mathbb{R}$ has the expected utility form if and only if it is *linear* in the probabilities, i.e.,

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_{k=1}^{K} \alpha_k \cdot U(L_k)$$

  for any K lotteries $L_k \in \mathcal{L}$, $k = 1,2, \ldots, K$, and probabilities $(\alpha_1, \alpha_2, \ldots, \alpha_K) \geq 0$, where $\sum_{k=1}^{K} \alpha_k = 1$.

- *Intuition*: the utility of the expected value of the K lotteries, $U\left(\sum_{k=1}^{K} \alpha_k L_k\right)$, coincides with the expected utility of the K lotteries, $\sum_{k=1}^{K} \alpha_k U(L_k)$.

Expected Utility Theory

- Note that the utility of the expected value of playing the K lotteries is

$$U\left(\sum_{k=1}^{K} \alpha_k L_k\right) = \sum_n u_n \cdot \left(\sum_k \alpha_k\, p_n^k\right)$$

where $\sum_k \alpha_k\, p_n^k$ is the total joint probability of outcome n occurring.

Expected Utility Theory

- Note that the expected utility from playing the K lotteries is

$$\sum_{k=1}^{K} \alpha_k \cdot U(L_k) = \sum_k \alpha_k \cdot \left(\sum_n u_n \, p_n^k \right)$$

where $\sum_n u_n \, p_n^k$ is the expected utility from playing a given lottery k.

Expected Utility Theory

- The EU property is a *cardinal property*:
  - Not only rank matters, the particular number resulting form $U: \mathcal{L} \to \mathbb{R}$ also matters.
- Hence, the EU form is preserved only under increasing linear transformations (a.k.a. affine transformations).
  - Hence, the expected utility function $\widetilde{U}: \mathcal{L} \to \mathbb{R}$ is another vNM utility function if and only if

$$\widetilde{U}(L) = \beta U(L) + \gamma$$

for every $L \in \mathcal{L}$, where $\beta > 0$.

Expected Utility Theory: Representability

- Suppose that the preference relation $\succsim$ satisfies rationality, continuity and independence. Then, $\succsim$ admits a utility representation of the EU form.
- That is, we can assign a number u_n to every outcome $n = 1,2, \dots, N$ in such a manner that for any two lotteries

  $$L = (p_1, p_2, \dots, p_N) \text{ and } L' = (p'_1, p'_2 \dots, p'_N)$$

  we have $L \succsim L'$ if and only if $U(L) \geq U(L')$, or

  $$\sum_{n=1}^{N} p_n u_n \geq \sum_{n=1}^{N} p'_n u_n$$

- *Notation*: u_n is the utility that the decision maker assigns to outcome n. It is usually referred as the Bernoulli utility function.

Expected Utility Theory: Indifference Curves

- Let us next analyze the effect of the IA on indifference curves over lotteries.

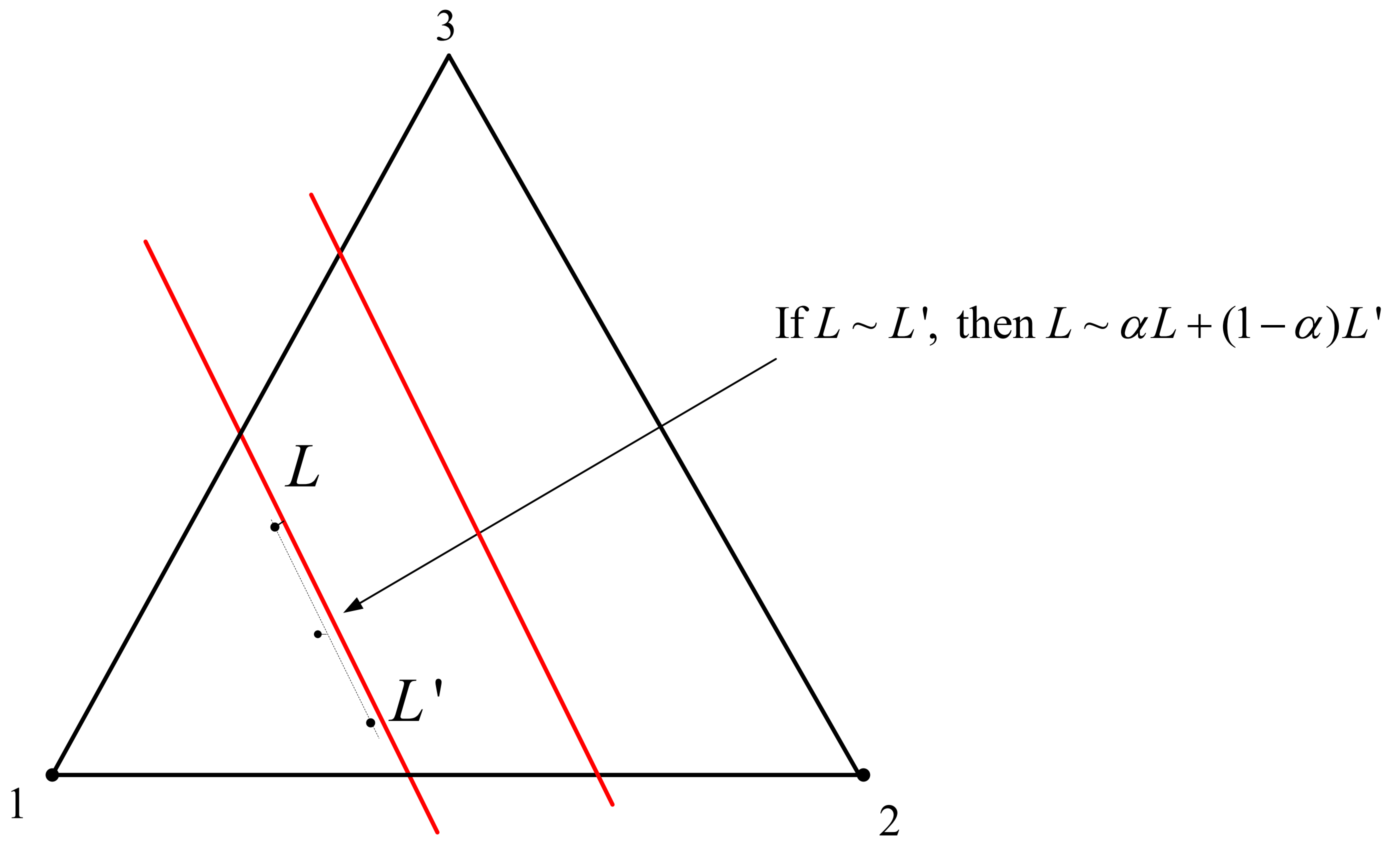
1) ***Indifference curves must be straight lines***:

   Recall that from the IA, $L \sim L'$ implies that

$$\underbrace{\alpha L + (1-\alpha)L}_{L} \sim \alpha L + (1-\alpha)L'$$

   for all $\alpha \in (0,1)$.

Expected Utility Theory: Indifference Curves



Straight indifference curves

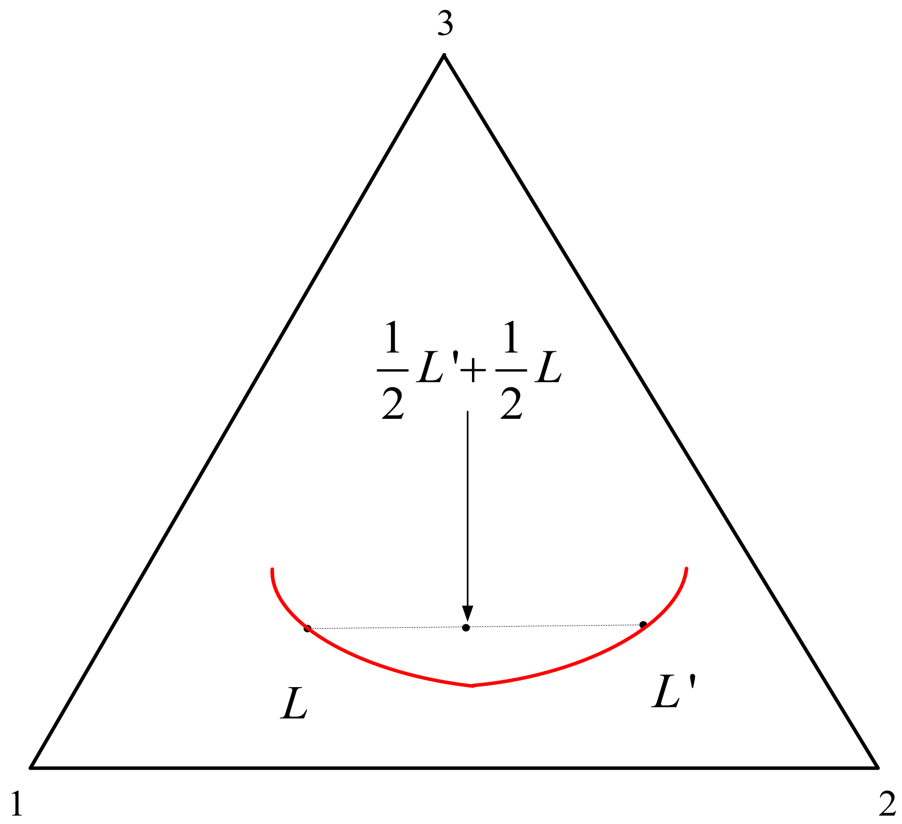
Expected Utility Theory: Indifference Curves

- Why indifference curves must be straight?
  - We have that $L \sim L'$, but $L \prec \frac{1}{2}L + \frac{1}{2}L'$. This is equivalent to
$$\frac{1}{2}L + \frac{1}{2}L \prec \frac{1}{2}L + \frac{1}{2}L'$$
  - But from the IA we must have
$$\frac{1}{2}L + \frac{1}{2}L \sim \frac{1}{2}L + \frac{1}{2}L'$$
  - Hence, indifference curves must be straight lines in order to satisfy the IA.

Expected Utility Theory: Indifference Curves

- Curvy indifference curves over lotteries are incompatible with the IA
  - The compound lottery $\frac{1}{2}L + \frac{1}{2}L'$ would not lie on the same indifference curve as lottery L and L'.
  - Hence, the decision maker is *not* indifferent between the compound lotteries $\frac{1}{2}L + \frac{1}{2}L$ and $\frac{1}{2}L + \frac{1}{2}L'$.

Expected Utility Theory: Indifference Curves



Curvy indifference curve

Expected Utility Theory: Indifference Curves

2) *Indifference curves must be parallel lines*:

If we have that $L \sim L'$, then by the IA

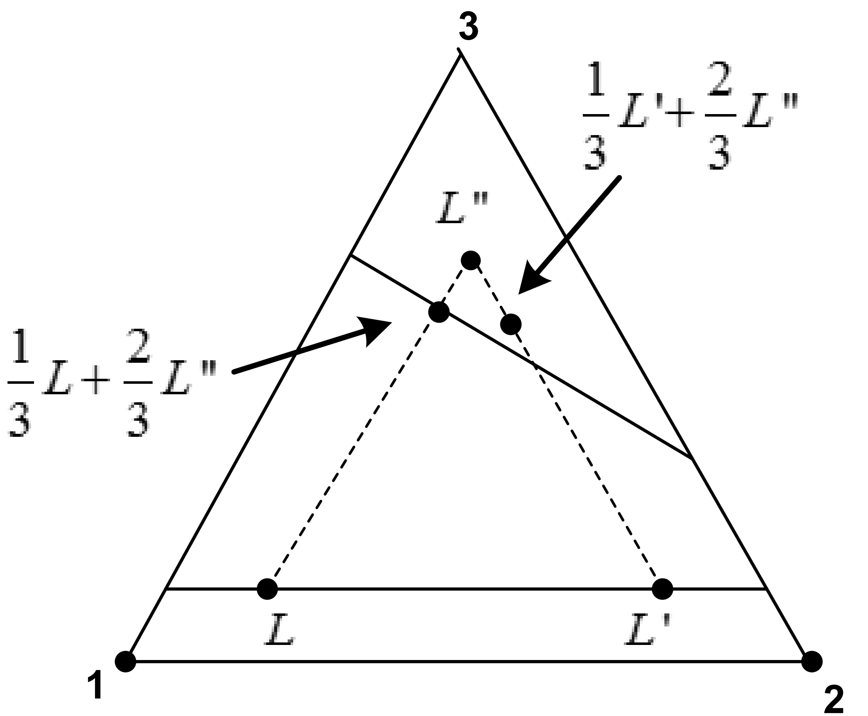
$$\frac{1}{3}L + \frac{2}{3}L'' \sim \frac{1}{3}L' + \frac{2}{3}L''$$

- That is, the convex combination of L and L' with a third lottery L'' should also lie on the same indifference curve.

- This implies that the indifference curves must be parallel lines in order to satisfy the IA.

Expected Utility Theory: Indifference Curves

- Nonparallel indifference curves are incompatible with the IA.
  - If compound lotteries $\frac{1}{3}L + \frac{2}{3}L''$ and $\frac{1}{3}L' + \frac{2}{3}L''$ lie on different (nonparallel) indifference curves, then
    $$\frac{1}{3}L + \frac{2}{3}L'' \prec \frac{1}{3}L' + \frac{2}{3}L''$$
    which violates the IA.



Expected Utility Theory:

Violations of the IA:

- Despite the intuitive appeal of the IA, we encounter several settings in which decision makers violate it.
- We next elaborate on these violations.

Expected Utility Theory: Violations of the IA

- ***Allais' paradox***:
  - Consider a lottery over three possible monetary outcomes:

1st prize	2nd prize	3rd prize
$2.5mln	**$500,000**	**$0**

  - First choice set:

$$L_1 = (0,1,0) \text{ and } L_1' = (\frac{10}{100}, \frac{89}{100}, \frac{1}{100})$$

  - Second choice set:

$$L_2 = (0, \frac{11}{100}, \frac{89}{100}) \text{ and } L_2' = (\frac{10}{100}, 0, \frac{90}{100})$$

Expected Utility Theory: Violations of the IA

- About 50% students surveyed expressed $L_1 \succ L_1'$ and $L_2' \succ L_2$.
- These choices violate the IA.
- To see this, consider that the decision maker's preferences over lotteries have a EU form. Hence, $L_1 \succ L_1'$ implies

$$u_5 > \frac{10}{100}u_{25} + \frac{89}{100}u_5 + \frac{1}{100}u_0$$

- By the IA, we can add $\frac{89}{100}u_0 - \frac{89}{100}u_5$ on both sides

$$u_5 + \left(\frac{89}{100}u_0 - \frac{89}{100}u_5\right) >$$

$$\frac{10}{100}u_{25} + \frac{89}{100}u_5 + \frac{1}{100}u_0 + \left(\frac{89}{100}u_0 - \frac{89}{100}u_5\right)$$

Expected Utility Theory: Violations of the IA

– Simplifying

$$\underbrace{\frac{11}{100}u_5 + \frac{89}{100}u_0}_{\text{EU of } L_2} > \underbrace{\frac{10}{100}u_{25} + \frac{90}{100}u_0}_{\text{EU of } L_2'}$$

which implies $L_2 \succ L_2'$.

– Did your own choices violate the IA?

Expected Utility Theory: Violations of the IA

- Reactions to the Allais' Paradox:
  - *Approximation to rationality*: people adapt their choices as they go.
  - *Little economic significance*: the lotteries involve probabilities that are close to zero and one.
  - *Regret theory*: the reason why $L_1 \succ L_1'$ is because I didn't want to regret a sure win of $500,000.
  - Give up the IA in favor of a weaker assumption: the *betweenness axiom*.

Expected Utility Theory: Violations of the IA

- ***Machina's paradox***:
  - Consider that

    Trip to Barcelona $\succ$ Movie about Barcelona $\succ$ Home

  - Now, consider the following two lotteries

    $$L_1 = (\frac{99}{100}, \frac{1}{100}, 0) \text{ and } L_2 = (0, \frac{99}{100}, \frac{1}{100})$$

  - From the previous preferences over certain outcomes, how can we know this individual's preferences over lotteries?
    - Using the IA.

Expected Utility Theory: Violations of the IA

- From $T \succ M$ and the IA, we can construct the compound lotteries
$$\frac{99}{100}T + \frac{1}{100}M \succ \frac{99}{100}M + \frac{1}{100}M$$
- From $M \succ H$ and the IA, we have
$$\frac{99}{100}M + \frac{1}{100}M \succ \frac{99}{100}M + \frac{1}{100}H$$
- By transitivity,
$$\underbrace{\frac{99}{100}T + \frac{1}{100}M}_{L_1} \succ \underbrace{\frac{99}{100}M + \frac{1}{100}H}_{L_2}$$
- Hence, $L_1 \succ L_2$.

Expected Utility Theory: Violations of the IA

- Therefore, for preferences over lotteries to be consistent with the IA, we need $L_1 \succ L_2$.

- Many subjects in experimental settings would rather prefer L_2, thus violating the IA.

- Many people explain choosing L_2 over L_1 on grounds of the disappointment they would experience in the case of losing the trip to Barcelona, and having to watch a movie instead.
  - Similar to regret theory.

Expected Utility Theory: Violations of the IA

- ***Dutch books***:
  - In the above two anomalies, actual behavior is inconsistent with the IA.
  - Can we then rely on the IA?
  - What would happen to individuals whose behavior violates the IA?
  - They would be weeded out of the market because they would be open to the acceptance of so-called *Dutch books*, leading them to a sure loss of money.

