EconS 594 - Theory of Industrial Organization
Homework #2 - Answer Key

1. **Vertical product differentiation and cost of quality.** A consumer with income m who consumes a product of quality s_i and pays p_i obtains the utility $\frac{s_i m}{6} - p_i$. If instead the consumer decides not buy the good, the resulting utility is zero. Consumer income m is uniformly distributed on the interval $[2, 8]$, so its density is $f(m) = \frac{1}{8-2} = \frac{1}{6}$. The total mass of consumers is equal to 1.

   There are two firms in the market, 1 and 2, offering qualities s_1 and s_2, respectively. Assume that $s_1, s_2 \in [1, 2]$. Label firms such that $s_1 \leq s_2$. Suppose that firm i has constant marginal cost equal to $c \times s_i$ where $c < 2$.

   (a) Derive the demand of firms 1 and 2, and calculate the best-response functions of the two firms presuming that first-order conditions hold with equality. Distinguish between full and partial market coverage.

   (b) Calculate the Nash Equilibrium in prices and find the equilibrium profits as a function of s_1 and s_2. Distinguish between full and partial market coverage.

   (c) What are the equilibrium quality choices of the two firms? (Again distinguish between the full and market coverage cases using first-oder conditions.)

   (d) Which firm is more profitable? Consider the two cases mentioned above.

   (e) How does in the partial coverage equilibrium an increase in cost c affect the profits of the two firms?

      - See scanned pages at the end of the answer key.

2. **Exercise 6.1,** from Belleflamme and Peitz's book (see page 159).

   - See scanned pages at the end of the answer key.

3. **Surplus-increasing advertising in the Hotelling model.** Consider a horizontally differentiated product market in which firms are located at the extreme points of the unit interval. Firms produce at marginal costs equal to zero. A continuum of consumers of mass 1 are uniformly distributed on the unit interval. They have unit demand and have an outside utility of $-\infty$. A consumer located at $x \in [0, 1]$ obtains indirect utility $v_1 = r_1 - tx - p_1$ if she buys one unit from firm 1 and $v_2 = r_2 - t(1 - x) - p_2$ if she buys from firm 2. Firms have marginal costs equal to zero.

   (a) Suppose that firms have set prices at p_1 and p_2 respectively. Determine the demand function for each firm for each admissible price pair.

   (b) Suppose that the social planner chooses first-best optimal prices. Which price pairs would be socially optimal?

   (c) Suppose that the two firms simultaneously set prices. Determine the market equilibrium for all possible combinations of (r_1, r_2).

(d) From now on consider the special case that $t = 1$. Suppose that each firm i can use advertising to increase the willingness to pay from $r_i = 1$ to $r_i = 2$. Consider the two-stage game in which firms choose advertising at the first stage and price at the second stage. Characterize the subgame-perfect equilibrium of the game depending on the advertising cost A. Evaluate your results when $A = \frac{2}{9}$, $A = \frac{3}{9}$, and $A = \frac{4}{9}$. What is the welfare ranking?

(e) What are the equilibria for $A = \frac{5}{18}$ and $A = \frac{7}{18}$?

(f) What are the welfare consequences of a reduction in the advertising cost from $A = \frac{5}{18} + \varepsilon$ to $A = \frac{5}{18} - \varepsilon$ where $\varepsilon \to 0$ (determine whether total surplus increases or decreases and by how much)? Comment on your result in one sentence.

(g) What are the welfare consequences of a reduction in advertising the advertising cost from $A = \frac{7}{18} + \varepsilon$ to $A = \frac{7}{18} - \varepsilon$ where $\varepsilon \to 0$ (determine whether total surplus increases or decreases and by how much)? Comment on your result in one sentence.

- See scanned pages at the end of the answer key.

Homework #2

Exercise #1

Assumptions:

- $v_i = \frac{s_i m}{6} - p_i$ (indirect utility) and $v_0 = 0$ if no good is consumed
- Income m is uniformy distributed on $[2;8]$. Thus $f(m) = \frac{1}{8-2} = \frac{1}{6}$
- 2 firms with qualities such that $s_1 \leq s_2, s_i \in [1;2]$
- $MC_i = cs_i$
- There may be full or partial market coverage.

