EconS 503 - Final Exam
June 2019 - Answer Key

1. **Cheap talk vs. Delegation, based on Dessein (2002).**[1] Consider the Crawford and Sobel (1982) cheap talk game, where an expert privately observes the realization of parameter $\theta \sim U[0,1]$, sends a message $m \in [0,1]$ to the uninformed politician who updates his beliefs about θ, and responds with a policy $p \in [0,1]$. The politician's utility function is given by $u_P(p,\theta) = -(p-\theta)^2$, while the expert's utility function is $u_E(p,\theta) = -(p-(\theta+\delta))^2$, where parameter $\delta > 0$ represents the divergence between the preferences of the politician and the expert (also known as the expert's bias parameter).

   Consider now an alternative communication setting, where the politician delegates the decision of policy p to the expert. Intuitively, the politician seeks to minimize the information loss from the cheap talk setting, at the cost of having the expert implementing a policy that considers his own bias. We next examine which policy the expert implements in under delegation, and in which cases the politician is better off delegating the decision to the expert rather than operating in the standard cheap talk context.

   (a) *Delegation.* Which policy the expert chooses under delegation? Which are the expected utilities for expert and politician?

   - After observing the realization of θ, the expert chooses the policy p that maximizes his utility, that is,

   $$\max_{p \geq 0} \; -(p-(\theta+\delta))^2$$

   Differentiating with respect to p, and solving for p, we obtain $p^{Del} = \theta + \delta$, where the superscript Del denotes "delegation."

   - Since the expert chooses policy $p^{Del} = \theta + \delta$ under delegation, his utility becomes

   $$u_E^{Del} = -(p^{Del} - (\theta+\delta))^2 = -(\underbrace{\theta+\delta}_{p^{Del}} - (\theta+\delta))^2 = 0$$

   Intuitively, he chooses his ideal policy $\theta + \delta$, so his utility is maximized. However, the politician's expected utility from p^{Del} becomes

   $$u_P^{Del} = -E\left[(p^{Del} - \theta)^2\right] = -E[(\underbrace{\theta+\delta}_{p^{Del}} - \theta)^2] = -\delta^2.$$

[1] Dessein, W. (2002) "Authority and communication in organizations." *Review of Economic Studies*, 69, pp. 811–38.

(b) *Do-it-yourself.* Consider now that the receiver (politician) ignores the sender's messages, which is often known as if the politician took a "do-it-yourself" approach. Find the policy he responds with, and his expected utility in this setting. Under which value of the preference divergence parameter δ the politician prefers the "do-it-yourself" approach than the delegation approach studied in part (a) of the exercise?

- Since the politician ignores the messages from the expert in this setting, he chooses the policy that maximizes his expected utility, as follows

$$\max_{p\geq 0} \; -E\left[(p-\theta)^2\right]$$

$$\text{subject to } \; \theta \sim U[0,1]$$

Then, the politician solves the following expected utility maximization problem:

$$\begin{aligned}\max_{p\geq 0} \; -E\left[(p-\theta)^2\right] &= -\int_0^1 (p-\theta)^2 f(\theta)\, d\theta \\ &= \frac{1}{3}\left[(p-\theta)^3\right]_0^1 \\ &= \frac{1}{3}\left[(p-1)^3 - p^3\right] \\ &= -\frac{1}{3}\left(3p^2 - 3p + 1\right)\end{aligned}$$

Differentiating the politician's expected utility with respect to policy p, yields

$$\frac{dEU[p]}{dp} = 1 - 2p$$

Solving for $\frac{dEU[p]}{dp} = 0$, the politician chooses a policy $p^{DIY} = \frac{1}{2}$ that maximizes his expected utility, where the superscript DIY denotes the "do-it-yourself" approach. (Note that the above expected utility is concave in policy p since $\frac{d^2EU[p]}{dp^2} = -2 < 0$; confirming that $p^{DIY} = \frac{1}{2}$ maximizes the politician's expected utility.)

