EconS 594 - Theory of Industrial Organization
Homework #1 - Answer Key

1. **Cournot with asymmetric fixed costs.** Consider a Cournot duopoly, allowing firms to face fixed costs. In particular, assume that firm 1 faces a total cost function $TC_1(q_1) = F_1 + cq_1$, where $F_1 > 0$ denotes its fixed cost and $1 > c > 0$ represents its marginal cost. Firm 2's total cost function is similar, $TC_2(q_2) = F_2 + cq_2$, where $F_2 > 0$ denotes its fixed cost, and satisfies $F_2 > F_1$, and $c > 0$ is the same marginal cost as firm 1's. Firms face a linear inverse demand function $p(Q) = 1 - Q$, where Q denotes aggregate output.

   (a) Find the best response functions of each firm and the equilibrium output.

   - *Firm 1's best response function.* In this setting, firm 1 chooses its output level q_1 to solve

     $$\max_{q_1} \ \pi_1 = (1 - q_1 - q_2)q_1 - cq_1 - F_1$$

     Differentiating with respect to q_1, we obtain

     $$\frac{\partial \pi_1}{\partial q_1} = 1 - 2q_1 - q_2 - c = 0$$

     rearranging, $1 - q_2 - c = 2q_1$, and solving for q_1 yields

     $$q_1(q_2) = \frac{1-c}{2} - \frac{1}{2}q_2$$

     which is firm 1's best response function. The best response function coincides with that in a Cournot duopoly where firms face no fixed costs. Intuitively, when choosing its optimal output, firm 1 only considers its marginal revenues and costs, but ignores its fixed cost. The fixed cost should only impact its profits and, as a consequence, the conditions under which this firm produces a positive output.

   - *Firm 2's best response function.* In this setting, firm 2 chooses its output level q_2 to solve

     $$\max_{q_2} \ \pi_2 = (1 - q_1 - q_2)q_2 - cq_2 - F_2$$

     Differentiating with respect to q_2, we obtain

     $$\frac{\partial \pi_2}{\partial q_2} = 1 - 2q_2 - q_1 - c = 0$$

     rearranging, $1 - q_1 - c_2 = 2q$, and solving for q_2 yields

     $$q_2(q_1) = \frac{1-c}{2} - \frac{1}{2}q_1$$

     which is firm 2's best response function. This best response function is symmetric to firm 1's.

- The best response functions of firm 1 and firm 2 do not differ from the Cournot duopoly with symmetric marginal costs. Therefore, the equilibrium quantity each firm produces is the same as in that situation, that is

$$q_1^* = q_2^* = \frac{1-c}{3}$$

(b) How are the equilibrium results affected? Interpret.

- Given that the best response functions and output do not change relative to the case of symmetric firms, the price in the market will not change, i.e. $p = \frac{1+2c}{3}$. Profits for the two firms will be altered slightly by the fixed cost:

$$\pi_1 = \frac{(1-c)^2}{9} - F_1$$
$$\pi_2 = \frac{(1-c)^2}{9} - F_2$$

For firm 1, if its fixed cost F_1 satisfies $F_1 > \frac{(1-c)^2}{9}$, the firm chooses to remain inactive as its overall profits from participating in this market would be negative. A similar argument applies for firm 2 if its fixed cost F_2 satisfies $F_2 > \frac{(1-c)^2}{9}$. Therefore, four cases arise, each of them depicted in one of the regions of figure 1, with F_1 in the horizontal axis and F_2 in the vertical axis:

- *Both firms active.* If $F_1, F_2 < \frac{(1-c)^2}{9}$, both firms produce the equilibrium output found in part (a) and make positive profits.
- *Only firm 1 is active.* If only firm 1 faces a sufficiently low fixed cost, that is, $F_1 < \frac{(1-c)^2}{9}$ but $F_2 > \frac{(1-c)^2}{9}$, then firm 1 produces the monopoly output $\frac{1-c}{2}$, earning monopoly profit $\frac{(1-c)^2}{4}$, while its rival remains inactive.
- *Only firm 2 is active.* If only firm 2 faces a sufficiently low fixed cost, that is, $F_2 < \frac{(1-c)^2}{9}$ but $F_1 > \frac{(1-c)^2}{9}$, then firm 2 produces the monopoly output $\frac{a-c}{2}$, earning monopoly profit $\frac{(1-c)^2}{4}$, while its rival remains inactive.
- *No active firms.* If both firms have relatively high fixed costs, that is

$F_1, F_2 > \frac{(1-c)^2}{9}$, then neither firm is active in the market.

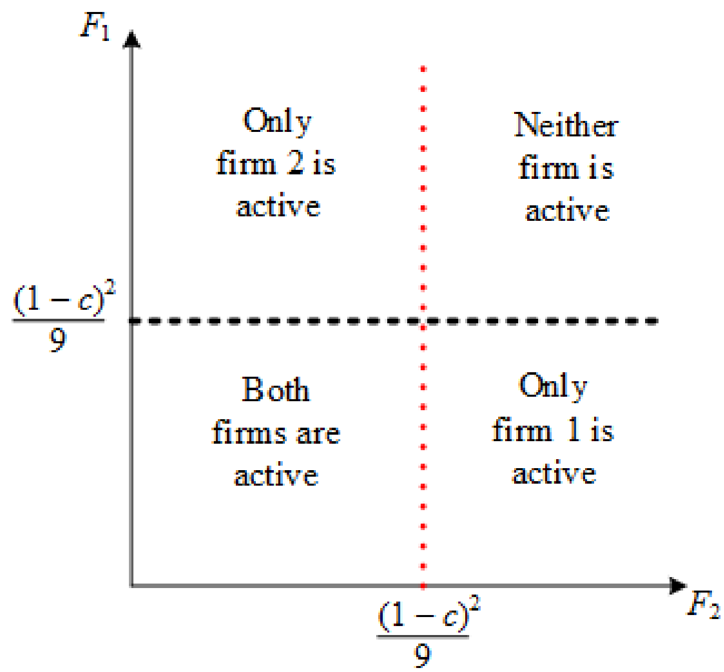

Figure 1. Production decisions.