1. Find consumer $\hat{m}$ who is indifferent between low and high quality.

$$\frac{s_1\hat{m}}{6} - p_1 = \frac{s_2\hat{m}}{6} - p_2$$

Thus,

$$\hat{m} = \frac{6(p_2 - p_1)}{s_2 - s_1}$$

Under full market coverage (condition for all consumers to buy: even for $\hat{m} = 2$, $v_1 \geq 0$):

$$\frac{2s_1}{6} - p_1 \geq 0 \quad \Leftrightarrow \quad p_1 \leq \frac{s_1}{3}$$

Find demand for given s_1, s_2.

Case 1: Full coverage with $p_1 \leq \frac{s_1}{3}$.

$$Q_1(p_1, p_2) = \int_2^{\hat{m}} \frac{1}{6} dm = \frac{\hat{m}}{6} - \frac{1}{3} = \frac{p_2 - p_1}{s_2 - s_1} - \frac{1}{3}$$

$$Q_2(p_1, p_2) = \int_{\hat{m}}^{8} \frac{1}{6} dm = \frac{4}{3} - \frac{p_2 - p_1}{s_2 - s_1}$$

Case 2: Partial coverage with $p_1 > \frac{s_1}{3}$. Marginal consumer $\tilde{m}$ such that $v_1(p_1) = 0$

$$\frac{s_1\tilde{m}}{6} - p_1 = 0 \quad \Leftrightarrow \quad \tilde{m} = \frac{6p_1}{s_1}$$

$$\text{thus,} Q_1(p_1, p_2)\Big|_{p_1 > \frac{s_1}{3}} = \int_{\tilde{m}}^{\hat{m}} \frac{1}{6} dm = \frac{\hat{m}}{6} - \frac{\tilde{m}}{6} = \frac{p_2 - p_1}{s_2 - s_1} - \frac{p_1}{s_1}$$

Find best reply for both cases:

$$\max_{p_i} \quad \pi_i = (p_i - cs_i)Q_i(p_i, p_j)$$

$$
\begin{aligned}
p_1^{br}(p_2) &= \begin{cases} \frac{p_2+cs_1}{2} - \frac{1}{6}(s_2 - s_1) & \text{if foc with full coverage holds} \\ \frac{p_2\frac{s_1}{s_2}+cs_1}{2} & \text{if foc with partial coverage holds} \end{cases} \\
p_2^{br}(p_1) &= \frac{p_1 + cs_2}{2} + \frac{2}{3}(s_2 - s_1)
\end{aligned}
$$

Firm 1 exhibits a kinked demand curve. Best-response function is reported only when foc's holds with equality. When the price at the kink is profit maximizing, we simple have as the best response $p_1 = s_1/3$; this case is ignored below.

2. Case 1 (using best response with full coverage)

   Equilibrium at the second stage: Solve $p_1^{br}(p_2^{br}(p_1))$ for p_1.

$$
p_1^* = \frac{1}{3}c(s_2 + 2s_1) + \frac{2}{9}(s_2 - s_1) \qquad Q_1^* = \frac{1}{3}c + \frac{2}{9}
$$

$$
p_2^* = \frac{1}{3}c(s_1 + 2s_2) + \frac{7}{9}(s_2 - s_1) \qquad Q_2^* = -\frac{1}{3}c + \frac{7}{9}
$$

$$
\pi_1^*(s_1, s_2) = (\frac{1}{3}c + \frac{2}{9})^2(s_2 - s_1)
$$

$$
\pi_2^*(s_1, s_2) = (\frac{1}{3}c - \frac{7}{9})^2(s_2 - s_1)
$$

   Case 2 (using best response with full coverage)

   Equilibrium at the second stage:

$$
\begin{aligned}
p_1^* &= \frac{s_1((4+9c)s_2 - 4s_1)}{3(4s_2 - s_1)} \\
p_2^* &= \frac{s_2((8+6c)s_2 - (8-3c)s_1))}{3(4s_2 - s_1)} \\
Q_1^* &= \frac{s_2(4-3c)}{3(4s_2 - s_1)} \\
Q_2^* &= \frac{2s_2(4-3c)}{3(4s_2 - s_1)} \\
\tilde{\pi}_1^*(s_1, s_2) &= \frac{(4-3c)^2 s_1 s_2 (s_2 - s_1)}{9(4s_2 - s_1)^2} \\
\tilde{\pi}_2^*(s_1, s_2) &= \frac{4(4-3c)^2 s_2^2 (s_2 - s_1)}{9(4s_2 - s_1)^2}
\end{aligned}
$$

   Note: This is not a full equilibrium characterization, as firm 1 has kinked demand and may set price at the kink.