- The politician's expected utility in the "do-it-yourself" approach then becomes

$$u_P^{DIY} = -E\left[\left(\frac{1}{2} - \theta\right)^2\right] = -Var(\theta) = -\frac{1}{12}.$$

where the second equality follows because the expected value of θ is $\frac{1}{2}$ given that $\theta \sim U[0,1]$; and the last equality follows from the property that the variance of a uniform distribution $U[a,b]$ is $-\frac{1}{12}(b-a)^2$, which in the case of $U[0,1]$ yields a variance of $-\frac{1}{12}(1-0)^2 = -\frac{1}{12}$. Then, the politician prefers the "do-it-yourself" approach over delegation if $u_P^{DIY} \geq u_P^{Del}$, entailing

$$-\frac{1}{12} \geq -\delta^2$$

which simplifies to $\delta \geq \frac{1}{2\sqrt{3}} \simeq 0.28$. Intuitively, when the expert's preference bias δ is sufficiently strong, the politician prefers to ignore the expert's messages and choose a policy that coincides with the expected value of random parameter θ. Otherwise, the politician prefers to delegate the policy decision to the expert since his preferences and the expert's are relatively similar.

(c) From the previous exercises, we found that expected utilities in the Crawford and Sobel (1982) cheap talk game are $u_P^{CT} = -\frac{1}{12N^2} - \frac{\delta^2(N^2-1)}{3}$ for the politician, where the superscript CT indicates "cheap talk"; and $u_E^{CT} = u_P^{CT} - \delta^2$ for the expert. In addition, recall that N denotes the number of partitions in the $[0,1]$ interval, or different messages that the expert sends; where this N-partition equilibrium can be sustained if the preference divergence parameter is sufficiently small, that is, $\delta \leq \frac{1}{2N(N-1)}$. Show that the expert prefers delegation rather than sending messages to the politician in the cheap talk game, and that this result holds under all parameter conditions. Then show that the politician prefers delegation only if the expert's bias, as captured by δ, is sufficiently small.

- *Expert.* The expert's expected utility from delegation is $u_E^{Del} = 0$, while his expected utility from playing the cheap talk game is negative, that is,

$$u_E^{CT} = \underbrace{-\frac{1}{12N^2} - \frac{\delta^2(N^2-1)}{3}}_{u_P^{CT}} - \delta^2 < 0.$$

  He then prefers delegation regardless of the number of partitions, N, and independently on his preference bias, δ.

- *Politician.* The politician's expected utility from delegation is $u_P^{Del} = -\delta^2$, which exceeds his expected expected utility from playing the cheap talk game

$$u_P^{CT} = -\frac{1}{12N^2} - \frac{\delta^2(N^2-1)}{3},$$

  if

$$-\delta^2 \geq -\frac{1}{12N^2} - \frac{\delta^2(N^2-1)}{3}$$

  After rearranging, we obtain

$$\frac{1}{12N^2} \geq \delta^2 \left(\frac{4-N^2}{3}\right)$$

  and solving for preference divergence parameter, δ, we find

$$\delta \leq \frac{1}{2N\sqrt{4-N^2}}$$

  We can now evaluate condition $\delta \leq \frac{1}{2N\sqrt{4-N^2}}$ at different numbers of partitions:

  - When $N = 1$ (meaning that the pooling equilibrium arises, which entails uninformative messages for the politician), we obtain that $\delta \leq \frac{1}{2\sqrt{3}} \simeq 0.28$. In this case, the politician prefers delegation to receiving a cheap-talk message from the expert if the bias parameter is sufficiently small, that is, $\delta \leq 0.28$.

  - In contrast, for any larger number of partitions, $N \geq 2$, we find that $\delta \leq +\infty$. In words, delegation is optimal for the politician when there is an informative PBE with two or more partitions in the cheap-talk game for all values of the divergence parameter δ.
- Therefore, delegation is preferred over cheap talk for the politician under relatively large parameter values because the welfare loss caused by self-interested communication in the cheap talk game is larger than the cost that the politician experiences from letting the expert choose a relatively biased policy under delegation.