Expected Utility Theory: Violations of the IA

– Consider that $L \succ L'$. By the IA, we should have

$$\underbrace{\alpha L + (1-\alpha)L}_{L} \succ \alpha L + (1-\alpha)L'$$

– If, instead, the IA is violated, then

$$L \prec \alpha L + (1-\alpha)L'$$

– Consider an individual with these preferences, who initially owns lottery L.

– If we offer him the compound lottery $\alpha L + (1-\alpha)L'$, for a small fee \$$x$, he would accept such a trade.

Expected Utility Theory: Violations of the IA

– After the realization stage, he owns either L or L'
  - If L', then we offer L again for \$$y$.
  - If L, then we offer $\alpha L + (1-\alpha)L'$ for \$$y$.
– Either way, he is at the same position as he started (owning L or $\alpha L + (1-\alpha)L'$), but having lost \$$x$ + \$$y$ in the process.
– We can repeat this process ad infinitum.
– Hence, individuals with preferences that violate the IA would be exploited by microeconomists (they would be a “money pump”).

Expected Utility Theory: Violations of the IA

- Further reading:
  - "Developments in non-expected utility theory: The hunt for a descriptive theory of choice under risk" (2000) by Chris Starmer, *Journal of Economic Literature*, vol. 38(2)
  - *Choices, Values and Frames* (2000) by Nobel prize winners Daniel Kahneman and Amos Tversky, Cambridge University Press.
  - *Theory of Decision under Uncertainty* (2009) by Itzhak Gilboa, Cambridge University Press.

Theories Modifying Expected Utility Theory

1) Weighted utility theory:

- The payoff function from playing lottery L is

$$V(L) = \sum_{x \in C} w_i \cdot u(x_i)$$

  where

$$w_i = \frac{g(x_i)p(x_i)}{\sum_{x \in C} g(x_i)p(x_i)} \text{ and } g: C \to \mathbb{R}$$

- The utility of outcome $x_i \in C$ is weighted according to:
  - a) its probability $p(x_i)$
  - b) outcome x_i itself through function $g: C \to \mathbb{R}$

Theories Modifying Expected Utility Theory

– ***Example***: Consider a lottery with two payoffs x_1 and x_2 with probabilities p and $1-p$. Then, the weighted utility is

$$V(L) = w_1 u(x_1) + w_2 u(x_2)$$
$$= \frac{g(x_1)p}{g(x_1)p + g(x_2)(1-p)} u(x_1)$$
$$+ \frac{g(x_2)(1-p)}{g(x_1)p + g(x_2)(1-p)} u(x_2)$$

If $g(x_i) = g(x_j)$ for any $x_i \neq x_j$, then

$$V(L) = pu(x_1) + (1-p)u(x_2)$$

which is a standard expected utility function.

Theories Modifying Expected Utility Theory

- The *weighted utility theory* relies on the same axioms as *expected utility theory*, except for the IA, which is relaxed to the "weak independence axiom."
  - ***Weak independence axiom***: if we have that $L_1 \sim L_2$, we can find a pair of probabilities α and α' such that
    $$\alpha L_1 + (1-\alpha)L_3 \sim \alpha' L_2 + (1-\alpha')L_3$$
  - The IA becomes a special case if $\alpha = \alpha'$.

Theories Modifying Expected Utility Theory

2) Rank dependent utility theory:

- First, rank the outcomes $x_1, x_2, \ldots, x_n$ from worst (x_1) to best (x_n)
- Second, apply a probability weighting function

$$w_i = \pi(p_i + \cdots + p_n) - \pi(p_{i+1} + \cdots + p_n)$$
$$w_n = \pi(p_n)$$

  where $\pi(\cdot)$ is a non-decreasing *transformation function*, with $\pi(0) = 0$ and $\pi(1) = 1$.

- Finally, a rank-dependent utility is

$$V(L) = \sum_{x \in C} w_i \cdot u(x_i)$$

Theories Modifying Expected Utility Theory

– For a lottery with two outcomes, x_1 and x_2 where $x_2 > x_1$, the rank-dependent utility is

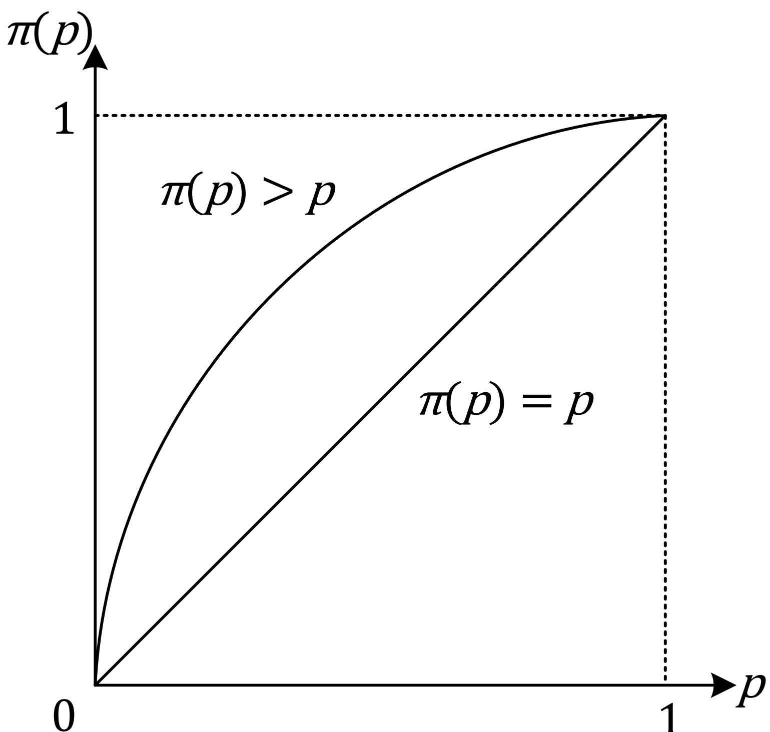
$$V(L) = w(p)u(x_1) + (1 - w(p))u(x_2)$$

where p is the probability of outcome x_1.

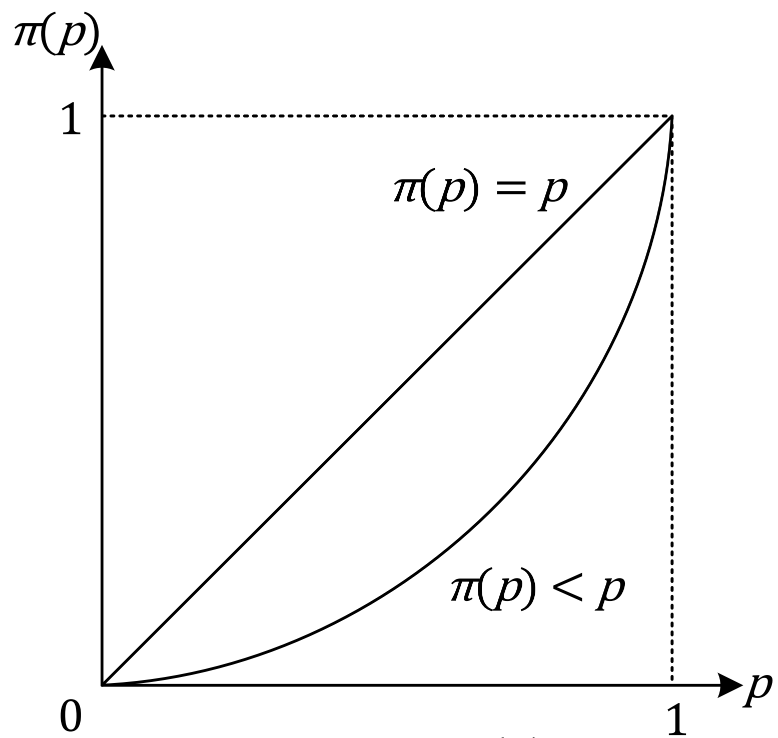
– This model allows for *different weight* to be attached to each outcome, as opposed to expected utility theory models in which the *same utility weight* is attached to all outcomes.

Theories Modifying Expected Utility Theory

– Transformation function $\pi(\cdot)$



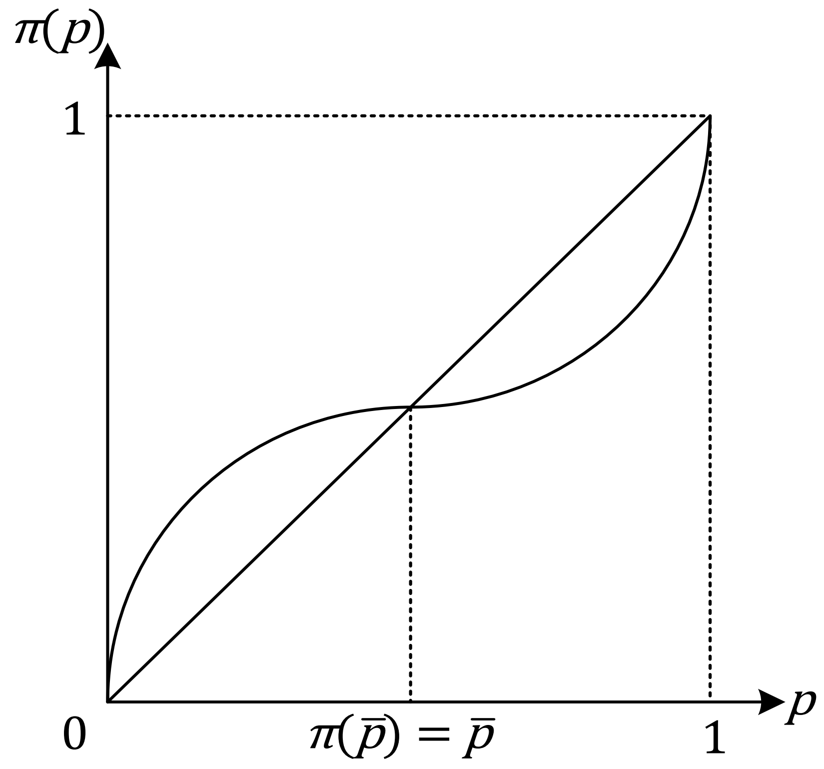
Pessimistic $\pi(p)$



Optimistic $\pi(p)$

Theories Modifying Expected Utility Theory

- Empirical evidence suggests an S-shaped transformation function.
- *Intuition*: individuals are pessimistic in rare outcomes (i.e., $p < \overline{p}$), but become optimistic for outcomes they have frequently encountered.



Theories Modifying Expected Utility Theory

- The *rank-dependent utility theory* relies on the same axioms as *expected utility theory*, except for the IA, which is replaced by co-monotonic independence.

Money Lotteries

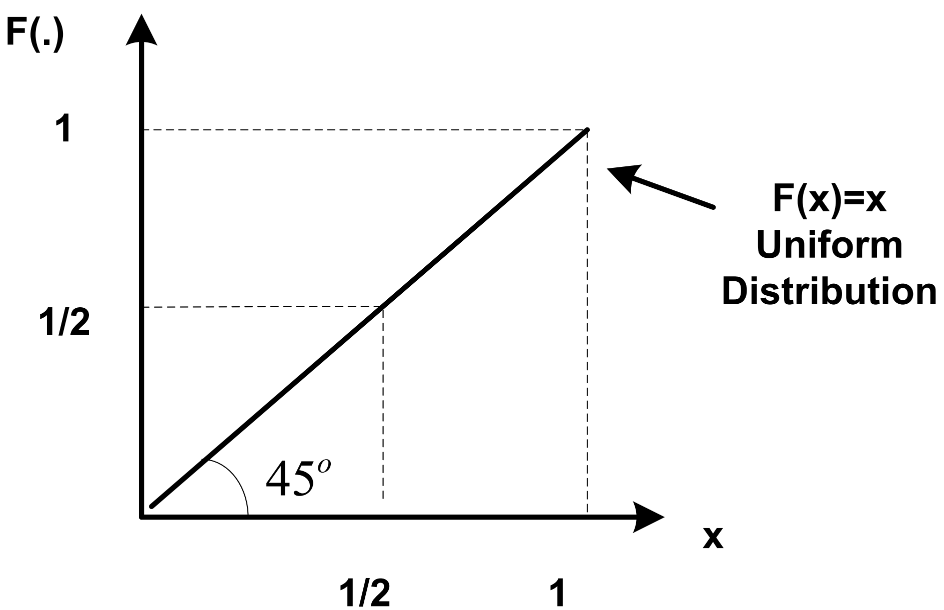
Money Lotteries

- We now restrict our attention to lotteries over *monetary* amounts, i.e., $C = \mathbb{R}$.
- Money is continuous variable, $x \in \mathbb{R}$, with cumulative distribution function (CDF)

$$F(x) = Prob\{y \leq x\} \text{ for all } y \in \mathbb{R}$$

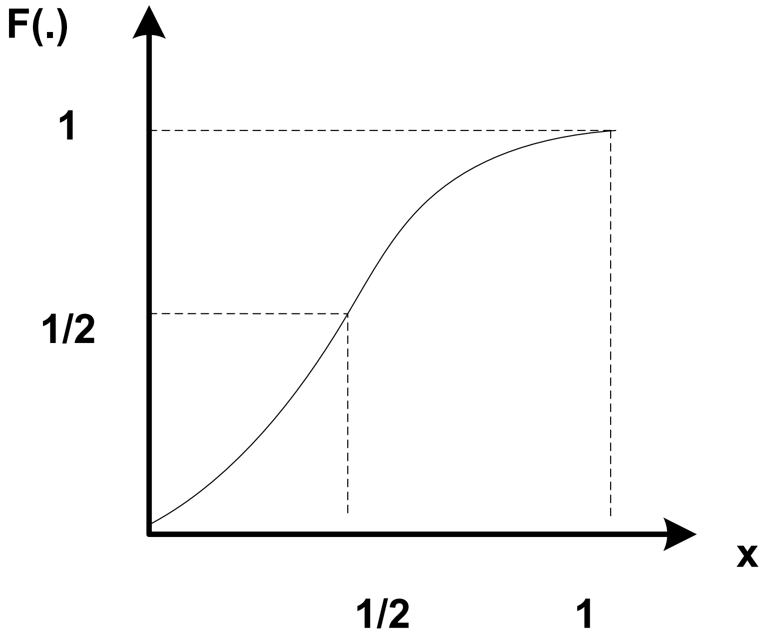
Money Lotteries

- A uniform, continuous CDF, $F(x) = x$
  - Same probability weight to every possible payoff



Money Lotteries

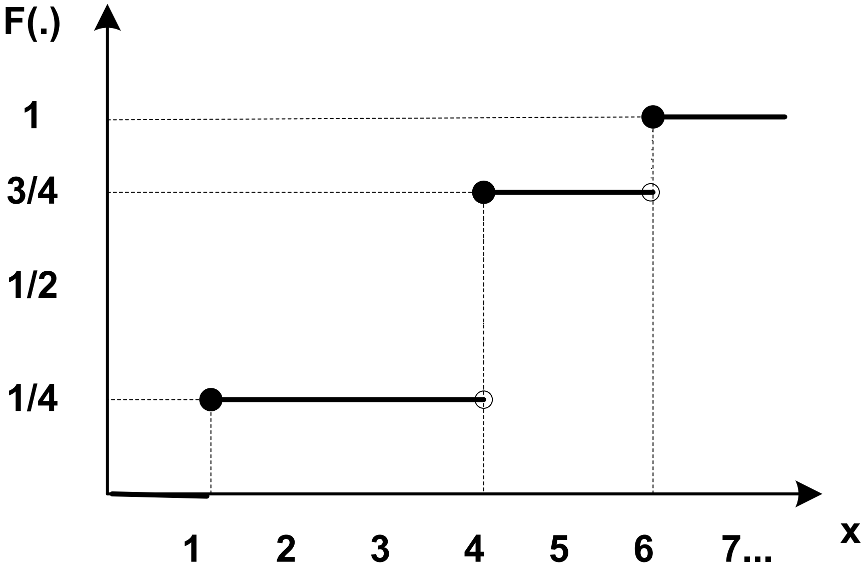
- A non-uniform, continuous CDF, $F(x)$



Money Lotteries

- A non-uniform, discrete CDF

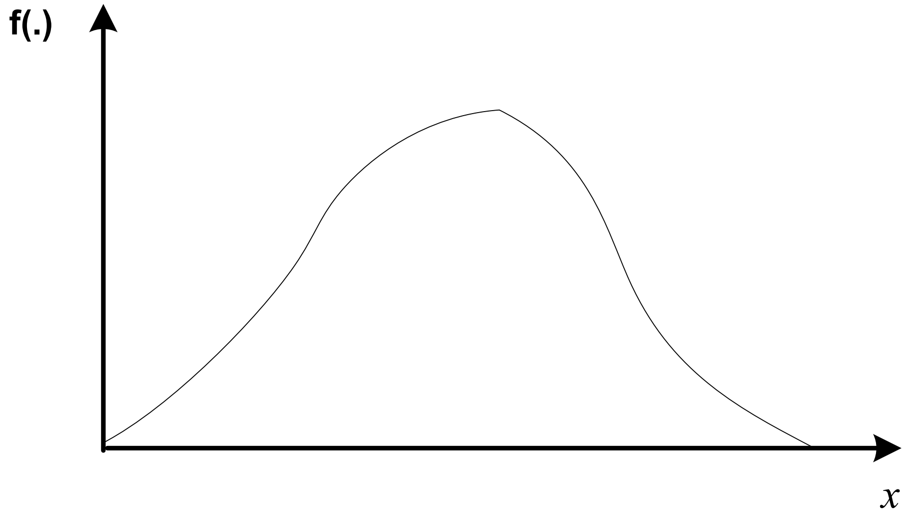
$$F(x) = \begin{cases} 0 \text{ if } x < 1 \\ \frac{1}{4} \text{ if } x \in [1, 4) \\ \frac{3}{4} \text{ if } x \in [4, 6) \\ 1 \text{ if } x \geq 6 \end{cases}$$