(c) *Numerical example.* Evaluate your equilibrium results in part (b) at $c = 1/2$. What happens if c decreases to $c = 1/10$? Interpret.

- When $c = 1/2$, the cutoffs of F_1 and F_2 in figure 1 become

$$\frac{(1-\frac{1}{2})^2}{9} = \frac{1}{36} \simeq 0.03.$$

When marginal costs decrease to $c = 1/10$, these cutoffs increase to $\frac{(1-\frac{1}{10})^2}{9} = \frac{9}{100} = 0.09$. Graphically, the horizontal cutoff at the vertical axis shifts upward and the vertical cutoff in the horizontal axis shifts rightward as marginal costs decrease, intuitively describing that the region where both firms are active expands (in the southwest of figure 1), and that the regions where only one firm is active also expand (at the northwest or southeast of the figure). In contrast, the region of (F_1, F_2)-pairs where neither firm is active (at the northeast of figure 1) shrinks.

2. **Can fewer firms decrease prices?** Consider an industry with $n \geq 2$ firms competing à la Cournot, facing an inverse demand function $p(Q) = 1 - Q$, where $Q \geq 0$ denotes aggregate output. Firms in this industry are asymmetric in their marginal costs. Specifically, a share $\alpha \in [0, 1]$ of them have marginal cost c_H, which we regard as "inefficient," and the remaining share, $1 - \alpha$, are firms with marginal cost c_L, which we refer as "efficient," where $a > c_H > c_L$.

(a) Find each type of firm's best response function. Interpret.

- The demand function is

$$p = 1 - Q = 1 - \sum_{i \in L} q_i - \sum_{j \in H} q_j,$$

where the last two terms represent the total production from low- and high-cost firms, respectively. The profit function of every high-cost firm $j \in H$ is

$$\pi_j = (p(Q) - c_H)q_j$$

Differentiating with respect to q_j, we find

$$1 - \left(q_j + \sum_{i \in L} q_i + \sum_{j \in H} q_j + c_H \right) = 0.$$

Similarly, the profit function of every low-cost firm $i \in L$ is

$$\pi_i = (p(Q) - c_L)q_i$$

Differentiating with respect to q_i, we find

$$1 - \left(q_i + \sum_{i \in L} q_i + \sum_{j \in H} q_j + c_L \right) = 0.$$

In a strategy profile where all low-cost firms produce the same output level q_i, we must have

$$\sum_{i \in L} q_i = (1 - \alpha)nq_i$$

since there are $(1 - \alpha)n$ firms with low costs in this industry. Similarly, in a strategy profile where all high-cost firms produce the same output q_j, we must have that

$$\sum_{j \in H} q_j = \alpha n q_j$$

since there are αn firms with high costs. Plugging these results into the above first-order conditions, we obtain

$$\begin{aligned} 1 - (q_j + (1 - \alpha)nq_i + \alpha n q_j + c_H) &= 0, \text{ and} \\ 1 - (q_i + (1 - \alpha)nq_i + \alpha n q_j + c_L) &= 0. \end{aligned}$$

Solving for q_i in the last expression, we find the best response function of every low-cost firm $i \in L$, as follows

$$q_L(q_H) = \frac{1 - c_L}{1 + (1 - \alpha)n} - \frac{\alpha n}{1 + (1 - \alpha)n} q_H$$

which, as expected, decreases in the production of each high-cost rival, q_H. Similarly, we can solve for q_j in the first expression above, obtaining the best response function of every high-cost firm $j \in H$, as follows

$$q_H(q_L) = \frac{1 - c_H}{1 + \alpha n} - \frac{(1 - \alpha)n}{1 + \alpha n} q_L$$

which is decreasing in the production of each low-cost rival, q_L.

(b) Find equilibrium output for every high-cost firm and every low-cost firm.

- *Equilibrium output.* Inserting best response function $q_H(q_L)$ into $q_L(q_H)$, yields

$$q_L = \frac{1-c_L}{1+(1-\alpha)n} - \frac{\alpha n}{1+(1-\alpha)n}\underbrace{\left(\frac{1-c_H}{1+\alpha n} - \frac{(1-\alpha)n}{1+\alpha n}q_L\right)}_{q_H}.$$

Solving for q_L, yields an equilibrium output of

$$q_L^* = \frac{1-c_L+n\alpha(c_H-c_L)}{n+1}$$

Therefore, the equilibrium output of every high-cost firm is

$$\begin{aligned} q_H^* &= \frac{1-c_H}{1+\alpha n} - \frac{(1-\alpha)n}{1+\alpha n}\underbrace{\frac{1-c_L+n\alpha(c_H-c_L)}{n+1}}_{q_L^*} \\ &= \frac{1-c_H-n(1-\alpha)(c_H-c_L)}{n+1} \end{aligned}$$

- *Equilibrium price.* Inserting the above equilibrium outputs, q_L^* and q_H^*, into the inverse demand function, we obtain an equilibrium price

$$\begin{aligned} p^* &= 1-(1-\alpha)nq_L^* - \alpha n q_H^* \\ &= 1 - \frac{(1-\alpha)n(1-c_L+n\alpha(c_H-c_L))}{n+1} - \frac{\alpha n(1-c_H-n(1-\alpha)(c_H-c_L))}{n+1} \\ &= \frac{1+n\alpha c_H+n(1-\alpha)c_L}{1+n}. \end{aligned}$$

(c) Find under which parameter conditions do high-cost firms produce a positive output level. Examine how this parameter condition is affected by the number of firms in the industry, n, and by the proportion of high-cost firms, α.

- The denominator of equilibrium output q_H^* is positive given that $1+n>0$ by definition. Therefore, q_H^* is positive as long as the numerator is positive, $1-c_H-n(1-\alpha)(c_H-c_L)>0$ which, solving for c_H, yields

$$c_H \leq \frac{1+n(1-\alpha)c_L}{1+n(1-\alpha)} \equiv \bar{c}_H.$$

Intuitively, condition $c_H \leq \bar{c}_H$ says that every high-cost firm produces a positive output only when its cost c_H is not too high, relative to its low-cost rivals. Figure 2 illustrates cutoff $\bar{c}_H$ in the (c_H, c_L)-quadrant. Cutoff $\bar{c}_H$ originates at $c_H = \frac{1}{1+n(1-\alpha)}$ when $c_L = 0$, and reaches a height of $c_H = 1$ when $c_L = 1$, thus lying above the 45^0-line where $c_H = c_L$. Therefore, among all admissible (c_H, c_L)-pairs (those above the 45^0-line so $c_H > c_L$), only those below cutoff $\bar{c}_H$ induce every high-cost firm to produce a positive output.