2. **Reputation on the job, based on Bénabou and Tirole (2006).**[2] Consider a continuum of risk-neutral agents, each of whom can make a decision, $a = \{0, 1\}$, where $a = 1$ corresponding to taking a prosocial action (such as donation to charities, community services, or working assiduously in a group project, etc.), and $a = 0$ corresponding to not taking such an action. Those agents who participate in prosocial activities incur a cost of c but receive a financial reward of w (such as free meals, tax credits, gift cards for volunteering work) and an intrinsic satisfaction of v that is privately observable by every agent. However, the distribution of $v \in [0, 1]$, $G(v)$, with density $g(v)$, is common knowledge among players. We consider participation decisions to be monotonic in v, such that exists a critical agent with intrinsic satisfaction v^* who is indifferent between participation or not, that is

$$a = \begin{cases} 1 & v \geq v^* \\ 0 & v < v^*. \end{cases}$$

In this context, each participant enjoys a positive utility from good reputation, defined as the difference between the expected value of intrinsic satisfaction among those who participate and the intrinsic satisfaction of the critical agent v^*. On the contrary, each non-participant receives a disutility from bad reputation, defined as the difference between the expected value of intrinsic satisfaction among those who do not participate and the intrinsic satisfaction of the critical agent v^*. In this exercise, we analyze whether giving out more or less financial reward w would impact agents' willingness to participate in socially desirable activities.

   (a) Assume that each agent places a common weight of μ on reputation (that is, $\mu = \mu_i = \mu_j$), what is the typical agent i's utility from taking or not taking the prosocial action?.

   - Agent i's utility from participation is

$$\begin{aligned} U(1) &\equiv U(a=1) \\ &= w - c + v + \mu\left(E\left[v|v \geq v^*\right] - v^*\right) \end{aligned}$$

   which is the sum of his financial reward w (for instance, wage), intrinsic valuation v, and good reputation $\mu\left(E\left[v|v \geq v^*\right] - v^*\right)$ for being in the mass of agents who participate in pro-social activities, minus the cost of action c.

[2] Bénabou R. and Tirole J. (2006). Incentives and Prosocial Behavior. *American Economic Review*, 96(5), 1652-78.

- Agent i's utility from non-participation is

$$\begin{aligned} U(0) &\equiv U(a=0) \\ &= \mu\left(E\left[v|v<v^*\right]-v^*\right) \end{aligned}$$

which comes from his bad reputation $\mu\left(E\left[v|v<v^*\right]-v^*\right)$ for being in the mass of agents who do not participate in pro-social activities.

(b) Solve for the critical agent v^*.

- Agent i would take the prosocial action a when his utility from participation exceeds that of non-participation, that is, $U(1) \geq U(0)$. Defining this utility diference as $\Delta \equiv U(1) - U(0)$, we obtain

$$\begin{aligned} \Delta &= \overbrace{w-c+v+\mu\left(E\left[v|v\geq v^*\right]-v^*\right)}^{U(1)} - \overbrace{\mu\left(E\left[v|v<v^*\right]-v^*\right)}^{U(0)} \\ &= w-c+v+\mu\left(E\left[v|v\geq v^*\right]-E\left[v|v<v^*\right]\right) \\ &= w-c+v+\mu\left(\underbrace{\frac{1}{1-G(v^*)}\int_{v^*}^{1} vdG(v)}_{\text{Term A}} - \underbrace{\frac{1}{G(v^*)}\int_{0}^{v^*} vdG(v)}_{\text{Term B}}\right) \end{aligned}$$

Evaluating Term A, and integrating by parts,

$$\begin{aligned} \frac{1}{1-G(v^*)}\int_{v^*}^{1} vdG(v) &= \frac{1}{1-G(v^*)}\left[\left[vG(v)\right]_{v^*}^{1} - \int_{v^*}^{1} G(v)\,dv\right] \\ &= \frac{1}{1-G(v^*)}\left[1-v^*G(v^*) - \int_{v^*}^{1} G(v)\,dv\right] \\ &= \frac{1}{1-G(v^*)}\left[v^*\left[1-G(v^*)\right] + \int_{v^*}^{1} \left[1-G(v)\right]dv\right] \\ &= v^* + \int_{v^*}^{1} \frac{1-G(v)}{1-G(v^*)}dv \end{aligned}$$

which means that the good reputational effect of taking prosocial action is above the intrinsic valuation of the critical agent v^*.
Evaluating Term B, and integrating by parts,

$$\begin{aligned} \frac{1}{G(v^*)}\int_{0}^{v^*} vdG(v) &= \frac{1}{G(v^*)}\left[\left[vG(v)\right]_{0}^{v^*} - \int_{0}^{v^*} G(v)\,dv\right] \\ &= \frac{1}{G(v^*)}\left[v^*G(v^*) - \int_{0}^{v^*} G(v)\,dv\right] \\ &= v^* - \int_{0}^{v^*} \frac{G(v)}{G(v^*)}dv \end{aligned}$$

which means that the bad reputational effect of not taking prosocial action is less than the intrinsic valuation of the critical agent v^*.