Money Lotteries

- If $f(x)$ is a density function associated with the *continuous* CDF $F(x)$, then

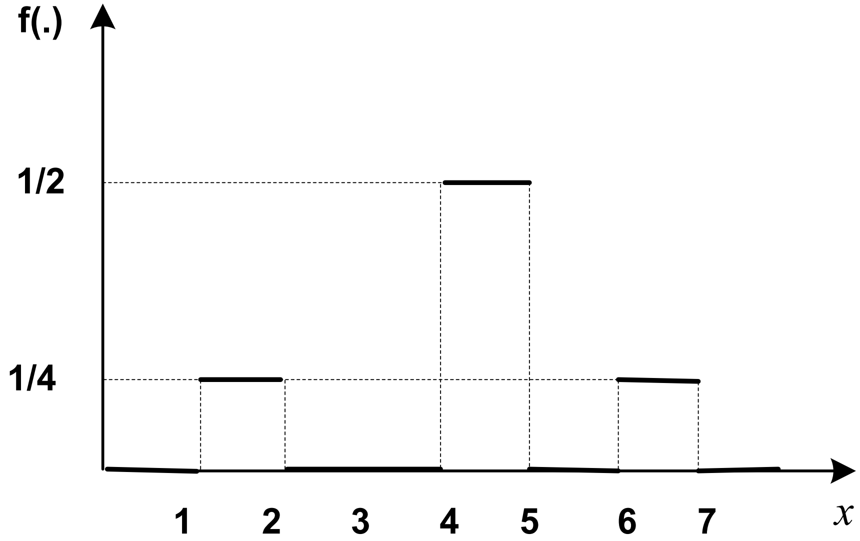
$$F(x) = \int_{-\infty}^{x} f(t)dt$$



Money Lotteries

- If $f(x)$ is a density function associated with the *discrete* CDF $F(x)$, then

$$F(x) = \sum_{t<x} f(t)$$



Money Lotteries

- We can represent *simple lotteries* by $F(x)$.
- *For compound lotteries*:
  - If the list of CDF's $F_1(x)$, $F_2(x)$, ..., $F_K(x)$ represent K simple lotteries, each occurring with probability $\alpha_1, \alpha_2, \ldots, \alpha_K$, then the compound lottery can be represented as

$$F(x) = \sum_{k=1}^{K} \alpha_k F_k(x)$$

  - For simplicity, assume that CDF's are distributed over non-negative amounts of money.

Money Lotteries

– We can express EU as

$$EU(F) = \int u(x)f(x)dx \text{ or } \int u(x)dF(x)$$

where $u(x)$ is an assignment of utility value to every non-negative amount of money.

– If there is a density function $f(x)$ associated with the CDF $F(x)$, then we can use either of the expressions. If there is no, we can only use the latter.

– *Note*: we do not need to write down the limits of integration, since the integral is over the full range of possible realizations of x.

Money Lotteries

- $EU(F)$ is the mathematical expectation of the values of $u(x)$, over all possible values of x.
- $EU(F)$ is linear in the probabilities
  - In the discrete probability distribution,

$$EU(F) = p_1(u_1) + p_2(u_2) + \cdots$$

- The EU representation is sensitive not only to the *mean* of the distribution, but also to the *variance*, and *higher order moments* of the distribution of monetary payoffs.
  - Let us next analyze this property.

Money Lotteries

- ***Example***: Let us show that if $u(x) = \beta x^2 + \gamma x$, then EU is determined by the mean and the variance alone.
  - Indeed,

$$EU(x) = \int u(x)dF(x) = \int [\beta x^2 + \gamma x]dF(x)$$

$$= \beta \underbrace{\int x^2 dF(x)}_{E(x^2)} + \gamma \underbrace{\int x\, dF(x)}_{E(x)}$$

  - On the other hand, we know that

$$Var(x) = E(x^2) - \big(E(x)\big)^2 \Longrightarrow$$

$$E(x^2) = Var(x) + \big(E(x)\big)^2$$

Money Lotteries

- ***Example*** (continued):
  - Substituting $E(x^2)$ in $EU(x)$,

$$EU(x) = \underbrace{\beta Var(x) + \beta\left(E(x)\right)^2}_{\beta E(x^2)} + \gamma E(x)$$

  - Hence, the EU is determined by the mean and the variance alone.

Money Lotteries

- Recall that we refer to $u(x)$ as the Bernoulli utility function, while $EU(x)$ is the vNM function.
- We imposed few assumptions on $u(x)$:
  – Increasing in money and continuous
- We must impose an additional assumption:
  – $u(x)$ is bounded
  – Otherwise, we can end up in relatively absurd situations (*St. Petersburg-Menger paradox*).

Money Lotteries

- ***St. Petersburg-Menger paradox***:
  - Consider an unbounded Bernoulli utility function, $u(x)$. Then, we can always find an amount of money x_m such that $u(x_m) > 2^m$, for every integer m.
  - Now consider a lottery in which we toss a coin repeatedly until tails come up. We give a monetary payoff of x_m if tails is obtained at the mth toss.
  - The probability that tails comes up in the *m*-th toss is $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \ldots (m \text{ times}) = \frac{1}{2^m}$.

Money Lotteries

– Then, the EU of this lottery is

$$EU(x) = \sum_{m=1}^{\infty} \frac{1}{2^m} u(x_m)$$

– But, because of $u(x_m) > 2^m$, we have that

$$EU(x) = \sum_{m=1}^{\infty} \frac{1}{2^m} u(x_m) \geq \sum_{m=1}^{\infty} \frac{1}{2^m} 2^m$$

$$= \sum_{m=1}^{\infty} 1 = +\infty$$

which implies that this individual would be willing to pay infinite amounts of money to be able to pay this lottery.

– Hence, we assume that the Bernouilli utility function is bounded.

Measuring Risk Preferences

Measuring Risk Preferences

- An individual exhibits ***risk aversion*** if

$$\int u(x)dF(x) \leq u\left(\int xdF(x)\right)$$

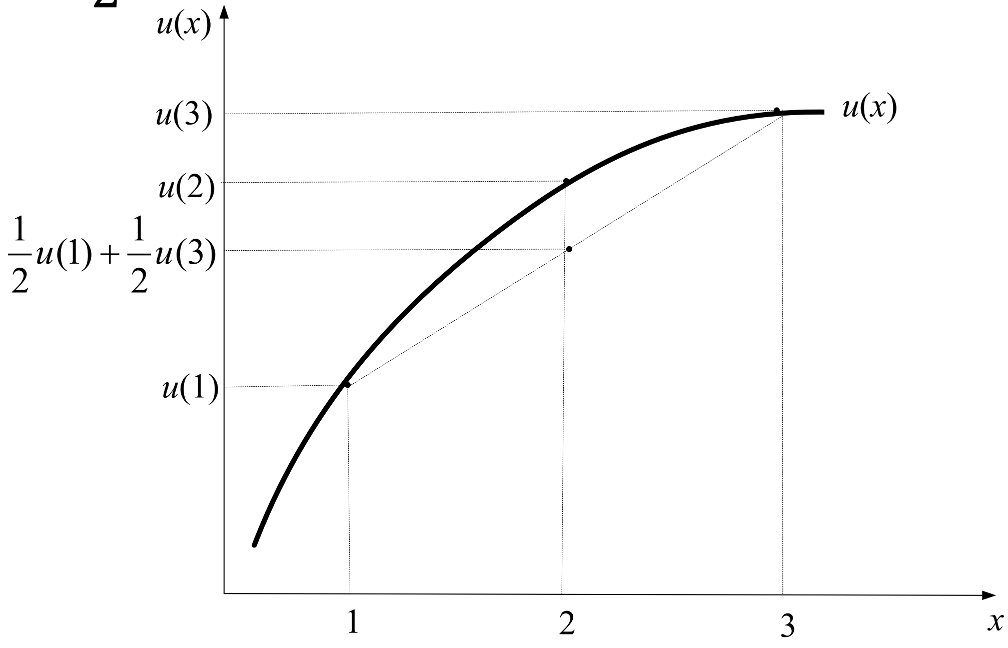
  for any lottery $F(\cdot)$
- *Intuition*:
  - The utility of receiving the expected monetary value of playing the lottery (right-hand side) is higher than…
  - The expected utility from playing the lottery (left-hand side).
- If this relationship happens with
  - *a)* $=$, we denote this individual as **risk neutral**
  - *b)* $<$, we denote him as **risk averter**
  - *c)* $\geq$, we denote him as **risk lover**.

Measuring Risk Preferences

- Graphical illustration:
  - Consider a lottery with two equally likely outcomes, \$1 and \$3, with associated utilities of $u(1)$ and $u(3)$, respectively.
  - *Expected value of the lottery* is $EV = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 3 = 2$, with associated utility of $u(2)$.
  - *Expected utility of the lottery* is $\frac{1}{2}u(1) + \frac{1}{2}u(3)$.

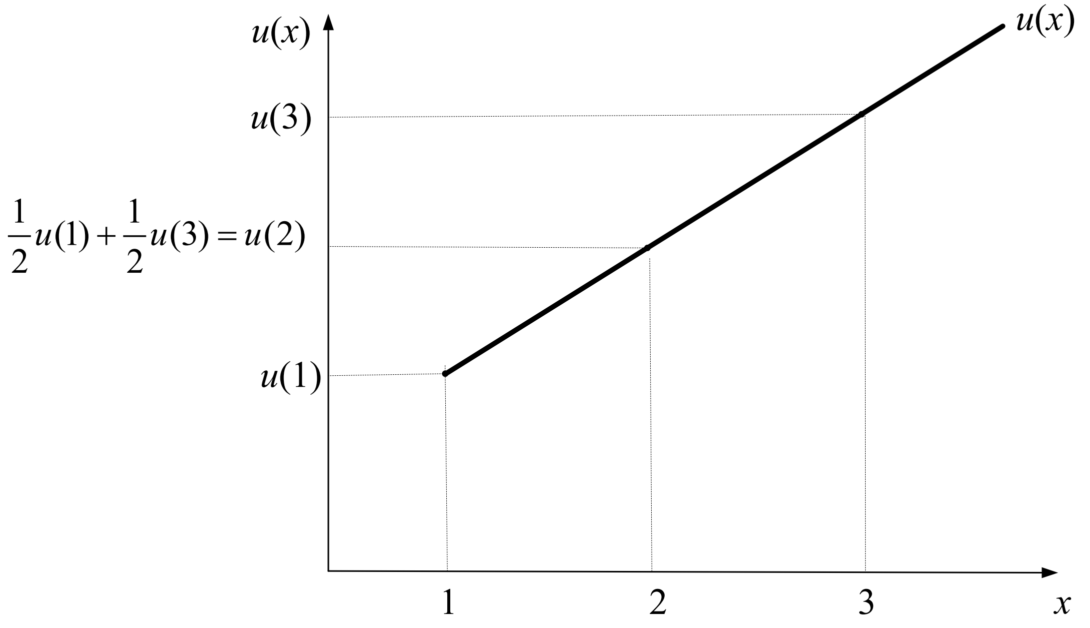
Measuring Risk Preferences

- *Risk averse individual*
  - Utility from the expected value of the lottery, $u(2)$, is **higher** than the EU from playing the lottery, $\frac{1}{2}u(1) + \frac{1}{2}u(3)$.



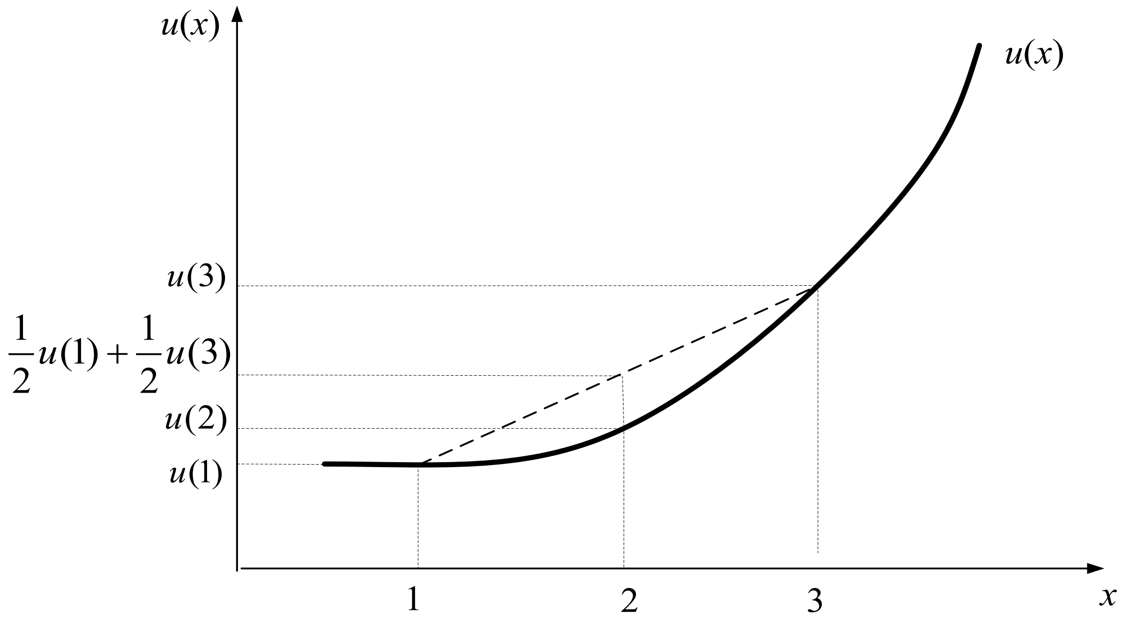
Measuring Risk Preferences

- *Risk neutral individual*
  - Utility from the expected value of the lottery, $u(2)$, **coincides** with the EU of playing the lottery, $\frac{1}{2}u(1) + \frac{1}{2}u(3)$.



Measuring Risk Preferences

- *Risk loving individual*
  – Utility from the expected value of the lottery, $u(2)$, is **lower** than the EU from playing the lottery, $\frac{1}{2}u(1) + \frac{1}{2}u(3)$.



Measuring Risk Preferences

- ***Certainty equivalent***, $c(F, u)$:
  - An alternative measure of risk aversion
  - It is the amount of money that makes the individual indifferent between playing the lottery $F(\cdot)$, and accepting a certain amount $c(F, u)$. That is,

$$u(c(F, u)) = \int u(x) dF(x) \text{ or } \sum u(x) f(x)$$

  - $c(F, u)$ is below (above) the expected value of the lottery for risk averse (lover) individuals, and exactly coincides for risk neutral individuals.