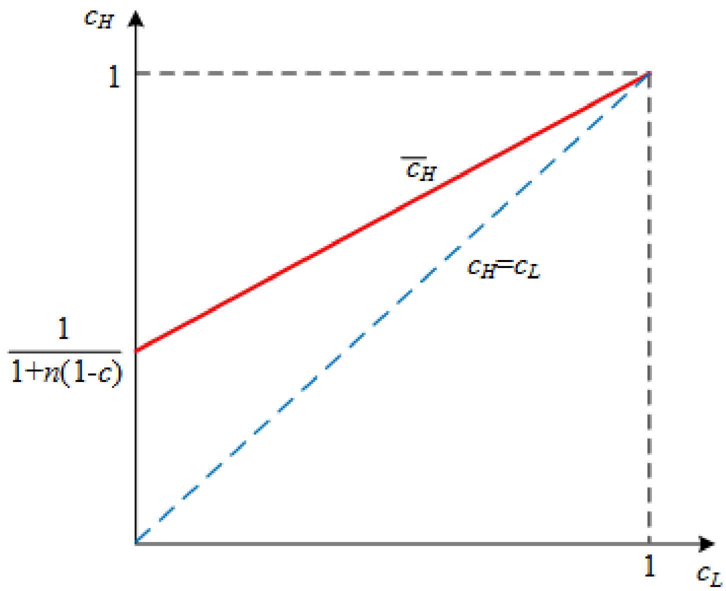

Figure 2. Cutoff $\bar{c}_H$.

- *Comparative statics.*
  - We can now check if cutoff $\bar{c}_H$ increases or decreases as more firms compete in this industry. In particular,

$$\frac{\partial \bar{c}_H}{\partial n} = -\frac{(1-\alpha)(1-c_L)}{[1+n(1-\alpha)]^2} < 0, \text{ where } a \geq 1 > c_L$$

    implying that, as more firms compete, the condition on high-cost firms to remain active, $c_H \leq \bar{c}_H$, becomes more stringent. Intuitively, as n increases, the number of low-cost firms, $(1-\alpha)n$, also increases, making it harder for every high-cost firm to compete. When n becomes sufficiently high, condition $c_H \leq \bar{c}_H$ does not hold, inducing all high-cost firms to exit the industry.
    Graphically, an increase in n produces a downward shift in the vertical intercept of cutoff $\bar{c}_H$ in figure 2.8, shrinking the region of (c_H, c_L)-pairs for which every high-cost firm produces a positive output.
  - We now check whether cutoff $\bar{c}_H$ increases or decreases in the proportion of high-cost firms in the industry, α, as follows

$$\frac{\partial \bar{c}_H}{\partial \alpha} = \frac{1-c_L}{[1+n(1-\alpha)]^2} > 0$$

    Intuitively, this result indicates that, as high-cost firms become a larger share of the industry, the condition on high-cost firms to remain active, $c_H \leq \bar{c}_H$, becomes less stringent.

(d) Find equilibrium output and prices if all high-cost firms exit the industry. Are consumers better off when all high-cost firms remain active or when they exit?

- If all high-cost firms exit, $(1-\alpha)n$ low-cost types of firms will remain in the market. The equilibrium quantity in that setting becomes

$$q_L^{**} = \frac{1-c_L}{1+n(1-\alpha)}$$

which does not depend on high-cost firm's marginal cost, c_H, since these firms are not active. The equilibrium price in this setting is

$$\begin{aligned} p^{**} &= 1-(1-\alpha)nq_L^* \\ &= 1-(1-\alpha)n\underbrace{\frac{1-c_L}{1+n(1-\alpha)}}_{q_L^{**}} \\ &= \frac{1+n(1-\alpha)c_L}{1+n(1-\alpha)} \end{aligned}$$

Comparing equilibrium prices when all high-cost firms are active, p^*, and when they exit the industry, p^{**}, we obtain that $p^* > p^{**}$ holds if

$$\frac{1+n\alpha c_H + n(1-\alpha)c_L}{1+n} > \frac{1+n(1-\alpha)c_L}{1+n(1-\alpha)}$$

Solving for c_H, yields

$$c_H > \frac{1+n(1-\alpha)c_L}{1+n(1-\alpha)} \equiv \bar{c}_H.$$

where cutoff $\bar{c}_H$ coincides with the one we found above. Therefore, when condition $c_H > \bar{c}_H$ holds, all high-cost firms exit the industry, and prices are lower than when these firms stay in the industry. Recall that figure 2 illustrates that this cutoff originates at $c_H = \frac{1}{1+n(1-\alpha)}$ and reaches a height of $c_H = 1$ when $c_L = 1$, thus lying above the 45^0-line where $c_H = c_L$.

- *Comparative statics.* The vertical intercept of cutoff $\bar{c}_H$ (see figure 2) is decreasing in the number of firms competing in the industry, n, but increasing in the share of firms with high marginal costs, α. Intuitively, as the industry becomes less competitive (low n) and a larger share of firms have a high marginal cost (high α), cutoff $\bar{c}_H$ shifts upward, expanding the region of (c_H, c_L)-pairs for which prices are lower after the exist of all high-cost firms.
- This type of example is often used to illustrate that, if an industry has a subset of inefficient firms with sufficiently high costs, it may actually better for consumers that these firms exit the industry, leaving the market with fewer firms, than if these inefficient firms stay.

3. **Price competition with heterogeneous goods and asymmetric costs.** Consider two firms competing à la Bertrand selling heterogeneous goods. The demand function of firm i, where $i, j \in \{1, 2\}$, is

$$q_i(p_i, p_j) = 1 - \gamma p_i + p_j$$

where $\gamma \geq 1$ represents the degree of product differentiation (homogeneous when $\gamma = 1$ but differentiated when $\gamma > 1$). Without loss of generality, assume that firm 1 has a lower marginal cost than firm 2 in producing every unit of the good, that is, $0 < c_1 < c_2 < 1$.