Summing up the two terms, the critical agent, who is indifferent between participation or not, derives zero net utility from participation, that is, $\Delta = 0$, yielding

$$w - c + v^* + \mu \left(\underbrace{v^* + \int_{v^*}^{1} \frac{1 - G(v)}{1 - G(v^*)} dv}_{\text{Term A}} - \underbrace{\left(v^* - \int_{0}^{v^*} \frac{G(v)}{G(v^*)} dv \right)}_{\text{Term B}} \right) = 0$$

which simplifies to

$$w + v^* + \mu \left(\underbrace{\int_{v^*}^{1} \frac{1 - G(v)}{1 - G(v^*)} dv + \int_{0}^{v^*} \frac{G(v)}{G(v^*)} dv}_{h(v^*)} \right) = c \qquad (\Delta_0)$$

which means that his cost c from taking a prosocial action (on the right-hand side of the above equality) offsets his benefit of doing so (on the left-hand side), which is the sum of his wage, intrinsic satisfaction, and reputation differential $h(v^*)$ (intuitively understood as the gain in good reputation and the elimination of bad reputation).

(c) *Comparative Statics.* Does giving out a higher financial reward w induce more agents to participate? For simplicity, evaluate your answer in the case of a uniformly distributed v, where $G(v) = v$. [*Hint*: Consider the labor supply function, $L(w) = 1 - G[v^*(w)]$, which is the mass of agents who participate in the prosocial activities as induced by wage w.]

- Totally differentiating expression Δ_0 with respect to w, and by the Implicit Function Theorem,

$$\frac{\partial v^*}{\partial w} = -\frac{\frac{\partial \Delta_0}{\partial w}}{\frac{\partial \Delta_0}{\partial v^*}} = -\frac{1}{1 + \mu (h'(v^*))}$$

- Differentiating the labor supply function with respect to w, and by the chain rule,

$$\frac{\partial L(w)}{\partial w} = -g[v^*(w)] \frac{\partial v^*}{\partial w} = \frac{g(v^*)}{1 + \mu (h'(v^*))}$$

which is positive if $1 + \mu h'(v^*) > 0$.

- Next, consider the reputation differential $h(v^*)$, where

$$\begin{aligned}
h'(v^*) &= \frac{\partial h(v^*)}{\partial v^*} \\
&= \frac{\partial}{\partial v^*}\left[\int_{v^*}^{1} \frac{1-G(v)}{1-G(v^*)}dv + \int_0^{v^*} \frac{G(v)}{G(v^*)}dv\right] \\
&= -\frac{-1}{1-G(v^*)}\left[1-G(v^*)\right]g(v^*) + \frac{g(v^*)}{\left[1-G(v^*)\right]^2}\int_{v^*}^{1}\left[1-G(v)\right]dv \\
&\quad + \frac{1}{G(v^*)}G(v^*)\,g(v^*) - \frac{g(v^*)}{\left[G(v^*)\right]^2}\int_0^{v^*} G(v)\,dv \\
&= 2g(v^*) + \frac{g(v^*)}{1-G(v^*)}\int_{v^*}^{1}\frac{1-G(v)}{1-G(v^*)}dv - \frac{g(v^*)}{G(v^*)}\int_0^{v^*}\frac{G(v)}{G(v^*)}dv
\end{aligned}$$

Evaluating this expression at a uniform distribution $G(v) = v$, where $g(v) = 1$, we find

$$\begin{aligned}
h'(v^*) &= 2 + \frac{1}{(1-v^*)^2}\int_{v^*}^{1}(1-v)\,dv - \frac{1}{(v^*)^2}\int_0^{v^*} v dv \\
&= 2 + \frac{1}{(1-v^*)^2}\left[1 - v^* - \frac{1}{2} + \frac{(v^*)^2}{2}\right] - \frac{1}{(v^*)^2}\left[\frac{v^2}{2}\right]_0^{v^*} \\
&= 2 + \frac{1}{1-v^*} - \frac{1+v^*}{2(1-v^*)} - \frac{1}{2} \\
&= \frac{3}{2} + \frac{2-1-v^*}{2(1-v^*)} \\
&= \frac{3-3v^*+1-v^*}{2(1-v^*)} \\
&= 2 > 0
\end{aligned}$$

Therefore, since $1 + \mu h'(v^*) = 1 + 2\mu > 0$, we have that $\frac{\partial L(w)}{\partial w} > 0$, implying that a higher wage would induce more agents to participate in prosocial activities.