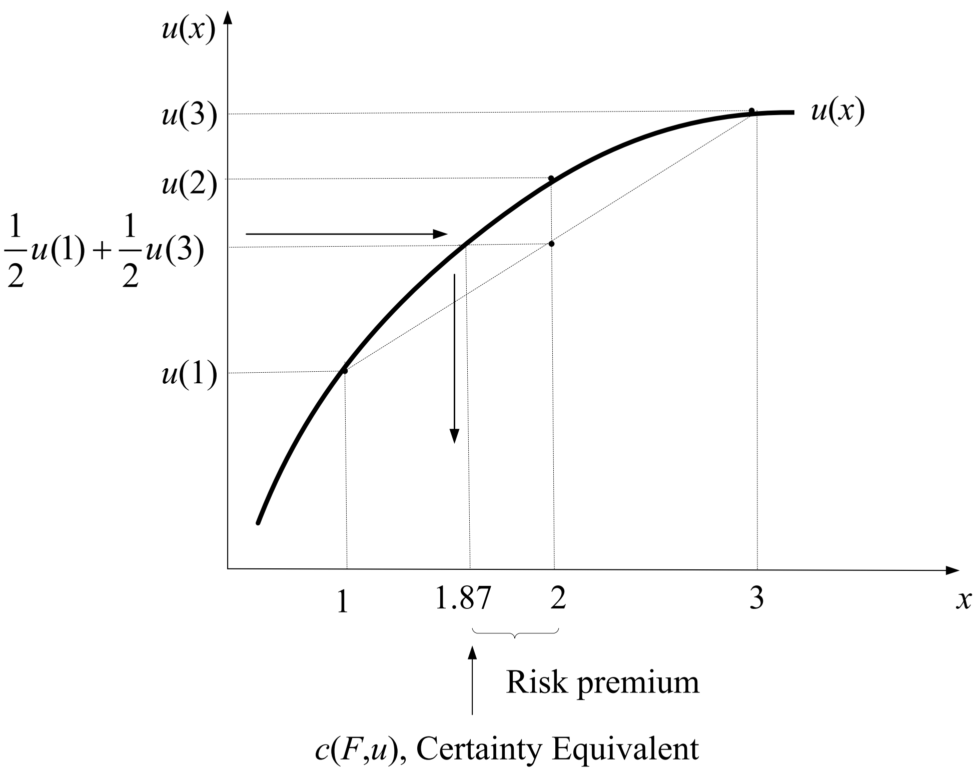
Measuring Risk Preferences

- Certainty equivalent for a risk-averse individual
  - $c(F,u)$ is the amount of money (x) for which utility is equal to the EU of the lottery

    $$u(c(F,u)) = \frac{1}{2}u(1) + \frac{1}{2}u(3)$$

  - ***Risk premium*** (RP): the amount that a risk-averse person would *pay* to avoid taking a risk:

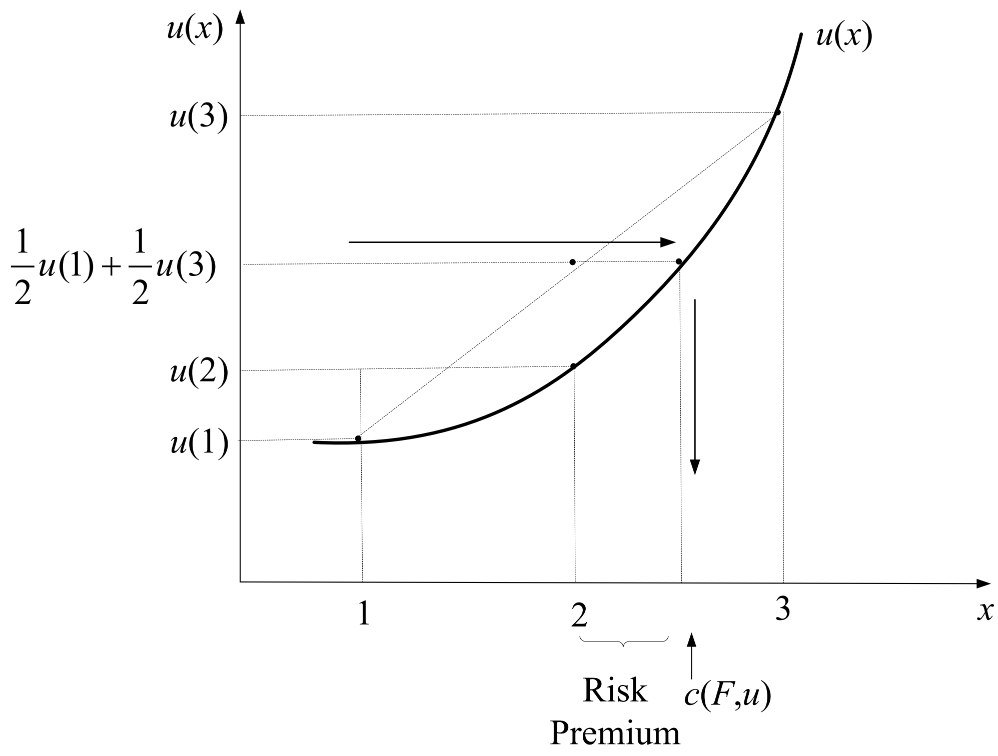
    $$RP = EV - c(F,u) > 0$$



Measuring Risk Preferences

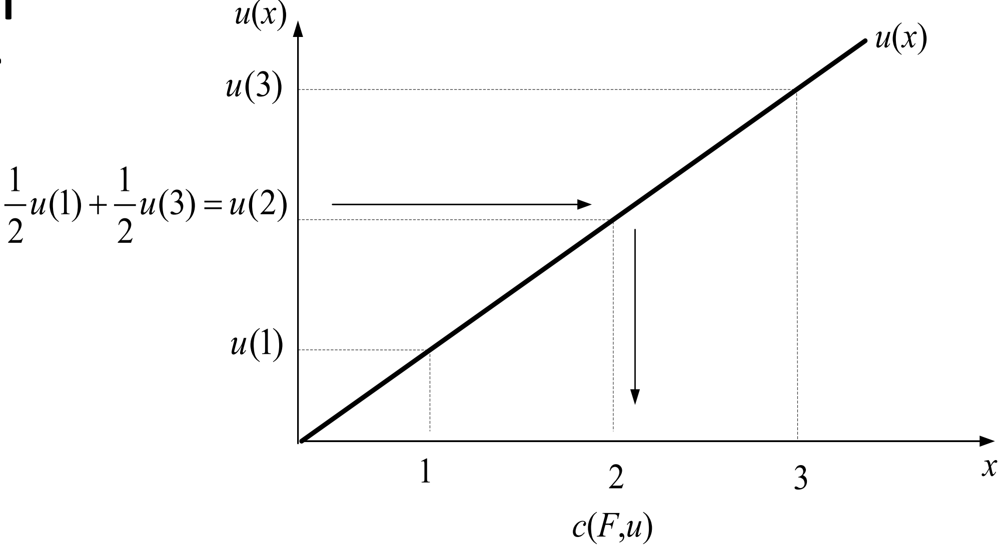
- Certainty equivalent for a risk lover
  - Individual would have to be given an amount of money *above* the expected value of the lottery in order to convince him to "stop playing" the lottery:

$$RP = EV - c(F, u) < 0$$



Measuring Risk Preferences

- Certainty equivalent for a risk neutral individual
  - The certainty equivalent $c(F, u)$ coincides with the expected value of the lottery.
  - Hence,
    $RP = EV - c(F, u) = 0$



Measuring Risk Preferences

- ***Probability premium***, $\pi(x, \varepsilon, u)$:
  - An alternative measure of risk aversion
  - It is the excess in winning probability over fair odds that makes the individual indifferent between the certainty outcome x and a gamble between the two outcomes $x + \varepsilon$ and $x - \varepsilon$:

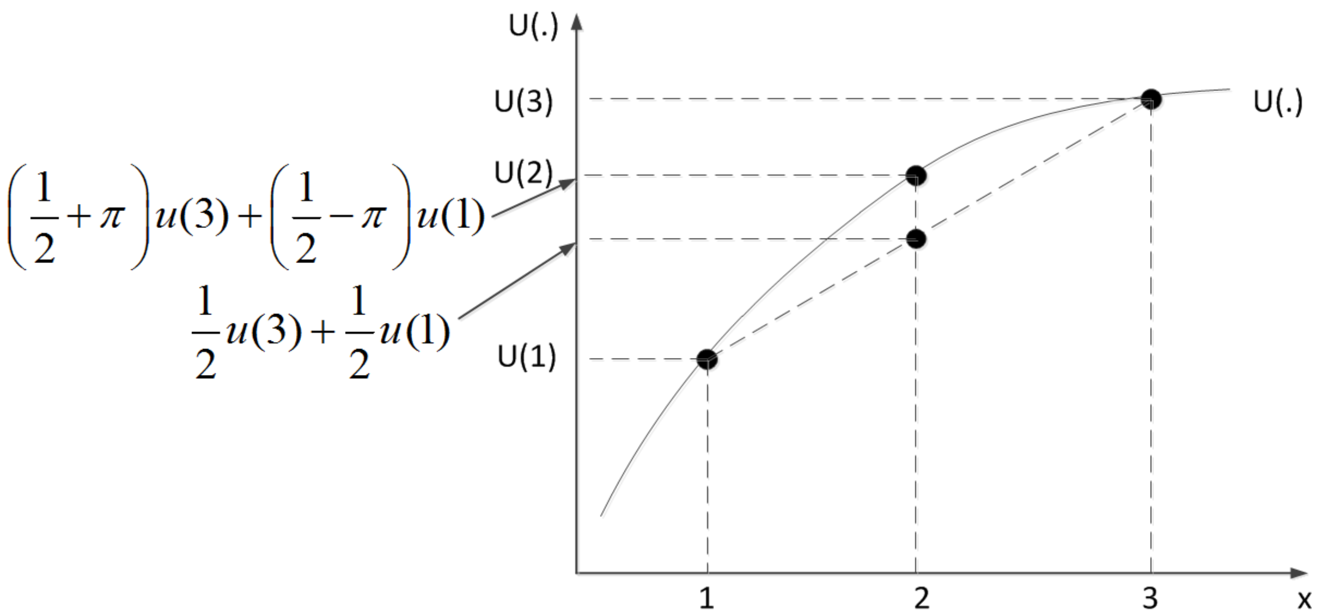
$$u(x) = \left[\frac{1}{2} + \pi(x, \varepsilon, u)\right] u(x + \varepsilon) + \left[\frac{1}{2} - \pi(x, \varepsilon, u)\right] u(x - \varepsilon)$$

  - *Intuition*: Better than fair odds must be given for the individual to accept the risk.

Measuring Risk Preferences

- The “extra probability” π that is needed to make the EU of the lottery coincides with the utility of the expected lottery:

$$u(2) = \left[\frac{1}{2} + \pi\right] u(3) + \left[\frac{1}{2} - \pi\right] u(1)$$



Measuring Risk Preferences

- The following properties are equivalent:
  1) The decision maker is risk averse.
  2) The Bernoulli utility function $u(x)$ is concave, $u''(x) \leq 0$.
  3) The certainty equivalent is lower than the expected value of the lottery, i.e., $c(F, u) \leq \int u(x) dF(x)$.
  4) The risk premium is positive, $RP = EV - c(F, u)$.
  5) The probability premium is positive for all x and ε, i.e., $\pi(x, \varepsilon, u) \geq 0$.

Measuring Risk Preferences

- ***Arrow-Pratt coefficient of absolute risk aversion***:

$$r_A(x) = -\frac{u''(x)}{u'(x)}$$

  - Clearly, the greater the curvature of the utility function, $u''(x)$, the larger the coefficient $r_A(x)$.
  - But, why do not we simply have $r_A(x) = u''(x)$?
    - Because it will not be invariant to positive linear transformations of the utility function, such as $v(x) = \beta u(x)$. That is, $v''(x) = \beta u''(x)$ is affected by the transformation, but the above coefficient of risk aversion is unaffected.

$$r_A(x) = -\frac{\beta u''(x)}{\beta u'(x)} = -\frac{u''(x)}{u'(x)}$$

Measuring Risk Preferences

- ***Example*** (CARA utility function).
  - Take $u(x) = -e^{-ax}$ where $a > 0$. Then
$$r_A(x) = -\frac{u''(x)}{u'(x)} = -\frac{-a^2 e^{-ax}}{a e^{-ax}} = a$$
    which is constant in wealth x.
  - The literature refers to this Bernoulli utility function as the *Constant Absolute Risk Aversion* (CARA).

Measuring Risk Preferences

- If $r_A(x)$ decreases as we increase wealth x, then we say that such Bernoulli utility function satisfies *decreasing absolute risk aversion* (DARA)

$$\frac{\partial r_A(x)}{\partial x} < 0$$

- *Intuition*: wealthier people are willing to bear more risk than poorer people.
  – This is NOT due to different utility functions, but because the same utility function is evaluated at higher/lower wealth levels.

Measuring Risk Preferences

- A sufficient (but not necessary) condition for DARA is $u'''(x) > 0$, that is,

$$r_A'(x) < 0 \underset{\nLeftarrow}{\Rightarrow} u'''(x) > 0$$

- For example, when $u(x) = -e^{-ax}$, its third-order derivative is $u'''(x) = a^3 e^{-ax} > 0$.

Measuring Risk Preferences

- ***Arrow-Pratt coefficient of relative risk aversion***:

$$r_R(x) = -x \cdot \frac{u''(x)}{u'(x)} \quad \text{or} \quad r_R(x) = x \cdot r_A(x)$$

  - $r_R(x)$ does not vary with the wealth level at which it is evaluated.
  - We can show that

$$\frac{\partial r_R(x)}{\partial x} = \underbrace{r_A(x)}_{+} + x \cdot \frac{\partial r_A(x)}{\partial x}$$

  - Therefore,

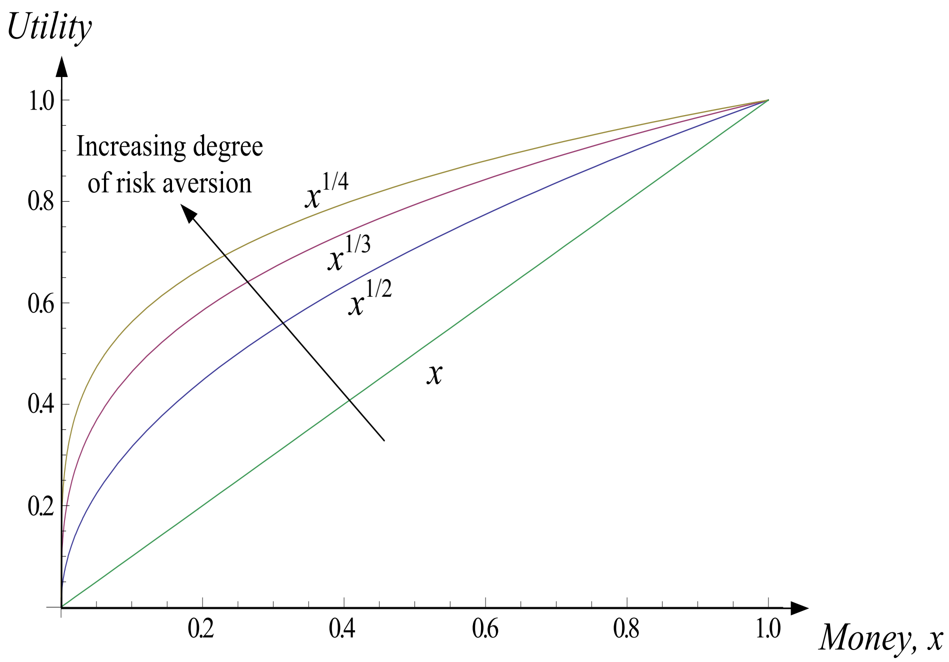
$$\frac{\partial r_R(x)}{\partial x} < 0 \ \substack{\Rightarrow \\ \not\Leftarrow} \ \frac{\partial r_A(x)}{\partial x} < 0$$

Measuring Risk Preferences

- ***Example***:
  - Take $u(x) = x^b$. Then
    $$r_R(x) = -x \cdot \frac{b(b-1)x^{b-2}}{bx^{b-1}} = 1 - b$$
    for all x.
  - The literature refers to this Bernoulli utility function as the *Constant Relative Risk Aversion* (CRRA).

Measuring Risk Preferences

- ***Example*** (continued):
  - Consider a CRRA utility function $u(x) = x^b$ for $b = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.
  - $r_R(x)$ increases, respectively, to $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, making utility function more concave.



Measuring Risk Preferences

- A utility function $u_A(\cdot)$ exhibits more *strong risk aversion* than another utility function $u_B(\cdot)$ if, there is a constant $\lambda > 0$,

$$\frac{u_A''(x_1)}{u_B''(x_1)} \geq \lambda \geq \frac{u_A'(x_2)}{u_B'(x_2)}$$

- In addition, if $x_1 = x_2$, the above condition can be re-written as

$$\frac{u_A''(x_1)}{u_A'(x_1)} \geq \frac{u_B''(x_1)}{u_B'(x_1)}$$

- Then, $u_A(\cdot)$ also exhibits more risk aversion than $u_B(\cdot)$.

Measuring Risk Preferences

- For two utility functions u_1 and u_2, where u_2 is a concave transformation of u_1, the following properties are equivalent:
  1) There exists an increasing concave function $\varphi(\cdot)$ such that $u_2(x) = \varphi(u_1(x))$ for any x. That is, $u_2(\cdot)$ is more concave than $u_1(\cdot)$.
  2) $r_A(x, u_2) \geq r_A(x, u_1)$ for any x.
  3) $c(F, u_2) \leq c(F, u_1)$ for any lottery $F(\cdot)$.
  4) $\pi(x, \varepsilon, u_2) \geq \pi(x, \varepsilon, u_1)$ for any x and ε.