3. Case 1: At the first stage, determine s_1^*, s_2^*.

$\frac{\partial \pi_1^*}{\partial s_1} < 0$ and $\frac{\partial \pi_2^*}{\partial s_2} > 0$
Firms will choose s_1, s_2 to maximally differentiate $s_2^* = 2$ and $s_1^* = 1$.

$$\pi_1^{**}(c) = \pi_1^*(1,2) = (\frac{1}{3}c + \frac{2}{9})^2$$
$$\pi_2^{**}(c) = \pi_2^*(1,2) = (\frac{1}{3}c - \frac{7}{9})^2$$

Case 2: At the first stage, determine s_1^*, s_2^*.

First-order conditions w.r.t. s_i yields best replies for s_i

$s_1^*(s_2) = \frac{4}{7}s_2$.

$s_2^*(s_1)$ no interior solution $\longrightarrow$ use Kuhn-Tucker $\rightarrow s_2^* = 2$.

In equilibrium, $s_2^* = 2$ and $s_1^* = \frac{8}{7}$

$$\widetilde{\pi}_1^{**}(c) = \frac{(4-3c)^2}{216} \text{ and } \widetilde{\pi}_2^{**}(c) = \frac{7(4-3c)^2}{216}$$

4. Show that $\widetilde{\pi}_1 < \widetilde{\pi}_2$ in case 1:

$$\begin{aligned}
(\frac{1}{3}c + \frac{2}{9})^2 &< (\frac{1}{3}c - \frac{7}{9})^2 \\
\frac{c^2}{9} + \frac{4}{27}c + \frac{4}{81} &< \frac{c^2}{9} - \frac{14}{27}c + \frac{49}{81} \\
\frac{6}{9}c &< \frac{45}{81} \\
c &< \frac{5}{6}
\end{aligned}$$

and thus always holds when case 1 applies. Case 2: $\widetilde{\pi}_2 > \widetilde{\pi}_1$ always satisfied.

5.

$$\begin{aligned}
\frac{d\widetilde{\pi}_1^{**}}{dc} &= -\frac{(4-3c)}{36} < 0 \text{ if } c < \frac{4}{3} \\
\frac{d\widetilde{\pi}_2^{**}}{dc} &= -\frac{7(4-3c)}{36} < 0 \text{ if } c < \frac{4}{3}
\end{aligned}$$

$\widetilde{\pi}_2$ falls more strongly than $\widetilde{\pi}_1$. Differentiation becomes more costly for firm 2 ($s_2 = 2$) as c increases. Price p_2 increases more strongly than p_1 and firm 2 will loose consumers to firm 1. As c increases, firm 1 experiences higher demand because consumers switch from 2 to 1.

Exercise 8 *The quality-quantity trade-off under vertical differentiation*[2] *[included in 2nd edition of the book]*

[2]This exercise draws from McCannon, B.C.. (2008). The Quality-Quantity Trade-off, *Eastern Economic Journal* 34, 95-100.

$x) - p_2$. Her utility if she does not buy at all is 0. For simplicity, both firms are assumed to have zero costs.

The two firms compete by simultaneously setting their prices. Consumers fall into two categories: at every point on the line, a fraction λ with $\lambda \geq \underline{\lambda} > 0$ of consumers observe p_1 and p_2 and then decide whether to buy from firm 1, firm 2, or not to buy at all (these consumers behave as in a standard Hotelling model). A fraction $1-\lambda$ of consumers at every point x, do not observe p_1 and p_2 (i.e., they are "uninformed" about prices). Instead, each uninformed consumer forms an expectation about p_1 and p_2, and uses these expectations to choose whether to visit firm 1, firm 2, or none of the firms.