(d) Evaluate the intrinsic satisfaction of the critical agent v^*. How does a cost increase affect him? What if the agent places a higher emphasis on reputation (that is, μ increases)?

- Expression Δ_0 for the critical agent, evaluated at the uniform distribution $G(v) = v$, simplifies to

$$\begin{aligned}
c &= w + v^* + \mu\left(\int_{v^*}^{1}\frac{1-v}{1-v^*}dv + \int_0^{v^*}\frac{v}{v^*}dv\right) \\
&= w + v^* + \mu\left(\frac{1}{1-v^*}\int_{v^*}^{1}(1-v)\,dv + \frac{1}{v^*}\int_0^{v^*} vdv\right) \\
&= w + v^* + \mu\left(1 - \frac{1+v*}{2} + \frac{v^*}{2}\right) \\
&= w + v^* + \frac{\mu}{2}
\end{aligned}$$

Rearranging, we find

$$v^* = c - w - \frac{\mu}{2}$$

We are now ready to do comparative statics of v^*:

- If it becomes more costly to take prosocial action (higher c), v^* increases so that the critical agent needs a higher intrinsic satisfaction to participate in prosocial activities, thus yielding a lower participation rate.
- In contrast, if agents place a higher emphasis on reputation (higher μ), the critical agent now needs a lower intrinsic satisfaction to participate, thus yielding a higher participation rate.
- Similarly, if wage increases (higher w), the critical agent needs a lower intrinsic satisfaction to participate, thus yielding a higher participation rate.

3. **Emission fees and mechanisms.** Consider an industry with N polluting firms producing a homogenous good. Let the profit function of firm i be $\pi_i(q_i) = \ln q_i$, which is increasing and concave in its pollutants q_i. The social cost from pollution is

$$C(q_1, \cdots, q_n) = \sum_{i=1}^{n} \frac{\gamma_i}{2} q_i^2,$$

which is also increasing but convex in the pollutants q_i emitted by firm i. Finally, a regulator (e.g., government agency) considers the following welfare function

$$W(q_1, \cdots, q_n) = \sum_{i=1}^{n} \pi_i(q_i) - C(q_1, \cdots, q_n)$$

(a) *Complete information.* Assume that the regulator can observe pollution levels and sets an emission fee t_i per unit of emissions. Find the following: (i) firm i's profit-maximizing pollution level as a function of fee t_i, $q_i(t_i)$; (ii) the socially optimal pollution from firm i, q_i^{SO}; and (iii) the emission fee t_i that induces firm i to produce q_i^{SO}, i.e., the fee t_i that solves $q_i(t_i) = q_i^{SO}$.

- *Equilibrium pollution.* Firm i solves

$$\max_{q_i \geq 0} \pi_i(q_i) - t_i q_i = \ln q_i - t_i q_i$$

Differentiating with respect to q_i, we find

$$\frac{1}{q_i} = t_i$$

which, after solving for q_i, yields

$$q_i(t_i) = \frac{1}{t_i}.$$

- *Socially optimal pollution.* Differentiating with respect to q_i in the social welfare function, yields
$$\frac{1}{q_i} = \gamma_i q_i$$
which solving for q_i yields a socially optimal pollution of
$$q_i^{SO} = \frac{1}{\sqrt{\gamma_i}}.$$
- *Emission fee.* Hence, the emission fee t_i should be set to induce every firm i to produce the socially optimal amount of pollution, that is, $q_i(t_i) = q_i^{SO}$
$$\frac{1}{t_i} = \frac{1}{\sqrt{\gamma_i}}$$
which yields an emission fee
$$t_i = \sqrt{\gamma_i}.$$
Intuitively, the emission fee is set to make firm i's marginal profit from one more unit of pollution, t_i, to coincide with its marginal social cost, $\sqrt{\gamma_i}$.