Measuring Risk Preferences

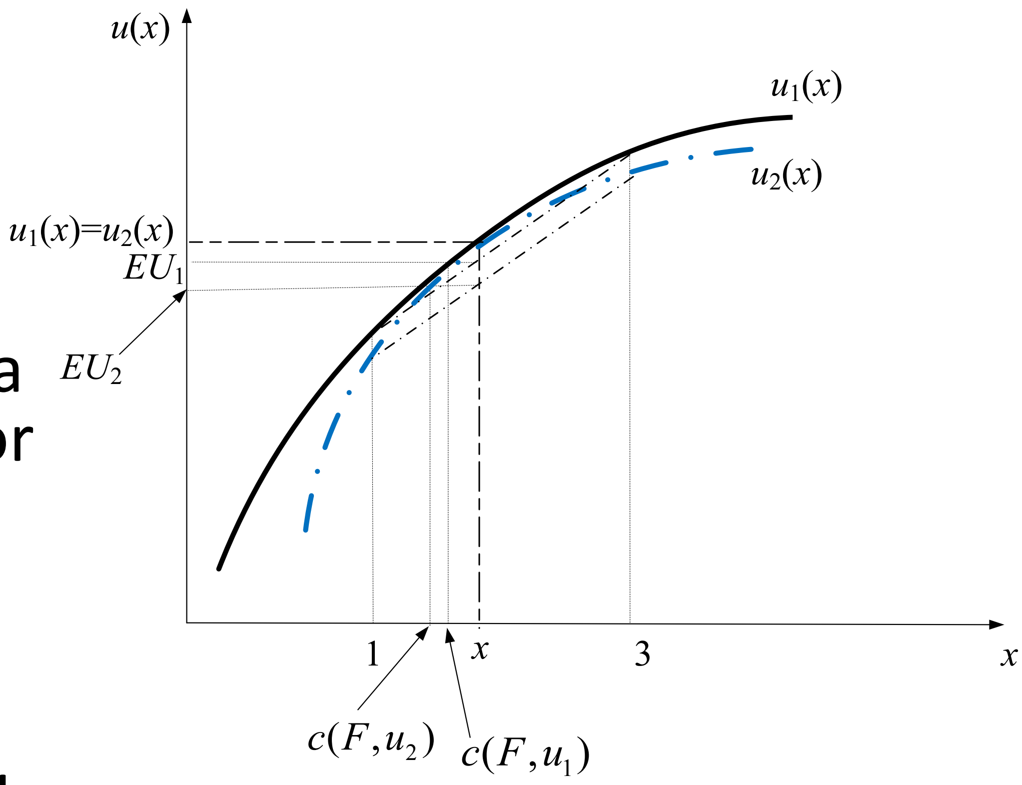
5) Whenever $u_2(\cdot)$ finds a lottery $F(\cdot)$ at least as good as a riskless outcome $\bar{x}$, then $u_1(\cdot)$ also finds such a lottery $F(\cdot)$ at least as good as $\bar{x}$. That is

$$EU_2 = \int u_2(x)dF(x) \geq u_2(\bar{x}) \Longrightarrow$$

$$EU_1 = \int u_1(x)dF(x) \geq u_1(\bar{x})$$

Measuring Risk Preferences

- Different degrees of risk aversion
- $u_1(\cdot)$ and $u_2(\cdot)$ are evaluated at the same wealth level x.
- The same lottery yields a larger expected utility for the individual with *less risk averse* preferences, $EU_1 > EU_2$.
- $c(F, u_2) < c(F, u_1)$, reflecting that individual 2 is more risk averse.



Measuring Risk Preferences

Prudence

- Introduced by Kimball (1990).
- It measures an individual's tendency to prepare for future risks.
- An individual is considered to be "prudent" if the third derivative of his utility function is positive, i.e., $u'''(x) > 0$.
- The formula for ***absolute*** *prudence* is:

$$K_A(x) = -\frac{u'''(x)}{u''(x)}$$

- The more prudent the individual becomes, the more precautionary saving will this individual prepare for.

Measuring Risk Preferences

Prudence

- The formula for ***relative*** *prudence* is given by

$$K_R(x) = -x\frac{u'''(x)}{u''(x)}$$

- It also measures the elasticity of concavity of the marginal utility function, as follows:

$$K_R(x) = x \cdot K_A(x) = -\frac{d\log u''(x)}{d\log x}$$

Measuring Risk Preferences

- The Arrow-Pratt coefficient of absolute risk aversion, $r_A(x)$, and the coefficient of relative prudence, $K_R(x)$, are closely related.
- Differentiating $r_A(x)$ yields

$$r_A'(x) = -\frac{u''' \cdot u' - (u'' \cdot u'')}{(u')^2}$$

$$= -\frac{u'''}{u'} + \left(\frac{u''}{u'}\right)^2$$

Dividing both sides by $r_A(x) = -u''/u'$ yields

$$\frac{r_A'(x)}{r_A(x)} = -\frac{u'''}{u'}\left(-\frac{u'}{u''}\right) + \left(\frac{u''}{u'}\right)^2 \left(-\frac{u'}{u''}\right)$$

$$= \frac{u'''}{u''} - \frac{u''}{u'}$$

$$= -K_A(x) + r_A(x)$$

Measuring Risk Preferences

- Hence, solving for $K_A(x)$, we obtain

$$K_A(x) = r_A(x) - \frac{r_A'(x)}{r_A(x)}$$

- Further multiplying both sides by x,

$$x \cdot K_A(x) = x \cdot r_A(x) - x \cdot \frac{r_A'(x)}{r_A(x)}$$

Since $K_R(x) = x \cdot K_A(x)$ and $r_R(x) = x \cdot r_A(x)$,

$$K_R(x) = r_R(x) - x \cdot \frac{r_A'(x)}{r_A(x)}$$

Measuring Risk Preferences

- ***Example*** (CARA utility function).
  - Take $u(x) = -e^{-ax}$ where $a > 0$.
  - Then, the relative prudence is

$$K_R(x) = x \cdot a - x \frac{0}{x \cdot a} = x \cdot a = r_R(x)$$

which coincides with relative risk aversion.

Measuring Risk Preferences

- ***Example***:
  - Take now $u(x) = x^b$.
  - Then $r_A(x) = \frac{1-b}{x}$ and $r_R(x) = 1 - b$, yielding a relative prudence of

$$K_R(x) = (1-b) - x \cdot \frac{-(1-b)/x^2}{(1-b)/x}$$
$$= 1 - b + 1$$
$$= 2 - b$$

  - implying that, in this case, $K_R(x) > r_R(x)$.

Measuring Risk Preferences

Cautiousness

- Cautiousness measures the individual's tendency to hedge against the downside risk of an investment.
- The formula for cautiousness is given by

$$C(x) = \frac{K_R(x)}{r_R(x)}$$

  and a ratio $C(x) > 1$ implies that $K_R(x) > r_R(x)$.
- *Examples*:
  - The CARA utility function yields $C(x) = 1$.
  - The CRRA utility function yields $C(x) > 1$.

Measuring Risk Preferences

Temperance

- Temperance measures the individual's tendency to reduce the total exposure to risks.
- The formula for temperance is

$$T(x) = -\frac{u''''(x)}{u'''(x)}$$

- An individual is deemed as "temperate" when the fourth derivative of his utility function is negative, i.e., $u''''(x) < 0$.

Prospect Theory and Reference-Dependent Utility

Prospect Theory

- ***Prospect theory***: a decision maker's total value from a list of possible outcomes $x = (x_1, x_2, \ldots, x_n)$ with associated probabilities $p = (p_1, p_2, \ldots, p_n)$ is

$$v(x, p) = \sum_{i=1}^{n} w(p_i) \cdot v(x_i)$$

  where
  - $w(p_i)$ is a "probability weighting function"
  - $v(x_i)$ is the "value function" the individual obtains from outcome x_i

Prospect Theory

- **Three main differences relative to standard expected utility theory**:
- First, $w(p_i) \neq p_i$:
  - if $w(p_i) > p_i$, individuals *overestimate* the likelihood of outcome x_i
  - if $w(p_i) < p_i$, individuals *underestimate* the likelihood of outcome x_i
  - if $w(p_i) = p_i$, the model coincides with standard expected utility theory.

Prospect Theory

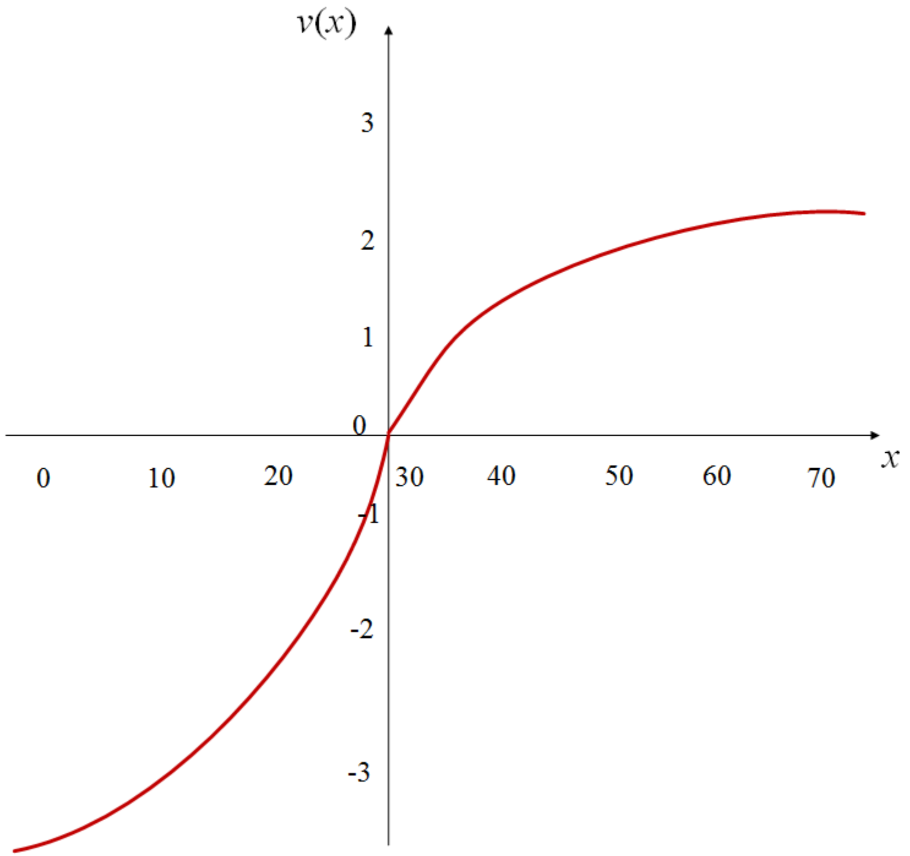
- Second, every payoff x_i is evaluated relative to a "reference point" x_0, with the value function $v(x_i)$, which is
  – Increasing and concave, $v''(x_i) < 0$, for all $x_i > x_0$,
    - That is, the individual is risk averse for gains.
  – Decreasing and convex, $v''(x_i) > 0$, for all $x_i < x_0$
    - That is, the individual is risk lover for losses.
  – *Extremes*:
    - if $x_0 = 0$, the individual is risk averse for all payoffs;
    - if $x_0 = +\infty$, he is risk lover for all payoffs.

Prospect Theory

- Third, value function $v(x_i)$ has a kink at the reference point x_0.
  - The curve becomes steeper for losses (to the left of x_0) than for gains (to the right of x_0).
    - Loss aversion:
    - A given loss of $\$a$ produces a larger disutility than a gain of the same amount.

Prospect Theory

- Value function in prospect theory



Prospect Theory

- ***Example***:
  - Consider as in Tversky and Kahneman (1992)

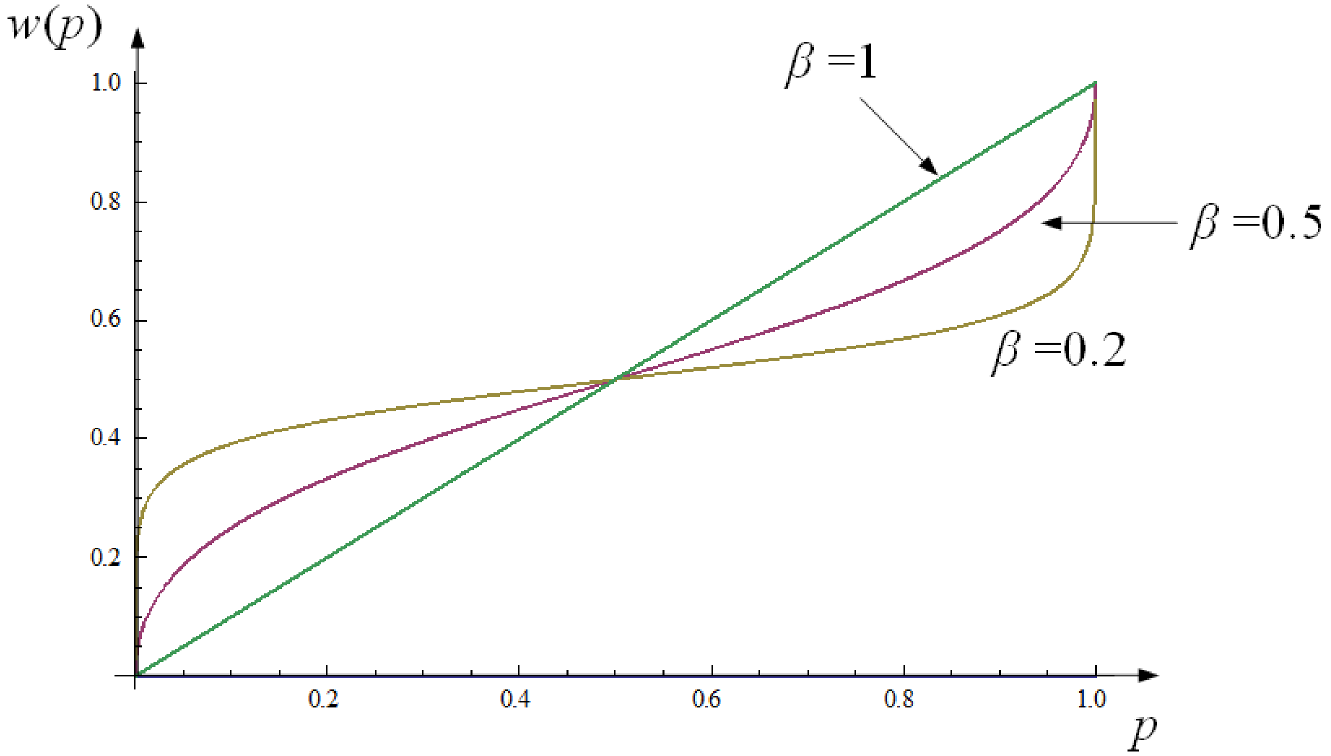
$$w(p) = \frac{p^{\beta}}{\left[p^{\beta} + (1-p)^{\beta}\right]^{\frac{1}{\beta}}} \quad \text{and} \quad v(x) = x^{\alpha}$$

  where $0 < \alpha < 1$ and $0 < \beta < 1$.

    - Note that this implies probability weighting, but does not consider a value function with loss aversion relative to a reference point.

Prospect Theory

- ***Example*** (continued):
  - Depicting the probability weighting function



Prospect Theory

- ***Example***:
  - A common value function is

$$v(x_i) = {x_i}^{\alpha} \quad \text{if } x_i \geq x_0 \text{, and}$$
$$= -\lambda(-x_i)^{\alpha} \quad \text{if } x_i < x_0$$

where $0 < \alpha \leq 1$, and $\lambda \geq 1$ represents loss aversion.

  - If $\lambda = 1$ the individual does not exhibit loss aversion.

Prospect Theory

- ***Example***:
  - Average estimates $\lambda = 2.25$ and $\beta = 0.88$
  - Common simplifications, assume $\alpha = \beta = 1$ (which implies no probability weighting, and linear value functions), to estimate λ.

Prospect Theory

- Further reading:
  - Nicholas Barberis (2013) “Thirty Years of Prospect Theory in Economics: A Review and Assessment,” *Journal of Economic Perspectives*, 27(1), pp. 173-96.
  - R. Duncan Luce and Peter C. Fishburn (1991) “Rank and sign-dependent linear utility models for binary gambles.” *Journal of Economic Theory*, 53, pp. 75–100.
  - Daniel Kahneman and Amos Tversky (1992) “Advances in prospect theory: Cumulative representation of uncertainty” *Journal of Risk and Uncertainty*, 5(4), pp. 297–323.
  - Peter Wakker and Amos Tversky (1993) “An axiomatization of cumulative prospect theory.” *Journal of Risk and Uncertainty*, 7, pp. 147–176.

Reference-Dependent Utility

- Individual preferences are affected by *reference points*. Thus, gains and losses can be evaluated differently.
- Consider a consumption vector $x \in \mathbb{R}^n$ which is evaluated against a n-dimensional reference vector $r \in \mathbb{R}^n$. Utility function is

$$u(x|r) = m(x) + n(x|r)$$

where $n(x_k|r_k) = \mu\big(m_k(x_k)\big) - m_k(r_k)$ measures the gain/loss of consuming x_k units of good k relative to its reference amount r_k.

Reference-Dependent Utility

- For lotteries with cumulative distribution function $F(x)$,

$$U(F|r) = \int u(x|r)dF(x)$$

- For lotteries over the set of reference points

$$u(F|G) = \int \int u(x|r)dG(r)dF(x)$$

Reference-Dependent Utility

- Further reading:
  - "Reference-Dependent Consumption Plans" (2009) by Koszegi and Rabin, *American Economic Review*, vol. 99(3).
  - "Rational Choice with Status Quo Bias" (2005) by Masatlioglu and Ok, *Journal of Economic Theory,* vol. 121(1).
  - "On the complexity of rationalizing behavior" (2007) Apesteguia and Ballester, *Economics Working Papers* 1048.