(a) Characterize the firms' best response functions and graphically illustrate your results.

- Firm i, where $i \in \{1, 2\}$, chooses p_i to solve the following profit maximization problem,

$$\max_{p_i > 0} \pi_i(p_i) = (p_i - c_i)(1 - \gamma p_i + p_j)$$

Differentiating with respect to p_i, and assuming interior solutions, we obtain

$$\frac{\partial \pi_i(p_i)}{\partial p_i} = 1 - 2\gamma p_i + p_j + \gamma c_i = 0$$

The best response function of firm i becomes

$$p_i(p_j) = \frac{1 + \gamma c_i}{2\gamma} + \frac{1}{2\gamma} p_j$$

which originates at $\frac{1+\gamma c_i}{2\gamma}$ and increases in p_j at a rate of $\frac{1}{2\gamma}$, as depicted in figure 3.

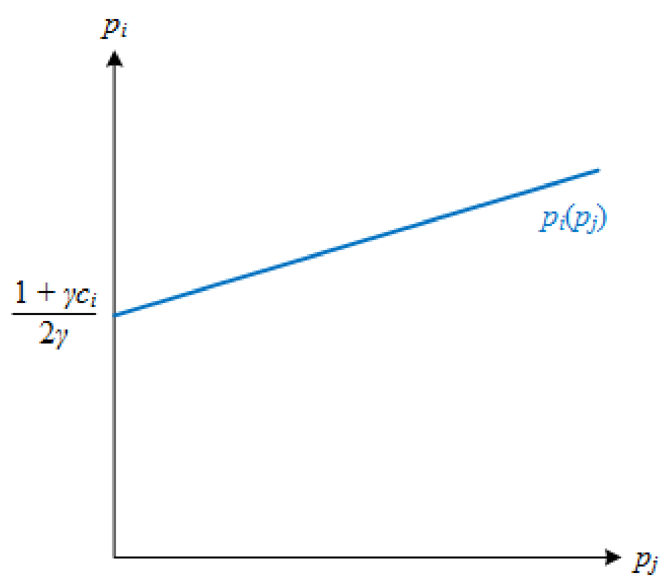

Figure 3. Firm i's best response function.

- *Comparative statics.* When we differentiate with respect to γ, we check that

$$\frac{\partial}{\partial \gamma}\left[\frac{1 + \gamma c_i}{2\gamma}\right] = \frac{2\gamma c_i - 2 - 2\gamma c_i}{4\gamma^2} = -\frac{1}{2\gamma^2} < 0$$

$$\frac{\partial}{\partial \gamma}\left[\frac{1}{2\gamma}\right] = -\frac{1}{4\gamma^2} < 0$$

so that both the intercept and the slope of the best response function decreases as goods become more differentiated, indicating that competition becomes less intense.

(b) What are the equilibrium price, output, and profit of each firm? Find the *sufficient* condition on γ in which both firms produce output, and the output level if every firm sets its price at the marginal cost.

- Rearranging the best response functions, we obtain

$$2\gamma p_1 - p_2 = 1 + \gamma c_1$$
$$2\gamma p_2 - p_1 = 1 + \gamma c_2$$

represented in matrix form as follows:

$$\begin{bmatrix} 2\gamma & -1 \\ -1 & 2\gamma \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1+\gamma c_1 \\ 1+\gamma c_2 \end{bmatrix}$$

Solving by Cramer's rule, we obtain

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \frac{1}{4\gamma^2 - 1} \begin{bmatrix} 2\gamma & 1 \\ 1 & 2\gamma \end{bmatrix} \begin{bmatrix} 1+\gamma c_1 \\ 1+\gamma c_2 \end{bmatrix} = \frac{1}{4\gamma^2 - 1} \begin{bmatrix} 2\gamma^2 c_1 + \gamma c_2 + 2\gamma + 1 \\ 2\gamma^2 c_2 + \gamma c_1 + 2\gamma + 1 \end{bmatrix}$$

which yields equilibrium prices:

$$p_1^* = \frac{2\gamma^2 c_1 + \gamma c_2 + 2\gamma + 1}{4\gamma^2 - 1}$$
$$p_2^* = \frac{2\gamma^2 c_2 + \gamma c_1 + 2\gamma + 1}{4\gamma^2 - 1}$$

Differentiating the equilibrium price p_1^* with respect to γ, we obtain

$$\frac{\partial p_1^*}{\partial \gamma} = \frac{(4\gamma c_1 + c_2 + 2)(4\gamma^2 - 1) - 8\gamma(2\gamma^2 c_1 + \gamma c_2 + 2\gamma + 1)}{(4\gamma^2 - 1)^2} = -\frac{4\gamma c_1 + (4\gamma^2 + 1) c_2 + 2(4\gamma^2 + 1) + 8\gamma}{(4\gamma^2 - 1)^2} < 0$$

so, as goods become more differentiated, equilibrium price of firm 1 decreases. A symmetric result applies if we differentiate p_2^* with respect to γ.

- Substituting p_1^* and p_2^* into the demand function, equilibrium output for each firm becomes

$$\begin{aligned}
q_1^* &= 1 - \gamma p_1^* + p_2^* \\
&= 1 - \gamma \left(\frac{2\gamma^2 c_1 + \gamma c_2 + 2\gamma + 1}{4\gamma^2 - 1} \right) + \left(\frac{2\gamma^2 c_2 + \gamma c_1 + 2\gamma + 1}{4\gamma^2 - 1} \right) \\
&= \frac{\gamma}{4\gamma^2 - 1} \left[\gamma c_2 - \left(2\gamma^2 - 1 \right) c_1 + 2\gamma + 1 \right] \\
q_2^* &= 1 - \gamma p_2^* + p_1^* \\
&= 1 - \gamma \left(\frac{2\gamma^2 c_2 + \gamma c_1 + 2\gamma + 1}{4\gamma^2 - 1} \right) + \left(\frac{2\gamma^2 c_1 + \gamma c_2 + 2\gamma + 1}{4\gamma^2 - 1} \right) \\
&= \frac{\gamma}{4\gamma^2 - 1} \left[\gamma c_1 - \left(2\gamma^2 - 1 \right) c_2 + 2\gamma + 1 \right]
\end{aligned}$$