(b) *Incomplete information.* Assume that the level of pollution is unobservable to the regulator but observable among all firms. Then, the regulator can devise a circular monitoring mechanism, in which firm i reports the observed pollution level of firm $i-1$, $\overline{q}_{i-1}$, firm $i-1$ reports the observed pollution of firm $i-2$, $\overline{q}_{i-2}$, and firm 1 reports that of firm n, $\overline{q}_n$. This allows the regulator to set an emission fee per unit of pollution
$$t_i = \frac{\partial C\left(\overline{q}_i, q_{-i}\right)}{\partial q_i},$$
where $\overline{q}_i$ denotes firm i's pollution (reported by firm $i+1$), and q_{-i} represents the true pollution level of all other firms. In addition, firm i faces a penalty of $\left(\overline{q}_{i-1} - q_{i-1}\right)^2$ for misporting his neighbor's pollution level not at q_{i-1}.

1. Will firm i misreport the output of firm $i-1$? Why or why not?
   - No, because firm i will face a penalty proportional to his misreporting, which is given by $\left(\overline{q}_{i-1} - q_{i-1}\right)^2$. As a consequence, firm i truly reports what it has observed from firm $i-1$ in order to avoid penalties. This applies to every firm $i \in \{1, \cdots, n\}$.
2. Write down firm i's profit-maximization problem and solve for its optimal output.
   - To find the pollution level that maximizes firm i's profit, we fix every firm j's report at truthtelling, $\overline{q}_j = q_j$, since firms have no incentives to misreport (see part a). Firm i chooses q_i to solve
$$\max_{q_i \geq 0} \quad \pi_i\left(q_i\right) - t_i q_i - \left(\overline{q}_{i-1} - q_{i-1}\right)^2$$
where the first term denotes firm i's profits, the second represents the emission fee payments, and the third captures the penalty from misreporting firm $i-1$'s pollution.

Since fee t_i is, by definition, $t_i = \frac{\partial C(\overline{q}_i, q_{-i})}{\partial q_i}$, and every firm i has no incentives to misreport (see part a), we can fix firms' reports at truthtelling, $\overline{q}_i = q_i$, the above problem becomes

$$\max_{q_i \geq 0} \ \pi_i(q_i) - \frac{\partial C(q_i, q_{-i})}{\partial q_i} q_i - (q_{i-1} - q_{i-1})^2$$

Differentiating with respect to q_i, yields

$$\frac{1}{q_i} = \gamma_i q_i$$

Therefore, every firm i chooses a pollution level $q_i = \frac{1}{\sqrt{\gamma_i}}$ which coincides with the socially optimal pollution q_i^{SO} found in part (a).

3. Find the tax revenue generated by the mechanism, and the social cost of pollution.
   - Total tax revenue is

$$\sum_{i=1}^{n} t_i q_i = \sum_{i=1}^{n} \sqrt{\gamma_i} \frac{1}{\sqrt{\gamma_i}} = n$$

   - The social cost of pollution, evaluated at the equilibrium output, $(q_1, \cdots, q_n)$, is

$$C(q_1, \cdots, q_n) = \sum_{i=1}^{n} \frac{\gamma_i}{2} \left(\frac{1}{\sqrt{\gamma_i}} \right)^2 = \frac{n}{2}.$$

4. **Designing Optimal Taxation using Mechanisms.** Consider a government needing to raise a fixed sum $\$S$ through income tax. There are two types of workers, high productivity (H) and low productivity (L), and the output (gross income) produced by each is given by

$$q^k = \theta^k e^k, \text{ where } k = H, L$$

where e^k is the amount of effort exerted by a worker of type k and the productivity parameter satisfies $\theta^H > \theta^L$. Hence, for a given effort level, the high-productivity worker generates a larger amount of output than the low-productivity worker. The utility function of a worker with type k is

$$v^k = q^k - t^k - g(e^k)$$

where t^k is the tax on a worker of type k, and $g(\cdot)$ is a strictly increasing and convex function in effort, i.e., $g' > 0$ and $g'' > 0$. The government has no interest in the inequality of utility outcomes and so just seeks to maximize the expected social welfare

$$W = pv^H + [1 - p]v^L$$

where p is the proportion of H-type workers.

(a) What is the government's budget constraint?