Visiting one firm is possible at zero costs, visiting both firms is infeasible or prohibitively costy. If an uniformed consumer chooses to visit one of the two firms, she learns its actual price, and then either buys from that firm or does not buy at all. In equilibrium, the beliefs of uninformed consumers are correct. For simplicity, assume that r is sufficiently high to ensure that the market is fully covered for all values of λ.

1. Solve for the equilibrium when firms 1 and 2 choose p_1 and p_2, respectively.
2. Let us interpret λ as "market transparency": An increase in λ makes the market "more transparent". What happens to prices and what happens to consumer surplus when the market becomes more transparent? What is the intuition for your answer?
3. Suppose that a policy maker maximizes total surplus as the sum of consumer surplus and profits. Should the policy maker enforce high transparency or not? Explain the intuition for your answer.
4. Now suppose that consumers always observe firm 2's price, p_2, but, as before, only a fraction λ of consumers observe p_1 while the others are uniformed and base their decision on their expectations regarding p_1, which are correct in equilibrium. Solve again for the Nash equilibrium. How does λ affect the profit of firm? Does it pay firm 1 to have non-transparent prices? Provide an intuition for your result.

Homework #2, EXERCISE 2

1. To solve for the Nash equilibrium, let us first determine the consumer who is indifferent between the two firms. Given prices p_1 and p_2, the location of the indifferent consumer satisfies

$$\tau x + p_1 = \tau(1-x) + p_2.$$

Thus,

$$x = \frac{1}{2} - \frac{p_1 - p_2}{2\tau}.$$

If a consumer is informed observes the prices, then p_1 and p_2 are the actual prices. If a consumer is uninformed then p_1 and p_2 are the expected prices (which, in equilibrium, are equal to the actual prices but differ from the actual price if a firm deviates from the equilibrium). The profits of firm i, $i = 1, 2$, is:

$$\pi_i = p_i \left[\lambda \left(\frac{1}{2} - \frac{p_1 - p_2}{2\tau} \right) + (1-\lambda) \left(\frac{1}{2} - \frac{p_1^e - p_2^e}{2\tau} \right) \right],$$

where $j \in \{1, 2\}$ and $j \neq i$ and p_i^e is the expected price of firm i. The first-order condition for firm 1's problem is:

$$\frac{\partial \pi_1}{\partial p_1} = \left[\lambda \left(\frac{1}{2} - \frac{p_1 - p_2}{2\tau} \right) + (1-\lambda) \left(\frac{1}{2} - \frac{p_1^e - p_2^e}{2\tau} \right) \right] - \lambda \frac{p_1}{2\tau} = 0.$$

In equilibrium, $p_1^e = p_1 = p_1^*$ and $p_2^e = p_2 = p_2^*$. Moreover, by symmetry, $p_1^* = p_2^* = p^*$. Hence, the equilibrium price satisfies

$$\frac{1}{2} + \lambda \frac{p^*}{2\tau} = 0,$$

and, therefore, $p^* = \frac{\tau}{\lambda}$.

2. It is easy to see that an increase in transparency, λ, leads to lower prices. The reason is simple: When some consumers are uninformed, the firm does not lose them when it increases its actual price. This can be seen by looking at the second term in the first-order condition which represents the cost to a firm from raising its price: the higher λ, the bigger the cost so the more reluctant the firm is to raise its price. When λ is low the cost of raising the price is low, so the firm raises it more. In equilibrium, both firms have the same market shares so consumers on the interval $[0, 1/2]$ buy from firm 1 and those on $[1/2, 1]$ buy from firm 2, independent of λ. Hence, less transparency leads to lower consumer surplus.

3. From a welfare perspective, the market is covered so there is no deadweight loss. The prices are then a wash (the firms gain and consumers lose but by the same amount), and total surplus is not affected by transparency. A policy maker who is only interested in total surplus should be indifferent to the degree of market transparency.