Differentiating equilibrium output q_1^* with respect to γ, we obtain

$$\begin{aligned}
\frac{\partial q_1^*}{\partial \gamma} &= \frac{\left[2\gamma c_2 - (6\gamma^2 - 1) c_1 + 4\gamma + 1 \right] (4\gamma^2 - 1) - 8\gamma^2 \left[\gamma c_2 - (2\gamma^2 - 1) c_1 + 2\gamma + 1 \right]}{(4\gamma^2 - 1)^2} \\
&= -\frac{2\gamma c_2 + (8\gamma^4 - 2\gamma^2 + 1) c_1 + (4\gamma^2 + 4\gamma - 1)}{(4\gamma^2 - 1)^2} < 0
\end{aligned}$$

so, as goods become more differentiated, equilibrium output of firm 1 decreases. A symmetric result holds if we differentiate q_2^* with respect to γ.

- Substituting q_1^* and q_2^* into the profit function, equilibrium profits become

$$\begin{aligned}
\pi_1^* &= (p_1^* - c_1)\, q_1^* \\
&= \frac{\gamma}{4\gamma^2 - 1} \left(\frac{2\gamma^2 c_1 + \gamma c_2 + 2\gamma + 1}{4\gamma^2 - 1} - c_1 \right) \left[\gamma c_2 - \left(2\gamma^2 - 1 \right) c_1 + 2\gamma + 1 \right] \\
&= \gamma \left(\frac{\gamma c_2 - (2\gamma^2 - 1) c_1 + 2\gamma + 1}{4\gamma^2 - 1} \right)^2 \\
\pi_2^* &= (p_2^* - c_2)\, q_2^* \\
&= \frac{\gamma}{4\gamma^2 - 1} \left(\frac{2\gamma^2 c_2 + \gamma c_1 + 2\gamma + 1}{4\gamma^2 - 1} - c_2 \right) \left[\gamma c_1 - \left(2\gamma^2 - 1 \right) c_2 + 2\gamma + 1 \right] \\
&= \gamma \left(\frac{\gamma c_1 - (2\gamma^2 - 1) c_2 + 2\gamma + 1}{4\gamma^2 - 1} \right)^2
\end{aligned}$$

- Next, let us also find the reservation price of the consumer (that is, the "choke price" where the inverse demand function crosses the vertical axis) by setting the inverse demand functions equal to zero, as follows:

$$\begin{aligned}
0 &= 1 - \gamma p_1 + p_2 \\
0 &= 1 - \gamma p_2 + p_1
\end{aligned}$$

which, by symmetry, yields

$$\gamma p = 1 + p$$

that is simplified to

$$\overline{p} = \frac{1}{\gamma - 1}$$

meaning that if $p \geq \overline{p}$, the consumer will consume zero units of output.

- For an interior solution to occur, where both firms produce output, we need

$$\begin{aligned} p_1^* &< \overline{p} \\ p_2^* &< \overline{p} \end{aligned}$$

that can be expressed as

$$\begin{aligned} \frac{2\gamma^2 c_1 + \gamma c_2 + 2\gamma + 1}{4\gamma^2 - 1} &< \frac{1}{\gamma - 1} \\ \frac{2\gamma^2 c_2 + \gamma c_1 + 2\gamma + 1}{4\gamma^2 - 1} &< \frac{1}{\gamma - 1} \end{aligned}$$

which is rearranged to yield

$$\begin{aligned} (2\gamma c_1 + c_2)(\gamma - 1) &< (2\gamma + 1) \\ (2\gamma c_2 + c_1)(\gamma - 1) &< (2\gamma + 1) \end{aligned}$$

so we find the *sufficient* condition by combining the above inequalities, as follows,

$$\begin{aligned} (\gamma - 1)(2\gamma + 1)\max\{c_1, c_2\} &< (2\gamma + 1) \\ \Rightarrow \max\{c_1, c_2\} &< \frac{1}{\gamma - 1} = \overline{p} \end{aligned}$$

As goods become more differentiated (γ increases), every firm i needs to be more efficient (lower c) in order to mitigate the stronger price effect on consumers' willingness-to-pay for its output. Otherwise, the less efficient firm may exit the market if $c_i \geq \overline{p}$ when marginal cost is above consumer's reservation price.

- When every firm sets its price at the marginal cost, output becomes

$$\begin{aligned} q_1(c_1, c_2) &= 1 - \gamma c_1 + c_2 \\ q_2(c_1, c_2) &= 1 - \gamma c_2 + c_1 \end{aligned}$$

so that $c < \overline{p}$ is the *necessary* condition for $q(c) \geq 0$.

(c) *Numerical example.* Evaluate equilibrium outcomes under $c_1 = 1/4$ and $c_2 = 1/2$ as a function of γ. Under which conditions of γ will both firms produce a positive output?

- Equilibrium outcomes are

$$p_1^* = \frac{2\gamma^2 \frac{1}{4} + \gamma \frac{1}{2} + 2\gamma + 1}{4\gamma^2 - 1} = \frac{\gamma^2 + 5\gamma + 2}{2\left(4\gamma^2 - 1\right)}$$

$$p_2^* = \frac{2\gamma^2 \frac{1}{2} + \gamma \frac{1}{4} + 2\gamma + 1}{4\gamma^2 - 1} = \frac{4\gamma^2 + 9\gamma + 4}{4\left(4\gamma^2 - 1\right)}$$

$$q_1^* = \frac{\gamma}{4\gamma^2 - 1}\left[\gamma \frac{1}{2} - \left(2\gamma^2 - 1\right)\frac{1}{4} + 2\gamma + 1\right] = \frac{\gamma\left(-2\gamma^2 + 10\gamma + 5\right)}{4\left(4\gamma^2 - 1\right)}$$

$$q_2^* = \frac{\gamma}{4\gamma^2 - 1}\left[\gamma \frac{1}{4} - \left(2\gamma^2 - 1\right)\frac{1}{2} + 2\gamma + 1\right] = \frac{\gamma\left(-4\gamma^2 + 9\gamma + 6\right)}{4\left(4\gamma^2 - 1\right)}$$