Comparison of Payoff Distributions

Comparison of Payoff Distributions

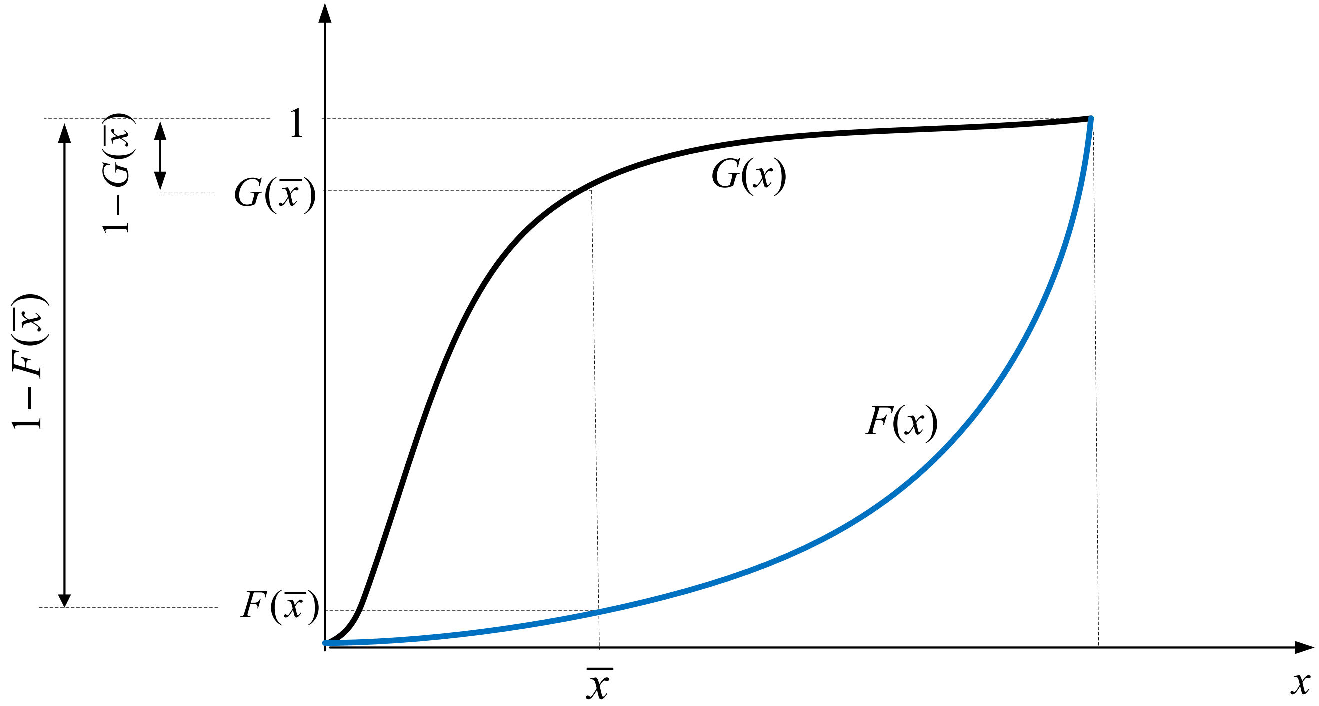
- So far we compared utility functions, but not the distribution of payoffs.
- Two main ideas:
  1) $F(\cdot)$ yields unambiguously *higher returns* than $G(\cdot)$. We will explore this idea in the definition of first order stochastic dominance (FOSD);
  2) $F(\cdot)$ is unambiguously *less risky* than $G(\cdot)$. We will explore this idea in the definition of second order stochastic dominance (SOSD).

Comparison of Payoff Distributions

- ***FOSD***: $F(\cdot)$ FOSD $G(\cdot)$ if, for every non-decreasing function $u: \mathbb{R} \to \mathbb{R}$, we have
$$\int u(x)dF(x) \geq \int u(x)dG(x)$$
- The distribution of monetary payoffs $F(\cdot)$ FOSD the distribution of monetary payoffs $G(\cdot)$ if and only if
$$F(x) \leq G(x) \text{ or } 1 - F(x) \geq 1 - G(x)$$
for every x.
- *Intuition*: For every amount of money x, the probability of getting at least x is higher under $F(\cdot)$ than under $G(\cdot)$.

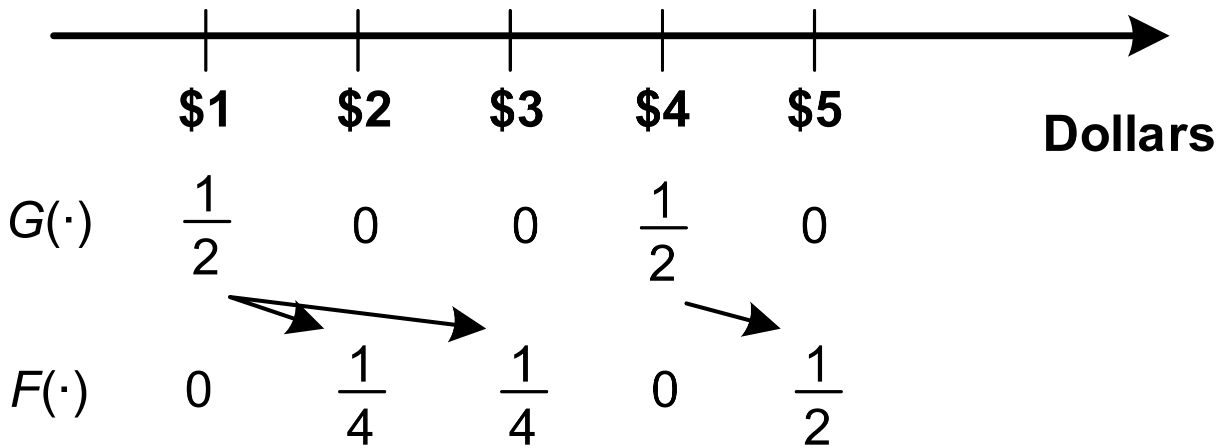
Comparison of Payoff Distributions

- At any given outcome x, the probability of obtaining prizes above x is higher with lottery $F(\cdot)$ than with lottery $G(\cdot)$, i.e., $1 - F(x) \geq 1 - G(x)$.



Comparison of Payoff Distributions

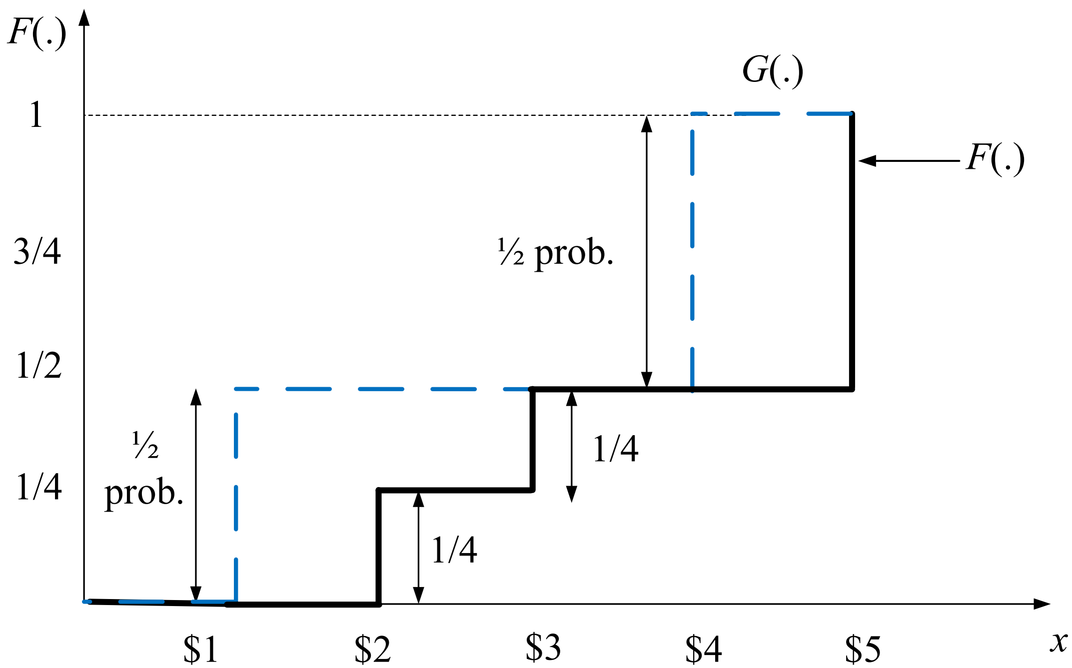
- ***Example***:
  - Let us take lotteries $F(\cdot)$ and $G(\cdot)$ over discrete outcomes.



How can we know if $F(\cdot)$ FOSD $G(\cdot)$?

Comparison of Payoff Distributions

- ***Example*** (continued):
  - $F(\cdot)$ lies below lottery $G(\cdot)$. Hence, $F(\cdot)$ concentrates more probability weight on higher monetary outcomes.
  - Thus, $F(\cdot)$ FOSD $G(\cdot)$.

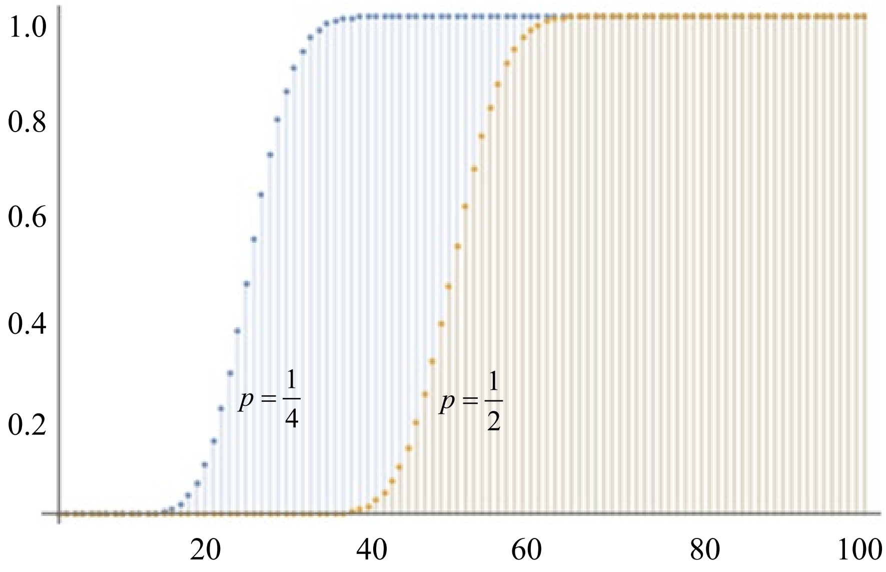


Comparison of Payoff Distributions

- ***Example*** (Binomial distribution):
  - Consider the binomial distribution

$$F(x; N, p) = \binom{N}{p} p^x (1-p)^{N-x}$$

  - where $x \in [0, N]$. Assuming $N = 100$ and parameter p increasing from $p = \frac{1}{4}$ to $p = \frac{1}{2}$. Then, $F(x; 100,1/2)$ FOSD $F(x; 100,1/4)$.



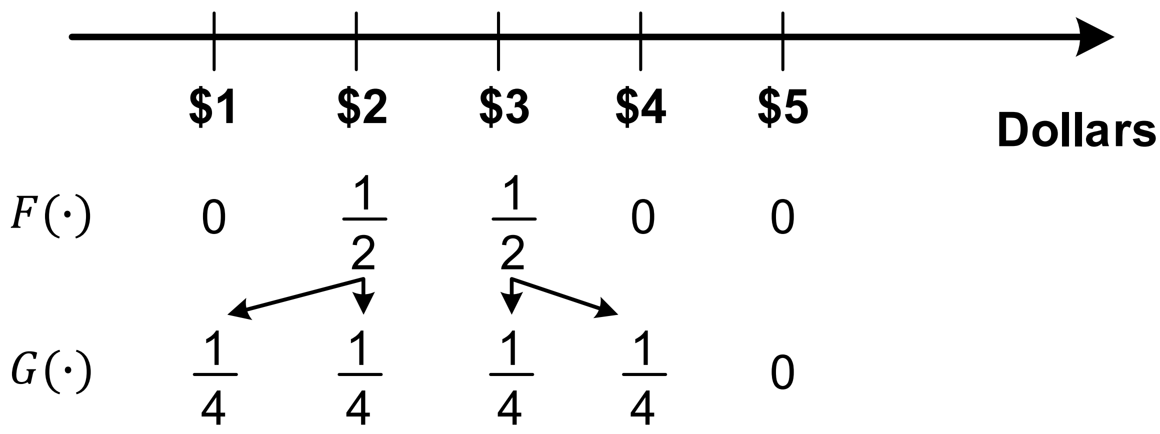
Comparison of Payoff Distributions

- We now focus on the *riskiness* or *dispersion* of a lottery, as opposed to higher/lower returns of lottery (FOSD).
- To focus on riskiness, we assume that the CDFs we compare have the *same mean* (i.e., same expected return).
- ***SOSD***: $F(\cdot)$ SOSD $G(\cdot)$ if, for every non-decreasing function $u: \mathbb{R} \to \mathbb{R}$, we have

$$\int u(x)dF(x) \geq \int u(x)dG(x)$$

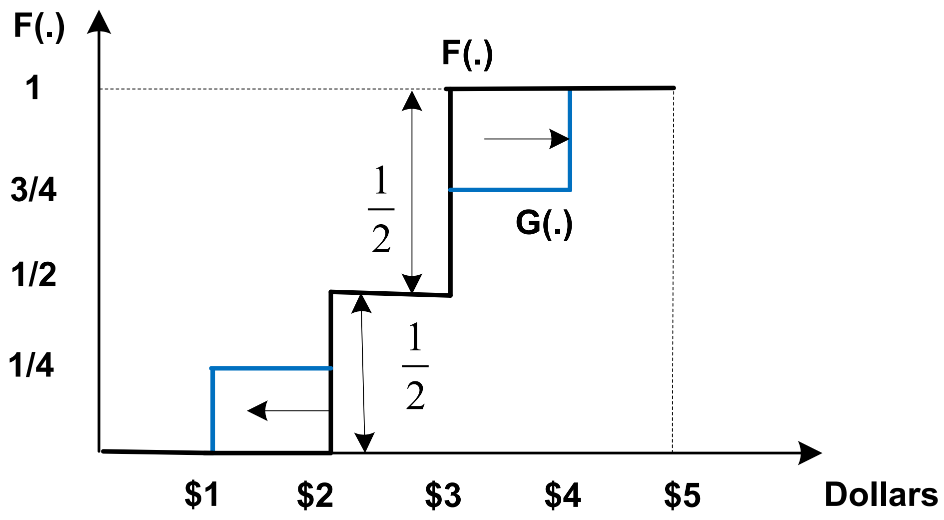
Comparison of Payoff Distributions

- ***Example*** (Mean-Preserving Spread):
  - Let us take lotteries $F(\cdot)$ and $G(\cdot)$ over discrete outcomes.
  - Lottery $G(\cdot)$ spreads the probability weight of lottery $F(\cdot)$ over a larger set of monetary outcomes.
  - The mean is nonetheless unaltered (2.5).
  - For these two reasons, we say that a CDF is a mean preserving spread of the other.



Comparison of Payoff Distributions

- $G(\cdot)$ is a mean-preserving spread of $F(\cdot)$, but it is riskier than $F(\cdot)$ in the SOSD sense.
- Note that neither FOSD the other
  - $F(\cdot)$ is not above/below $G(\cdot)$ for all x



Comparison of Payoff Distributions

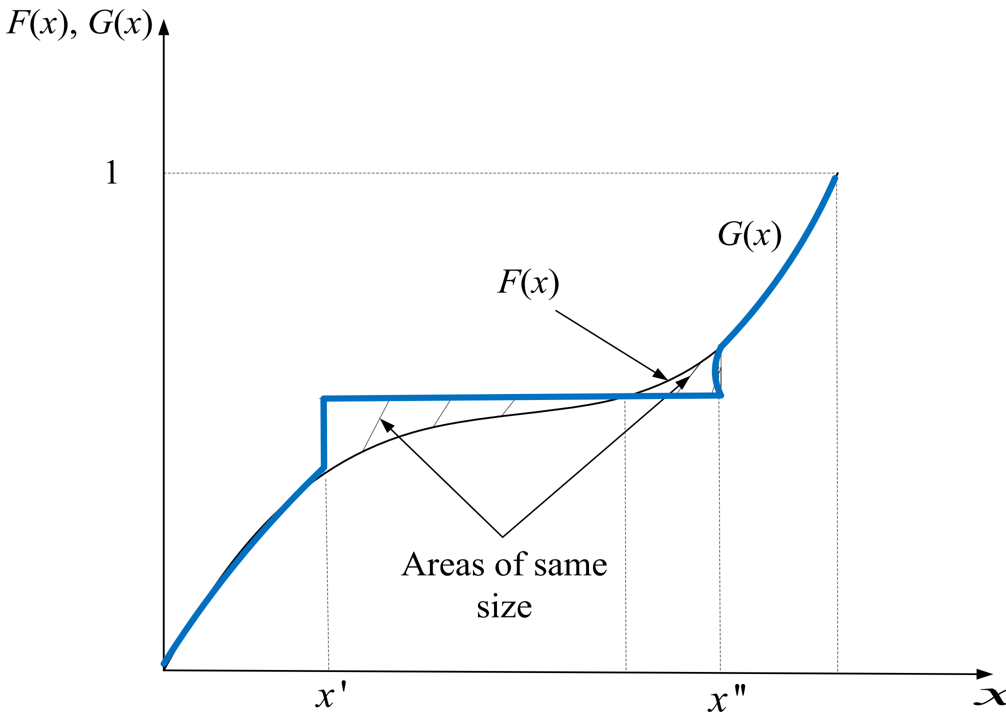
- ***Example*** (Elementary increase in risk):
  - $G(\cdot)$ is an *Elementary Increase in Risk* (EIR) of another CDF $F(\cdot)$ if $G(\cdot)$ takes all the probability weight of an interval $[x', x'']$ and transfers it to the *end points* of this interval, x' and x'', such that the mean of the original lottery is preserved.
  - EIR is a mean-preserving spread (MPS), but the converse is not necessarily true:

$$EIR \underset{\nLeftarrow}{\overset{\Rightarrow}{}} MPS$$

  - Hence, if $G(\cdot)$ is an EIR of $F(\cdot)$, then $F(\cdot)$ SOSD $G(\cdot)$.