4. When only the price of firm 1 may not be fully transparent, the profits are given by:

$$\begin{aligned}
\pi_1 &= p_1 \left[\lambda \left(\frac{1}{2} - \frac{p_1 - p_2}{2\tau} \right) + (1-\lambda) \left(\frac{1}{2} - \frac{p_1^e - p_2}{2\tau} \right) \right], \\
\pi_2 &= p_2 \left[\lambda \left(\frac{1}{2} + \frac{p_1 - p_2}{2\tau} \right) + (1-\lambda) \left(\frac{1}{2} + \frac{p_1^e - p_2}{2\tau} \right) \right].
\end{aligned}$$

The first-order conditions for profit maximization are:

$$\begin{aligned}
\frac{\partial \pi_1}{\partial p_1} &= \left[\lambda \left(\frac{1}{2} - \frac{p_1 - p_2}{2\tau} \right) + (1-\lambda) \left(\frac{1}{2} - \frac{p_1^e - p_2}{2\tau} \right) \right] - \lambda \frac{p_1}{2\tau} = 0, \\
\frac{\partial \pi_2}{\partial p_2} &= \left[\lambda \left(\frac{1}{2} + \frac{p_1 - p_2}{2\tau} \right) + (1-\lambda) \left(\frac{1}{2} + \frac{p_1^e - p_2}{2\tau} \right) \right] - \frac{p_2}{2\tau} = 0.
\end{aligned}$$

In equilibrium, $p_1^e = p_1 = p_1^*$ and $p_2^e = p_2 = p_2^*$. Hence, first-order conditions can be rewritten as

$$\left(\frac{1}{2} - \frac{p_1^* - p_2^*}{2\tau}\right) - \lambda\frac{p_1^*}{2\tau} = 0,$$
$$\left(\frac{1}{2} + \frac{p_1^* - p_2^*}{2\tau}\right) - \frac{p_2^*}{2\tau} = 0.$$

Solving this systen gives equilibrium prices

$$p_1^* = \frac{3\tau}{1+2\lambda},$$
$$p_2^* = \frac{(2+\lambda)\tau}{1+2\lambda}.$$

Notice that firm 1, which lacks transparency, charges a higher price than firm 2 ($\lambda < 1$ and, thus, $3 > 2+\lambda$). Hence, firm 1 has a market share of less than 1/2. Substituting the prices in firm 1's profit, yields

$$\pi_1^* = \frac{2\lambda\tau}{2(1+2\lambda)^2},$$

which is the profit maximum. Differentiating with respect to λ gives

$$\frac{\partial\pi_1^*}{\partial\lambda} = \frac{9(1-2\lambda)\tau}{2(1+2\lambda)^3}.$$

Hence, firm 1 benefits from having some non-transparency and the optimal degree of non transparency is $\lambda = 1/2$. Being non-transparent allows firm 1 to be less aggressive and maintain high prices. Firm 2 responds with high prices as well (strategic complements) and, hence, firm 1 benefits from facing a less aggressive rival. This is beneficial for firm 1 if $\lambda < 1/2$. When $\lambda > 1/2$, firm 1 loses more from being too soft than it gains by making firm 2 soft.

Exercise 13 *Advertising intensity*

Consider the elasticities reported in the table below. The easiest way to think about the advertising elasticities is the following: Total demand consists of demand today and tomorrow. The short-run elasticity is the effect that advertising today has on demand today whereas the long-run elasticity is the effect that advertising today has on demand tomorrow. In which industries do you expect advertising intensity to be high? Distinguish between short run and long run.

	Income elasticity	Price elasticity	Short-run advertising elasticity	Long-run advertising elasticity
Bakery products	0.7	0.3	0.2	0.3
Books	2.2	0.8	0.3	0.4
Drugs	0.7	1.1	0.7	1.0
Tobacco products	0.0	1.8	0.4	0.6

HOMEWORK #2, EXERCISE 3.