$$\pi_1^* = \gamma\left(\frac{\gamma \frac{1}{2} - \left(2\gamma^2 - 1\right)\frac{1}{4} + 2\gamma + 1}{4\gamma^2 - 1}\right)^2 = \frac{\gamma\left(-2\gamma^2 + 10\gamma + 5\right)^2}{16\left(4\gamma^2 - 1\right)^2}$$

$$\pi_2^* = \gamma\left(\frac{\gamma \frac{1}{4} - \left(2\gamma^2 - 1\right)\frac{1}{2} + 2\gamma + 1}{4\gamma^2 - 1}\right)^2 = \frac{\gamma\left(-4\gamma^2 + 9\gamma + 6\right)^2}{16\left(4\gamma^2 - 1\right)^2}$$

Firms will produce positive units of output if $q_1^* > 0$ and $q_2^* > 0$, where

$$2\gamma^2 - 10\gamma - 5 < 0$$
$$4\gamma^2 - 9\gamma - 6 < 0$$

Solving for γ, we obtain

$$\frac{5 - \sqrt{35}}{2} < \gamma < \frac{5 + \sqrt{35}}{2}$$
$$\frac{9 - \sqrt{177}}{8} < \gamma < \frac{9 + \sqrt{177}}{8}$$

Since γ, it suffices to say that both firms produce output if

$$1 < \gamma < 2.79$$

4. **Horizontal differentiation in two dimensions, based on Irmen and Thisse (1998).**[1] Consider the model of horizontally differentiated products discussed in class. However, assume now that, in the first stage, every firm i chooses its location, l_i, in the interval $[0, 1]$, where $i = \{1, 2\}$; and, similarly, its location h_i in the interval $[0, 1]$. Intuitively, this indicates that firms differentiate along two dimensions (e.g., sweetness and color), which implies that consumer preferences in this setting are uniformly distributed in a unit square (i.e., a square of side one). For simplicity, assume that consumer's per-unit disutility from purchasing a good that does not coincide with his ideal coincides across both dimensions.

   (a) *Third stage - Finding demand*. For given locations from the first stage (l_1, l_2, h_1, h_2), and given prices from the second stage (p_1, p_2), find the demand that each firm has in the third stage.

[1] Irmen, A. and J-F. Thisse (1998) "Competition in Multi-characteristics Spaces: Hotelling Was Almost Right," Journal of Economic Theory, 78, pp. 76–102.

- If a consumer purchases from firm 1, his utility is

$$r - p_1 - t(x - l_1)^2 - t(x - h_1)^2,$$

while purchasing from firm 2 yields

$$r - p_2 - t(x - l_2)^2 - t(x - h_2)^2.$$

Therefore, the indifferent consumer $\widehat{x}$ solves

$$r - p_1 - t(\widehat{x} - l_1)^2 - t(\widehat{x} - h_1)^2 = r - p_2 - t(\widehat{x} - l_2)^2 - t(\widehat{x} - h_2)^2$$

The demand of firm 1 is $\widehat{x}$, where

$$\widehat{x} = \frac{p_1 - p_2 + t\left(l_1^2 + h_1^2 - l_2^2 - h_2^2\right)}{2t\left(l_1 + h_1 - l_2 - h_2\right)}$$

In contrast, firm 2's demand is $1 - \widehat{x}$, where

$$1 - \widehat{x} = \frac{2t\left(l_1 + h_1 - l_2 - h_2\right) - p_1 + p_2 + t\left(l_2^2 + h_2^2 - l_1^2 - h_1^2\right)}{2t\left(l_1 + h_1 - l_2 - h_2\right)}$$

- When both firms set the same price, that is, $p_1 = p_2$, demand simplifies to

$$\widehat{x} = \frac{l_1^2 + h_1^2 - l_2^2 - h_2^2}{2\left(l_1 + h_1 - l_2 - h_2\right)} \quad \text{for firm 1}$$

$$1 - \widehat{x} = 1 - \frac{l_2^2 + h_2^2 - l_1^2 - h_1^2}{2\left(l_2 + h_2 - l_1 - h_1\right)} \quad \text{for firm 2}$$

(b) *Second stage - Prices.* Given locations from the first stage, find the price that each firm sets in the second stage.

- *Finding firm 1's best response function.* Firm 1 chooses price p_1 to solve

$$\max_{p_1} \; (p_1 - c) \underbrace{\left(\frac{p_1 - p_2 + t\left(l_1^2 + h_1^2 - l_2^2 - h_2^2\right)}{2t\left(l_1 + h_1 - l_2 - h_2\right)}\right)}_{\text{Demand, } \widehat{x}}$$

Differentiating with respect to p_1, we obtain

$$2p_1 - p_2 + t\left(l_1^2 + h_1^2 - l_2^2 - h_2^2\right) - c = 0$$

Solving for p_1, we find firm 1's best response function

$$p_1\left(p_2\right) = \frac{c + t\left(l_2^2 + h_2^2 - l_1^2 - h_1^2\right)}{2} + \frac{1}{2}p_2$$

with vertical intercept at $\frac{c+t\left(l_2^2+h_2^2-l_1^2-h_1^2\right)}{2}$ and slope $\frac{1}{2}$. Intuitively, when firm 2 increases its price by \$1, firm 1 responds by increasing its own by \$0.5.

- *Comparative statics of* $p_1\left(p_2\right)$:

– A marginal increase in firm 1's location, l_1 or h_1, or in firm 2's location, l_2 or h_2, yields the following changes in the above best response functions

$$\frac{\partial p_1(p_2)}{\partial l_1} = -tl_1 < 0 \quad \text{and} \quad \frac{\partial p_1(p_2)}{\partial l_2} = tl_2 > 0,$$
$$\frac{\partial p_1(p_2)}{\partial h_1} = -th_1 < 0 \quad \text{and} \quad \frac{\partial p_1(p_2)}{\partial h_2} = th_2 > 0.$$

Therefore, when firm 1 moves its position rightward, its best response function shifts downward in a parallel fashion, indicating that the firm charges less for its product. Intuitively, its position is closer to firm 2's, attenuating product differentiation, and ultimately decreasing the price that firm 1 can charge.