Comparison of Payoff Distributions

- ***Example*** (continued):
  - both CDFs $F(\cdot)$ and $G(\cdot)$ maintain the same mean.
  - $G(\cdot)$ concentrates more probability at the end points of the interval $[x', x'']$ than $F(\cdot)$.



Comparison of Payoff Distributions

- ***Hazard rate dominance***: The hazard rate of lottery $F(x)$ is

$$HR_F(x) = \frac{f(x)}{1 - F(x)}$$

  - *Intuition*: It measures the instantaneous probability of an event happening at time x given that it did not happen before x.
  - *Example*: a computer stops working at exactly x
  - If $HR_F(x) \leq HR_G(x)$, lottery $F(x)$ dominates $G(x)$ in terms of the hazard rate.

Comparison of Payoff Distributions

– Since $-HR_F(x)$ can be expressed as

$$-HR_F(x) = \frac{d}{dx}\ln(1 - F(x))$$

– Solving for $F(x)$,

$$F(x) = 1 - \exp\left(-\int_0^x HR_F(t)dt\right)$$

– Then,

$$F(x) = 1 - \exp\left(-\int_0^x HR_F(t)dt\right)$$
$$\leq 1 - \exp\left(-\int_0^x HR_F(t)dt\right) = G(x)$$

– Thus, $HR_F(x) \leq HR_G(x)$ implies that $F(x)$ FOSD $G(x)$.

Comparison of Payoff Distributions

- ***Reverse hazard rate***: The reverse hazard rate of lottery $F(x)$ is

$$RHR_F(x) = \frac{f(x)}{F(x)}$$

  - *Intuition*: It measures the probability that, conditional on the realized payoff in the lottery being equal or lower than x, the payoff you receive is exactly x.
  - If $RHR_F(x) \geq RHR_G(x)$, lottery $F(x)$ dominates $G(x)$ in terms of the reverse hazard sense.

Comparison of Payoff Distributions

- Integrating both sides, we obtain

$$\int_x^\infty RHR(t)dt = \int_x^\infty \frac{d}{dt}\ln(F(t))\,dt$$

$$= \ln F(\infty) - \ln F(x) = -\ln F(x).$$

- where the last steps use $F(\infty) = 1$ and $\ln(1) = 0$. Solving for $F(x)$,we have

$$F(x) = \exp\left(-\int_x^\infty RHR(t)dt\right)$$

- Therefore, if $RHR_F(x) \geq RHR_G(x)$, then

$$F(x) = \exp\left(-\int_x^\infty RHR_F(t)dt\right) \leq \exp\left(-\int_x^\infty RHR_G(t)dt\right) = G(x)$$

- which simplifies to $F(x) \leq G(x)$.
- That is, RHR dominance implies FOSD dominance; but the converse is not necessarily true.

Comparison of Payoff Distributions

- ***Likelihood ratio***: The likelihood ratio of a lottery $F(x)$ is

$$LR_F = \frac{f(y)}{f(x)}$$

  for any two payoffs x and y, where $y > x$.
  - $F(x)$ dominates $G(x)$ in terms of likelihood ratio if

$$\frac{f(x)}{g(x)} \leq \frac{f(y)}{g(y)}$$

Comparison of Payoff Distributions

- LR dominance implies HR dominance:
  - Let us rewrite LR dominance as
$$\frac{g(y)}{g(x)} \leq \frac{f(y)}{f(x)}$$
  - Then, for all x,
$$\int_x^\infty \frac{g(y)}{g(x)} dy \leq \int_x^\infty \frac{f(y)}{f(x)} dy$$
  - Simplifying
$$\frac{1-G(x)}{g(x)} \leq \frac{1-F(x)}{f(x)} \text{ or } \frac{f(x)}{1-F(x)} \leq \frac{g(x)}{1-G(x)}$$
which implies $HR_F(x) \leq HR_G(x)$.

Comparison of Payoff Distributions

- Summary:
  - LR dominance implies HR dominance
  - HR and RHR dominance imply FOSD.

Appendix 5.1:
State-Dependent Utility

State-Dependent Utility

- So far the decision maker only cared about the payoff arising from every outcome of the lottery.
- Now we assume that the decision maker cares not only about his monetary outcomes, but also about the ***state of nature*** that causes every outcome.
  - That is, $u_{\text{state 1}}(x) \neq u_{\text{state 2}}(x)$ for given x.

State-Dependent Utility

- Let us assume that each of the possible monetary payoffs in a lottery is generated by an underlying cause (i.e., an underlying state of nature).
- ***Examples***:
  - The monetary payoff of an insurance policy is generated by a car accident
    - State of nature = {car accident, no car accident}
  - The monetary payoff of a corporate stock is generated by the state of the economy
    - State of nature = {economic growth, economic depression}

State-Dependent Utility

- Generally, let $s \in S$ denote a state of nature, where S is a finite set.
- Every state s has a well-defined, objective probability $\pi_s \geq 0$.
- A random variable is function $g: S \to \mathbb{R}$, that maps states into monetary payoffs.

State-Dependent Utility

- ***Examples*** (revisited):
  - *Car accident*: the random variable assigns a monetary value to the state of nature car accident, and to the state of nature no accident.

State of nature	Probability	Monetary payoff
Car accident	π_{accident}	Damage + Deductible – Premium = $1,000
No car accident	$\pi_{\text{no accident}}$	Premium = -$50

State-Dependent Utility

- ***Examples*** (revisited):
  - *Corporate stock*: the random variable assigns a monetary value to the state of nature economic growth, and to the state of nature economic depression.

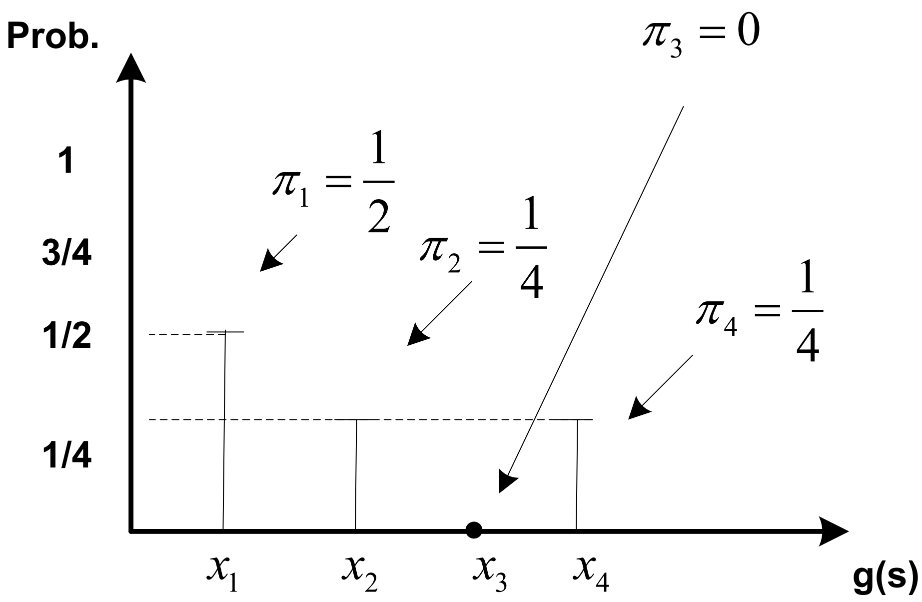
State of nature	Probability	Monetary payoff
Economic growth	π_{growth}	Dividends, higher price of shares = $250
Economic depression	$\pi_{\mathrm{depression}}$	No dividends, loss if we sell shares = -$125

State-Dependent Utility

- Every random variable $g(\cdot)$ can be used to represent lottery $F(\cdot)$ over monetary payoffs as
$$F(x) = \sum_{\{s:\, g(s) \leq x\}} \pi_s$$
where $\{s:\ g(s) \leq x\}$ represents all those states of nature s that generate a monetary payoff $g(s) \in \mathbb{R}$ below a cutoff payoff x.
- The random variable $g(\cdot)$ generates a monetary payoff for every state of nature $s \in S$, and since S is finite, we can represent this list of monetary payoffs as
$$(x_1, x_2, \dots, x_S) \in \mathbb{R}_+^S$$
where x_s is the monetary payoff corresponding to state of nature s.

State-Dependent Utility

- ***Example***:
  - A random variable $g(\cdot)$ describes the monetary outcome associated to the four states of nature $S = \{1,2,3,4\}$.
  - Outcomes are ordered from low to high, i.e., $x_1 \leq x_2 \leq x_3 \leq x_4$.



State-Dependent Utility

- ***Example*** (continued):
  - Hence,

$$F(x_1) = \pi_1 = \frac{1}{2}$$
$$F(x_2) = \pi_1 + \pi_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$
$$F(x_3) = \pi_1 + \pi_2 + \pi_3 = \frac{1}{2} + \frac{1}{4} + 0 = \frac{3}{4}$$
$$F(x_4) = \pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$$

- Disadvantage of $F(x)$:
  - For a given x, we cannot keep track of which state(s) of nature generated x.

State-Dependent Utility: Extended EU representation

- We now have a preference relation $\succsim$ ranks lists of monetary payoffs $(x_1, x_2, \dots, x_S) \in \mathbb{R}_+^S$.
- Note the similarity of this setting with that in consumer theory:
  - Preferences over bundles then, preferences over lists of monetary payoffs here.
  - Since $(x_1, x_2, \dots, x_S) \in \mathbb{R}_+^S$ specifies one payoff for each state of nature, this list is referred to as *contingent commodities*.

State-Dependent Utility: Extended EU representation

- Preference relation $\succsim$ has an ***Extended EU representation*** if for every $s \in S$, there is a function $u_S: \mathbb{R}_+ \to \mathbb{R}$ (mapping the monetary outcome of state s, x_s, into a utility value in $\mathbb{R}$), such that for any two lists of monetary outcomes $(x_1, x_2, \ldots, x_S) \in \mathbb{R}_+^S$ and $(x_1', x_2', \ldots, x_S') \in \mathbb{R}_+^S$,

$$(x_1, x_2, \ldots, x_S) \succsim (x_1', x_2', \ldots, x_S') \text{ iff}$$

$$\sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(x_s')$$

- The main difference with the previous sections is that now the Bernoulli utility function is *state-dependent*, $u_s(\cdot)$, whereas in the previous sections it was *state-independent*, $u(\cdot)$.

State-Dependent Utility: Extended EU representation

- Graphical representation:
  - First, at the "certainty line" the decision maker receives the same monetary amount, regardless the state of nature, $x_1 = x_2$.
  - Second, all the (x_1, x_2) pairs on a given ind. curve satisfy $\pi_1 \cdot u_1(x_1) + \pi_2 \cdot u_2(x_2) = \bar{U}$
  - Third, the upper contour set of an ind. curve that passes through point $(\bar{x}_1, \bar{x}_2)$ satisfy

$$\pi_1 \cdot u_1(x_1) + \pi_2 \cdot u_2(x_2)$$
$$\geq \pi_1 \cdot u_1(\bar{x}_1) + \pi_2 \cdot u_2(\bar{x}_2)$$

or, more generally, $\sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(\bar{x}_s)$.

State-Dependent Utility: Extended EU representation

- Graphical representation:
  - Fourth, movement along a given ind. curve does not change the decision maker's utility level. Hence, totally differentiating

$$\pi_1 \cdot \frac{\partial u_1(\bar{x}_1)}{\partial x_1} dx_1 + \pi_2 \cdot \frac{\partial u_2(\bar{x}_2)}{\partial x_2} dx_2 = 0$$

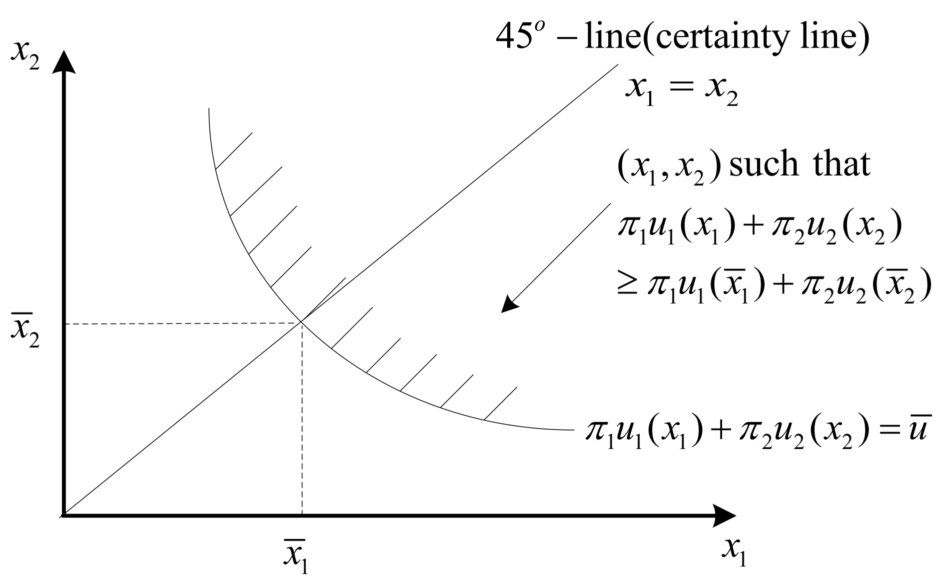
    and re-arranging,

$$\frac{dx_2}{dx_1} = -\frac{\pi_1 \cdot \frac{\partial u_1(\bar{x}_1)}{\partial x_1}}{\pi_2 \cdot \frac{\partial u_2(\bar{x}_2)}{\partial x_2}} = -\frac{\pi_1 \cdot u_1'(\bar{x}_1)}{\pi_2 \cdot u_2'(\bar{x}_2)}$$

    which represents the slope of the ind. curve, evaluated at point $(\bar{x}_1, \bar{x}_2)$. This is really similar to MRS.

State-Dependent Utility: Extended EU representation

- Graphical representation:
  - The slope of the ind. curve at $(\bar{x}_1, \bar{x}_2)$ is
$$\frac{dx_2}{dx_1} = -\frac{\pi_1 \cdot u_1'(\bar{x}_1)}{\pi_2 \cdot u_2'(\bar{x}_2)}$$
  - If the Bernoulli utility is state-independent, i.e., $u_1(\cdot) = u_2(\cdot) = \cdots = u_S(\cdot)$, then the slope is
$$\frac{dx_2}{dx_1} = -\frac{\pi_1}{\pi_2}$$

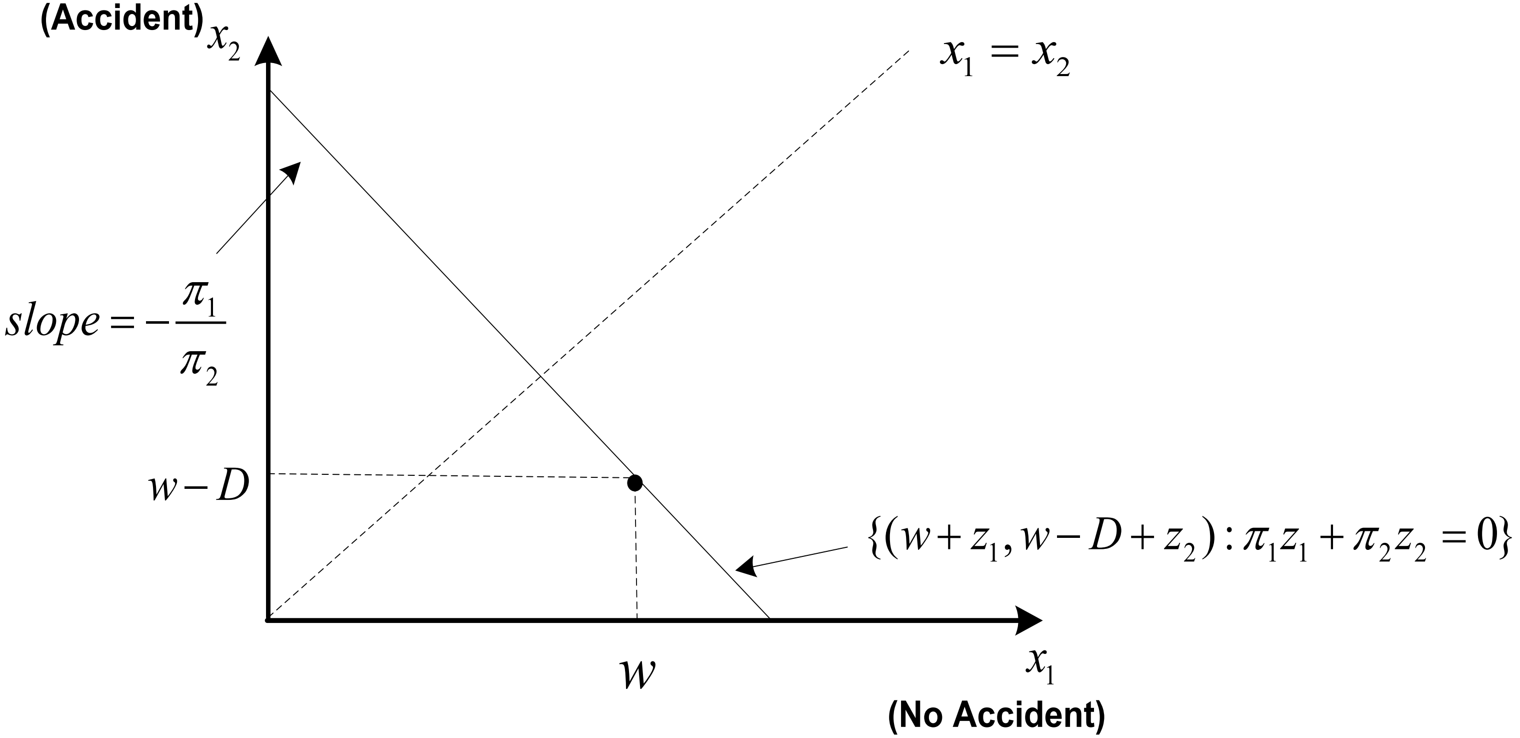


State-Dependent Utility: Extended EU representation

- ***Example*** (Insurance with state-dependent utility):
  - Start from an initial situation of $(w, w - D)$ without insurance, where D is loss from accident.
  - After insurance is purchased, the decision maker gets a payment of z_1 in state 1, and z_2 in state 2, where $z_1 \lesseqgtr 0$ and $z_2 \lesseqgtr 0$,
  $$(w + z_1, w - D + z_2)$$
  - Moreover, if the policy is actuarially fair, then its expected payoff is zero,
  $$\pi_1 z_1 + \pi_2 z_2 = 0$$