~~**Solutions to Exercise 15**~~

1. For prices such that demand is strictly positive for each firm, demand of firm 1 is
$$Q_1(p_1, p_2) = \frac{1}{2} + \frac{(r_1 - r_2) - (p_1 - p_2)}{2t}.$$

2. Maximize social surplus $\hat{x}r_1 - t\hat{x}^2/2 + (1-\hat{x})r_2 - t(1-\hat{x})^2/2$ by choosing $\hat{x}$. Clearly, price equal marginal costs implement the first best. Also, any prices $p_1 = p_2$ implement the first best. The optimal (interior) allocation is characterized by
$$\hat{x}^W = \frac{1}{2} + \frac{(r_1 - r_2)}{2t}.$$

3. Firm 1 solves $\max_{p_1} p_1 \left(\frac{1}{2} + \frac{(r_1-r_2)-(p_1-p_2)}{2t}\right)$, firm 2 solves $\max_{p_2} p_2 \left(\frac{1}{2} - \frac{(r_1-r_2)-(p_1-p_2)}{2t}\right)$. First-order conditions can be written as best-response functions $p_i = t/2 + (r_i - r_j + p_j)/2$. In equilibrium, firms set prices
$$p_i^* = t + (r_i - r_j)/3.$$
Thus, from a social point of view the high-quality firm i with $r_i > r_j$ sets a too high price, whereas the low-quality firm j sets a too low price. The equilibrium allocation is characterized by
$$\hat{x}^* = \frac{1}{2} + \frac{(r_1 - r_2)}{6t}$$
such that too few consumers buy from the high-quality firm. Equilibrium profits are
$$\pi_i^* = \frac{1}{2t}\left(t + \frac{r_i - r_j}{3}\right)^2.$$

4. Profits in asymmetric advertising case: advertising firm makes $(1/2)(4/3)^2 - A = 8/9 - A$.

	no ad	ad
no ad	$1/2, 1/2$	$2/9, 8/9 - A$
ad	$8/9 - A, 2/9$	$1/2 - A, 1/2 - A$

case $A = 2/9$:

	no ad	ad
no ad	$1/2, 1/2$	$2/9, 2/3$
ad	$2/3, 2/9$	$5/18, 5/18$

equilibrium in which both firms advertise.

case $A = 3/9$:

	no ad	ad
no ad	$1/2, 1/2$	$2/9, 5/9$
ad	$5/9, 2/9$	$1/6, 1/6$

two asymmetric equilibria in which one of the firm advertises

case $A = 4/9$:

	no ad	ad
no ad	$1/2, 1/2$	$2/9, 5/9$
ad	$5/9, 2/9$	$1/18, 1/18$

equilibrium in which none of the firm advertises.

welfare with $A = 2/9$: $TS = 2 - 1/4 - 4/9 = 47/36$

welfare with $A = 3/9$: $TS = 5/3 - 5/18 - 3/9 = 19/18$

welfare with $A = 4/9$: $TS = 1 - 1/4 = 3/4$

5. case $A = 5/18$:

	no ad	ad
no ad	$1/2, 1/2$	$2/9, 11/18^*$
ad	$11/18, 2/9^*$	$2/9, 2/9^*$

case $A = 7/18$:

	no ad	ad
no ad	$1/2, 1/2^*$	$2/9, 1/2^*$
ad	$1/2, 2/9^*$	$1/9, 1/9$

6. welfare if one firm advertises $5/3 - 5/18 - 5/18 = 20/18$

   welfare if both firms advertise: $2 - 1/4 - 10/18 = 43/36$

   Discontinuous increase in total surplus at a marginal decrease of advertising costs at 5/18. Here increase of total consumer valuations for one third of the population + reduction in transport cost more than offset the advertising cost (the first of the two positive effects alone already dominates).

7. welfare if none advertises $3/4$

   welfare if one firm advertises: $5/3 - 5/18 - 7/18 = 1$

   "quality" effect dominates transport cost and cost of advertising. Marginal decrease in advertising cost at 7/18 leads to a discontinuous upward jump of total surplus.

Exercise 16 *Negative advertising and information disclosure [included in 2nd edition of the book]*

Consider the linear Hotelling duopoly in which each firm produces a product with a firm-specific undesirable ingredient at zero marginal costs. Suppose that, absent advertising, consumers are not aware of this ingredient. In this case a consumer of type x derives utility $r - tx - p_1$ if she purchases product 1 and utility $r - t(1 - x) - p_2$ if she purchases product 2. If a consumer learns that product i has the undesirable ingredient utility is decreased by d. Suppose that parameter values are such that in the equilibria to be characterized below the market is fully covered. Firms set prices simultaneously.