– In contrast, when firm 2 moves its position rightward, both firms move further away from each other, entailing more differentiated products. In this case, firm 1's best response function shifts upwards, thus indicating a higher price p_1.

– Finally, when firms only differentiate their products in one dimension (i.e., $h_1 = h_2$), this best response function simplifies to $p_1(p_2) = \frac{c+t\left(l_2^2-l_1^2\right)}{2} + \frac{1}{2}p_2$, which coincides with that in the standard Hoteling model.

- *Finding firm 2's best response function.* Operating similarly for firm 2, we have that this firm chooses price p_2 that solves

$$\max_{p_2} \ (p_2 - c)\underbrace{\left(\frac{2t(l_1+h_1-l_2-h_2) - p_1 + p_2 + t(l_2^2+h_2^2-l_1^2-h_1^2)}{2t(l_1+h_1-l_2-h_2)}\right)}_{\text{Demand, } 1-\widehat{x}}$$

Differentiating with respect to p_2, we obtain

$$2t(l_1+h_1-l_2-h_2) - p_1 + 2p_2 + t\left(l_2^2+h_2^2-l_1^2-h_1^2\right) - c = 0$$

Solving for p_2, we find firm 2's best response function

$$\begin{aligned} p_2(p_1) &= \frac{c - t(l_2^2+h_2^2-l_1^2-h_1^2) - 2t(l_1+h_1-l_2-h_2)}{2} + \frac{1}{2}p_1 \\ &= \frac{c + t\left[l_2(2-l_2) + h_2(2-h_2) - l_1(2-l_1) - h_1(2-h_1)\right]}{2} + \frac{1}{2}p_1 \end{aligned}$$

with vertical intercept at $\frac{c+t[l_2(2-l_2)+h_2(2-h_2)-l_1(2-l_1)-h_1(2-h_1)]}{2}$ and slope of $\frac{1}{2}$ that exhibit the same intuition as the positively sloped best response function of firm 1.

- *Comparative statics of* $p_2(p_1)$:

– As in the case of firm 1, a marginal increase in firm 1's location, l_1 or h_1, or in firm 2's location, l_2 or h_2, yields the following changes in the above best response functions

$$\frac{\partial p_2(p_1)}{\partial l_1} = -t(1-l_1) < 0 \quad \text{and} \quad \frac{\partial p_2(p_1)}{\partial l_2} = t(1-l_2) > 0,$$
$$\frac{\partial p_2(p_1)}{\partial h_1} = -t(1-h_1) < 0 \quad \text{and} \quad \frac{\partial p_2(p_1)}{\partial h_2} = t(1-h_2) > 0$$

Therefore, when firm 1 moves its position rightward, it becomes closer to firm 2, attenuating product differentiation, and ultimately decreasing the price that firm 2 charges (downward shift in firm 2's best response function).

– However, when firm 2 moves its position rightward, both firms move further away from each other, entailing more differentiated products. In this case, firm 2's best response function shifts upwards.

– Finally, when firms only differentiate their products in one dimension (i.e., $h_1 = h_2$), this best response function simplifies to $p_2(p_1) = \frac{c+t[l_2(2-l_2)-l_1(2-l_1)]}{2} + \frac{1}{2}p_1$, which coincides with that in the standard Hoteling model.

- *Finding equilibrium prices.* Substituting the best response function of firm 2 into that of firm 1, we obtain

$$3p_1 = 3c + t\left(l_2^2 + h_2^2 - l_1^2 - h_1^2\right) - 2t\left(l_1 + h_1 - l_2 - h_2\right)$$

Rearranging, we obtain

$$p_1^*(l_1, l_2, h_1, h_2) = c + \frac{t}{3}\left[l_2(2+l_2) + h_2(2+h_2) - l_1(2+l_1) - h_1(2+h_1)\right]$$

Substituting $p_1^*(l_1, l_2, h_1, h_2)$ into $p_2(p_1)$, we find

$$p_2^*(l_1, l_2, h_1, h_2) = c + \frac{t}{3}\left[l_2(4-l_2) + h_2(4-h_2) - l_1(4-l_1) - h_1(4-h_1)\right]$$

- *Special cases.* When both firms locate at the same position, that is, $l_1 = l_2$ and $h_1 = h_2$, equilibrium prices simplify to marginal cost pricing,

$$p_1^*(l_1, l_2, h_1, h_2) = p_2^*(l_1, l_2, h_1, h_2) = c.$$

Our results also help us examine the case of pricing under exogenous product differentiation, such as $l_1 = h_1 = 0$ and $l_2 = h_2 = 1$ where firms are located at the two extremes of the unit square. In this setting, equilibrium prices become

$$\begin{aligned} p_1^*(l_1, l_2, h_1, h_2) &= c + 2t, \text{ and} \\ p_2^*(l_1, l_2, h_1, h_2) &= c + 2t. \end{aligned}$$

From the second-stage prices, we can find the demand for firm 1, as follows

$$\begin{aligned} \widehat{x} &= \frac{\begin{bmatrix} l_2(2+l_2) + h_2(2+h_2) - l_1(2+l_1) - h_1(2+h_1) \\ -l_2(4-l_2) - h_2(4-h_2) + l_1(4-l_1) + h_1(4-h_1) \end{bmatrix}}{6(l_1 + h_1 - l_2 - h_2)} + \frac{l_1^2 + h_1^2 - l_2^2 - h_2^2}{2(l_1 + h_1 - l_2 - h_2)} \\ &= \frac{l_2(2+l_2) + h_2(2+h_2) - l_1(2+l_1) - h_1(2+h_1)}{6(l_2 + h_2 - l_1 - h_1)} \end{aligned}$$

and similarly the demand for firm 2 becomes

$$\begin{aligned} 1 - \widehat{x} &= 1 - \frac{l_2(2+l_2) + h_2(2+h_2) - l_1(2+l_1) - h_1(2+h_1)}{6(l_2 + h_2 - l_1 - h_1)} \\ &= \frac{l_2(4-l_2) + h_2(4-h_2) - l_1(4-l_1) - h_1(4-h_1)}{6(l_2 + h_2 - l_1 - h_1)} \end{aligned}$$