State-Dependent Utility: Extended EU representation

- ***Example*** (continued):
  - The budget line is $z_2 = -\frac{\pi_1}{\pi_2} z_1$



State-Dependent Utility: Extended EU representation

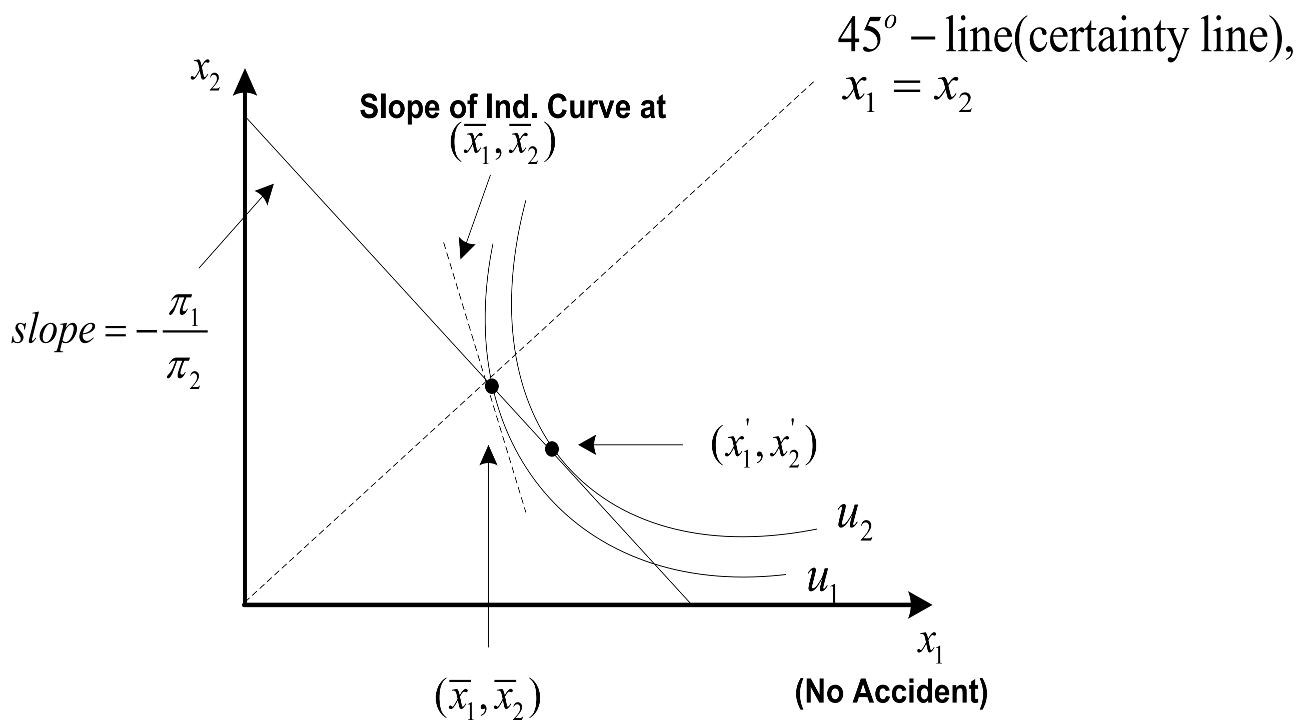
- Without state dependency:
  - Indifference curves are tangent to the budget line at the certainty line, since the slope of the indifference curve is $-\frac{\pi_1}{\pi_2}$.
  - Hence, the decision maker would insure completely since his consumption level is unaffected by the possibility of suffering an accident.

State-Dependent Utility: Extended EU representation

- With state dependency:
  - Indifference curves are NOT tangent to the budget line at the certainty line.
- ***Example*** (continued):
  - The decision-maker prefers a point such as (x_1', x_2') to the certain outcome $(\bar{x}, \bar{x})$.
  - That is, at $(\bar{x}, \bar{x})$ he prefers higher payoffs in state 1 than in state 2 if $u_1'(\bar{x}) > u_2'(\bar{x})$. Otherwise, he would prefer higher payoffs in state 2 than in state 1.

State-Dependent Utility: Extended EU representation

– Note that $u_1'(\bar{x}) > u_2'(\bar{x})$ implies that $\frac{u_1'(\bar{x})}{u_2'(\bar{x})} > 1$ and $-\frac{\pi_1 \cdot u_1'(\bar{x})}{\pi_2 \cdot u_2'(\bar{x})} < -\frac{\pi_1}{\pi_2}$.



State-Dependent Utility: Extended EU representation

- Let us now allow for the possibility that the monetary payoff under state s, x_s, is not a certain amount of money, but a random amount with distribution function $F_s(\cdot)$.
- Hence, all monetary outcomes arising from the S states of world can be described as a lottery $L = (F_1, F_2, \ldots, F_S)$.
- Given this "extended" definition of lotteries, we can then re-write the IA, as the "extended" IA.

State-Dependent Utility: Extended EU representation

- ***Extended IA***: The preference relation satisfies the extended IA if, for any three lotteries L, L', and L'' and $\alpha \in (0,1)$, we have that

$$L \succsim L' \text{ iff}$$
$$\alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$$

- Hence, "extended" IA is a mere extension of the standard IA to the case of "extended" lotteries $L = (F_1, F_2, \ldots, F_S)$.

State-Dependent Utility: Extended EU representation

- ***Extended EU theorem***: Suppose preferences relation satisfies continuity and the extended IA. Then we can assign a utility function $u_s(\cdot)$ for money in every state s such that for any two lotteries $L = (F_1, F_2, \dots, F_S)$ and $L' = (F_1', F_2', \dots, F_S')$ we have

$$L \gtrsim L' \text{ iff}$$

$$\sum_s \left(\int u_s(x_s) dF_s(x_s) \right) \geq \sum_s \left(\int u_s(x_s) dF_s'(x_s) \right)$$

Appendix 5.2:
Subjective Probability Theory

Subjective Probability Theory

- So far we were assuming that probabilities were objective and observable.
- This is not the case in certain cases. People might instead hold probabilistic *beliefs* about the likelihood of a certain event: ***subjective probability***.

Subjective Probability Theory

- Can we deduce subjective probability from actual behavior? Yes!
- Imagine a decision maker who prefers a gamble

$$(\$1 \text{ in state } 1, \$0 \text{ in state } 2) \succsim (\$0 \text{ in state } 1, \$1 \text{ in state } 2)$$

- If the value of money is the same across states, then he must be assigning a higher subjective probability to state 1 than to state 2.

Subjective Probability Theory

- Let us start with some definitions.
- First, we define state s preferences, $\succsim_s$, on state s lotteries $F_s(\cdot)$ by $F_s(\cdot) \succsim F_s'(\cdot)$ if

$$\int u_s(x_s) dF_s(x_s) \geq \int u_s(x_s) dF_s'(x_s)$$

- Hence, the state preferences $(\succsim_1, \succsim_2, \dots, \succsim_S)$ on state lotteries $(F_1, F_2, \dots, F_S)$ are ***state uniform*** if

$\succsim_s = \succsim_{s'}$ for any two states s and s'

Subjective Probability Theory

- That is, preferences over lotteries are state uniform if for any two states s and s', the ranking of any two lotteries $F_s(\cdot)$ and $F_s'(\cdot)$ coincides in both states, i.e.,

$$F_s(\cdot) \gtrsim F_s'(\cdot) \text{ or}$$

$$F_s'(\cdot) \gtrsim F_s(\cdot) \text{ or}$$

$$F_s(\cdot) \sim F_s'(\cdot)$$

Subjective Probability Theory

- With state uniformity, $u_s(\cdot)$ and $u_{s'}(\cdot)$ can differ only up to an increasing linear transformation.
- That is, there is a utility function $u(\cdot)$ such that
$$u_s(\cdot) = \pi_s u(\cdot) + \beta_s$$
$$u_{s'}(\cdot) = \pi_{s'} u(\cdot) + \beta_{s'}$$
for every state s and s', and for every $\pi_s, \pi_{s'} > 0$ and $\beta_s, \beta_{s'} > 0$.
- In words, the ranking between the expected utility of state s and s' remains unaffected.

Subjective Probability Theory

- ***Subjective probabilities EU theorem***:
  - Suppose that a preference relation satisfies continuity and the extended IA, and that preferences over lotteries are state uniform.
  - Then, there are subjective probabilities $(\pi_1, \pi_2, \ldots, \pi_S) \gg 0$ and a utility function $u(\cdot)$ on certain amounts of money, such that for any two lists of monetary amounts $(x_1, x_2, \ldots, x_S)$ and $(x_1', x_2', \ldots, x_S')$,

$$(x_1, x_2, \ldots, x_S) \succsim (x_1', x_2', \ldots, x_S') \text{ iff}$$

$$\sum_s \pi_s u_s(x_s) \geq \sum_s \pi_s u_s(x_s')$$

Subjective Probability Theory

- *Intuition*: a decision maker prefers the first list of monetary outcomes to the second if the "subjective" expected utility from the first list is larger than that from the second.

- The predictions of the subjective EU theorem are not necessarily satisfied in all experimental settings.
  – Example: *Ellsberg paradox*

Subjective Probability Theory

- ***Ellsberg paradox***:
  - An urn contains 300 balls: 100 are red and the remaining 200 are either blue or green.
  - We first present the following two gambles to a group of students, asking each of them to choose either gamble A or B.
    - *Gamble A*: $1000 if the ball is red
    - *Gamble B*: $1000 if the ball is blue
  - We next present the following two gambles to the same group of students, asking each of them to choose either gamble C or D.
    - Gamble C: $1000 if the ball is not red
    - Gamble D: $1000 if the ball is not blue

Subjective Probability Theory

- ***Ellsberg paradox*** (continued):
  - Common choices: people choose A to B, and C to D.
  - But these choices violate subjective EU theory!
  - We know that

$$p(\text{Red}) = 1 - p(\text{not Red})$$
$$p(\text{Blue}) = 1 - p(\text{not Blue})$$

  - If gamble A is preferred to B, then we must have

$$p(\text{Red})u(\$1000) > p(\text{Blue})u(\$1000) \Longrightarrow$$
$$p(\text{Red}) > p(\text{Blue})$$

  - And if gamble C is preferred to D, then we must have

$$p(\text{not Red})u(\$1000) > p(\text{not Blue})u(\$1000) \Longrightarrow$$
$$p(\text{not Red}) > p(\text{not Blue})$$

  - But the above two expressions are incompatible.

Appendix 5.3: Ambiguity and Ambiguity Aversion

Ambiguity and Ambiguity Aversion

- Alternative theories that account for the anomaly in the Ellsberg paradox:
  1) expected utility theory with multiple priors (also referred to as *maxmin* expected utility)
  2) rank-dependent expected utility (or *Choquet* expected utility)
- Individuals have ambiguous (unclear) beliefs, rather than objective or subjective beliefs.
- Let f denote an act $f: s \rightarrow x$ from the set of states to the set of outcomes.

Ambiguity and Ambiguity Aversion

- ***Maxmin expected utility*** (MEU):
  - If subjects have too little information to form their priors, one could alternatively allow them to consider a set of priors.
  - If an individual is uncertainty averse, he will choose lottery f over another lottery g if the former provides a higher expected utility than the latter according to his worst possible prior.

Ambiguity and Ambiguity Aversion

- ***Uncertainty aversion***: Consider an individual who is indifferent between two lotteries f and g. Then, he is *uncertainty averse* if he weakly prefers the compound lottery $\alpha f + (1-\alpha)g$ to lottery f, where $\alpha \in (0,1)$.
  - *Intuition*: a decision maker who is uncertainty averse has a preference for mixing (or hedging), since the compound lottery becomes at least as valuable as either of the two lotteries alone.

Ambiguity and Ambiguity Aversion

- ***Certainty-independence***: For any two lotteries f and g and a constant act k (i.e., a certain outcome or a lottery that remains constant across all states), the decision maker weakly prefers lottery f to g if and only if he prefers $\alpha f + (1-\alpha)k$ to $\alpha g + (1-\alpha)k$, where $\alpha \in (0,1)$.
  - Certainty-independence axiom relaxes the IA as it only requires that preferences over two lotteries to be unaffected when each lottery is mixed with a certain outcome k.

Ambiguity and Ambiguity Aversion

- A decision maker weakly prefers lottery f to g if and only if

$$\min_{p \in C} \int_S u(f(s))\,dp(s) \geq \min_{p \in C} \int_S u(g(s))\,dp(s)$$

- That is, the individual evaluates the expected utility of lotteries f and g according to each of his multiple priors $p \in C$, and then selects the lottery that yields the highest of the worst possible expected utilities.

Ambiguity and Ambiguity Aversion

- ***Example***:
  - Consider a decision maker with Bernoulli utility function $u(x) = \sqrt{x}$, where $x \geq 0$ denotes monetary amounts.
  - Assume that the decision maker faces two lotteries

$$L_A = (\$1, \$100)$$
$$L_B = (\$3, \$5)$$

  - Also, assume that the decision maker's priors are

$$(p_A, 1 - p_A) \text{ for } L_A$$
$$(p_B, 1 - p_B) \text{ for } L_B$$

Ambiguity and Ambiguity Aversion

- ***Example*** (continued):
  - According to MEU, the decision maker chooses lottery L_B if

$$\min_{p_B}[p_B\sqrt{3} + (1 - p_B)\sqrt{5}]$$
$$\geq \min_{p_A}[p_A\sqrt{1} + (1 - p_A)\sqrt{100}]$$

  - If the decision maker does not have any available information with which to update his priors, priors can take values $(p_A, p_B) \in [0,1]$.
  - It is possible that in his most pessimistic belief, he receives the lowest monetary amount with probability one.

Ambiguity and Ambiguity Aversion

- ***Example*** (continued):
  - Then, with argmin $p_B = 1$,
  $$\min_{p_B}[p_B\sqrt{3} + (1 - p_B)\sqrt{5}] = \sqrt{3}$$
  - Similarly, with argmin $p_A = 1$,
  $$\min_{p_A}[p_A\sqrt{1} + (1 - p_A)\sqrt{100}] = \sqrt{1}$$
  - Hence a decision maker with MEU preferences selects lotter L_B because $\sqrt{3} \geq \sqrt{1}$.

Ambiguity and Ambiguity Aversion

- ***Choquet expected utility*** (CEU):
  - Define beliefs with the use of capacities.
  - A capacity is defined as a real-valued function $v(\cdot)$ from a subset of the state space S to $[0,1]$, with the normalization $v(\emptyset) = 0$ and $v(S) = 1$.
  - If the capacity $v(\cdot)$ satisfies monotonicity, $v(A) \geq v(B)$, where A is a superset of B.
  - We cannot use a standard integral over states since the capacity $v(\cdot)$ does not correspond to our notion of beliefs.

Ambiguity and Ambiguity Aversion

– A decision maker weakly prefers f to g if the *Choquet integrals* satisfy

$$\int_S u\big(f(S)\big)dv(S) \geq \int_S u(g(S))dv(S)$$

– The CEU and MEU models are connected if we impose the uncertainty aversion axiom in CEU context. For that we need that capacity $v(\cdot)$ satisfies *supermodularity*, i.e.,

$$v(A \cup B) - v(B) \geq v(A \cup C) - v(C)$$

where C is a subset of B, i.e., $C \subset B$.

Ambiguity and Ambiguity Aversion

- ***Example***:
  - While the use of Choquet integrals is involved, the literature often uses “simple” capacities.
  - A simple capacity on state space S can be understood as a convex combination between two extreme capacities:
    1. a standard probability weight on A, $p(A) \in [0,1]$.
    2. the “complete ignorance” capacity w, where $w(S) = 1$ and $w(A) = 0$ for every $A \subseteq S$.

Ambiguity and Ambiguity Aversion

- ***Example*** (continued):
  - Formally, simple capacities are defined as
    $$v(A) = \lambda p(A) + (1 - \lambda) w(A)$$
    for every $A \subseteq S$ and where $\lambda \in [0,1]$.
  - Parameter λ denotes the individual's degree of confidence on $p(A)$, while $(1 - \lambda)$ captures his degree of ambiguity about $p(A)$.
  - For further reading, see Haller (2000) and Aflaki (2013).

Ambiguity and Ambiguity Aversion

- Further reading:

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