Therefore, second-stage profits are

$$\pi_1^*(l_1, l_2, h_1, h_2) = \frac{t\left[l_2(2+l_2) + h_2(2+h_2) - l_1(2+l_1) - h_1(2+h_1)\right]^2}{18(l_2 + h_2 - l_1 - h_1)}$$
$$\pi_2^*(l_1, l_2, h_1, h_2) = \frac{t\left[l_2(4-l_2) + h_2(4-h_2) - l_1(4-l_1) - h_1(4-h_1)\right]^2}{18(l_2 + h_2 - l_1 - h_1)}$$

which collapse to zero when both firms are exactly located at the same position, $l_1 = l_2 = h_1 = h_2$, and to $\pi_1^*(l_1, l_2, h_1, h_2) = \pi_2^*(l_1, l_2, h_1, h_2) = t$ when firms' locations are exogenously determined at $l_1 = h_1 = 0$ and $l_2 = h_2 = 1$.

(c) *First stage - Equilibrium location.* Anticipating equilibrium behavior in the second and third stages, find the equilibrium location choice of each firm in the first stage of the game.

- *Finding firm 1's best response functions.* In the first stage, firm 1 anticipates the equilibrium prices that firms charge in the second stage, and chooses its locations l_1 and h_1 to solve

$$\max_{l_1, h_1} \frac{t\left[l_2(2+l_2) + h_2(2+h_2) - l_1(2+l_1) - h_1(2+h_1)\right]^2}{18(l_2 + h_2 - l_1 - h_1)}$$

Differentiating with respect to l_1 and h_1, we find, respectively,

$$l_2(2+l_2) + h_2(2+h_2) - l_1(2+l_1) - h_1(2+h_1) - 4(1+l_1) = 0, \text{ and}$$
$$l_2(2+l_2) + h_2(2+h_2) - l_1(2+l_1) - h_1(2+h_1) - 4(1+h_1) = 0.$$

Subtracting these two first-order conditions, we obtain

$$4(1+l_1) = 4(1+h_1)$$

which implies that, in equilibrium, $l_1 = h_1$. Inserting this property into any of the above first-order conditions, yields

$$2\left(l_1^2 - 2\right) = l_2(2+l_2) + h_2(2+h_2)$$

which rearranging, we find firm 1's best response function in the first stage, as follows,

$$l_1(l_2, h_2) = h_1(l_2, h_2) = \sqrt{\frac{4 + l_2(2+l_2) + h_2(2+h_2)}{2}}.$$

- *Finding firm 2's best response functions.* Similarly, firm 2 chooses locations l_2 and h_2 to solve

$$\max_{l_2, h_2} \frac{t\left[l_2(4-l_2) + h_2(4-h_2) - l_1(4-l_1) - h_1(4-h_1)\right]^2}{18(l_2 + h_2 - l_1 - h_1)}$$

Differentiating with respect to l_2 and h_2, we find, respectively,

$$l_2(4-l_2) + h_2(4-h_2) - l_1(4-l_1) - h_1(4-h_1) - 2(2-l_2) = 0, \text{ and}$$
$$l_2(4-l_2) + h_2(4-h_2) - l_1(4-l_1) - h_1(4-h_1) - 2(2-h_2) = 0.$$

Subtracting these two first-order conditions, yields

$$2(2-l_2)=2(2-h_2)$$

which entails that, equilibrium, $l_2=h_2$. Inserting this result into any of the above first-order conditions, we obtain

$$2l_2^2-10l_2+4+l_1(4-l_1)+h_1(4-h_1)=0$$

which we can rearrange to yield firm 1's best response function in the first stage, as follows,

$$l_2(l_1,h_1)=h_2(l_1,h_1)=\frac{5+\sqrt{25-2\left[4+l_1(4-l_1)+h_1(4-h_1)\right]}}{2}.$$

- *Finding equilibrium location.* Inserting $l_2=h_2$ into the first-order condition of firm 1, yields

$$l_1^2-2=l_2(2+l_2)$$

Similarly, substituting $l_1=h_1$ into the first-order condition of firm 2, yields

$$l_2^2-5l_2+2+l_1(4-l_1)=0$$

Further substituting $l_1=\sqrt{2+l_2(2+l_2)}$ into the above expression, we find that

$$l_2^2-5l_2+2+\sqrt{2+l_2(2+l_2)}\left(4-\sqrt{2+l_2(2+l_2)}\right)=0$$

which can be rearranged into

$$33l_2^2-32l_2-32=0$$

or simplified as

$$l_2=\frac{32+\sqrt{32^2+(4\times 33\times 32)}}{2\times 33}=\frac{4\left(4+\sqrt{82}\right)}{33}\approx 1.58.$$

This result indicates that we are in a corner solution. Therefore, firms differentiate their products as much as possible. In the interval $[0,1]^2$, they locate at the vertexes of the unit square, $l_1=h_1=0$ and $l_2=h_2=1$. From our above discussion, we know that these positions yield equilibrium prices

$$p_1^*(0,1,0,1)=p_2^*(0,1,0,1)=c+2t,$$

and equilibrium profits

$$\pi_1^*(0,1,0,1)=\pi_2^*(0,1,0,1)=t.$$

(d) *Comparison.* Compare your equilibrium location and profit with those in the standard Hoteling model where firms differentiate their products in just one dimension.

- When firms differentiate their products in only one dimension, we showed that they locate at the endpoints of the line, $l_1 = 0$ and $l_2 = 1$, thus showing similar incentives as in the two-dimension setting considered in this exercise. Their equilibrium prices in that context become

  $$p_1^*(0,1) = p_2^*(0,1) = c + t,$$

  which are lower than when firms differentiate in two dimensions, $c + 2t$. Finally, their equilibrium profits in the standard Hotelling model (with differentiation in only one dimension) are

  $$\pi_1^*(0,1) = \pi_2^*(0,1) = \frac{t}{2}$$

  which are also lower than when firms have two dimensions to differentiate their products. Intuitively, the additional dimension to differentiate their products softens price competition and increases firm profits.