- If S is the amount to be raised per person in the economy the government's budget constraint is

$$S \leq pt^H + [1-p]t^L$$

which must hold with equality at the optimum. Indeed, if the sum of taxes exceeded S you could increase individual utility and therefore social welfare by cutting taxes. Hence, we can write the constraint as

$$S = pt^H + [1-p]t^L.$$

(b) *Complete information.* If the government was perfectly informed about the worker's type, find the socially optimal taxes, and the associated output levels.

- If the government was perfectly informed about the worker's type, the government maximizes each type's utility subject to its budget constraint and participation constraints of both types of worker. That is, since $q^k = \theta^k e^k$, then $e^k = \frac{q^k}{\theta^k}$, the planner's problem when observing a type-θ^k worker is

$$\max_{q^k \geq 0} \; q^k - t^k - g\left(\frac{q^k}{\theta^k}\right)$$

$$\text{subject to} \quad S = t^k$$

or, inserting the constraint into the objective function,

$$\max_{q^k \geq 0} \; q^k - S - g\left(\frac{q^k}{\theta^k}\right)$$

Taking FOCs with respect to q^k for type k's utility function, we obtain

$$1 - \frac{1}{\theta^k} g'\left(\frac{q^k}{\theta^k}\right) = 0$$

or, more compactly,

$$\theta^k = g'\left(\frac{q^k}{\theta^k}\right)$$

(c) *Parametric example (Complete information).* Assume that the cost of effort function is $g(e^k) = (e^k)^2$, so its derivatives are $g' = 2e^k \geq 0$ and $g'' = 2 > 0$; as required. Evaluate the FOCs found in part (b) for the complete information context assuming that productivity parameters are $\theta^H = 1$ and $\theta^L = \frac{1}{2}$. Find the optimal values of q^H and q^L.

- *Output.* In the case of a high-productivity worker, the FOC found in part (b), $\theta^H = g'\left(\frac{q^H}{\theta^H}\right) = 0$, yields

$$1 = 2\frac{q^H}{1} \Longleftrightarrow q^H = \frac{1}{2}$$

and in the case of a low-productivity worker, we obtain

$$\frac{1}{2} = 2\frac{q^L}{\frac{1}{2}} \Longleftrightarrow q^L = \frac{1}{8}$$

- *Effort.* Hence, optimal effort levels, $e^k = \frac{q^k}{\theta^k}$, are

$$e^H = \frac{1/2}{1} = \frac{1}{2} \quad \text{and} \quad e^L = \frac{1/8}{1/2} = \frac{1}{4}$$

- *Taxes.* Substituting the optimal output levels into each type's participation constraints, we obtain

$$y^H = g(e^H) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$y^L = g(e^L) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

Since $y^k \equiv q^k - t^k$ by definition, then taxes are $t^k \equiv q^k - y^k$, which helps us find the associated optimal taxes under complete information

$$\begin{aligned} t^H &= q^H - y^H = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}, \text{ and} \\ t^L &= q^L - y^L = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}. \end{aligned}$$

Intuitively, the output, effort, and tax on the high-productivity worker is higher than on the low-type of worker.

(d) *Incomplete information.* Assuming that the government cannot observe the worker's type, write the government's objective function in terms of q^H, q^L, p, and S.

- By definition, the objective function is $pv^H + [1-p]v^L$. Substituting the utility function of each type of worker, yields

$$p[q^H - t^H - g(e^H)] + (1-p)\left[q^L - t^L - g(e^L)\right]$$

which may be rewritten as

$$p[q^H - g(e^H)] + (1-p)\left[q^L - g(e^L)\right] - \underbrace{\left[pt^H + (1-p)\,t^L\right]}_{S}$$

Substituting the government's budget constraint in the last term, we obtain

$$p[q^H - g(e^H)] + (1-p)\left[q^L - g(e^L)\right] - S.$$

Finally, since $e^k = \frac{q^k}{\theta^k}$, we can express the government's objective function function of output and taxes alone (without including effort levels explicitly).

$$p\left[q^H - g\left(\frac{q^H}{\theta^H}\right)\right] + (1-p)\left[q^L - g\left(\frac{q^L}{\theta^L}\right)\right] - S$$

(e) Using the government's objective function you identified in part (d), write down the government's optimization problem. [*Hint*: You can use $y^k \equiv q^k - t^k$ to simplify your calculations. In that case, recall that the set of choice variables for the social planner changes from (q^H, q^L, t^H, t^L) to (q^H, q^L, y^H, y^L).]

- The government's problem is to maximize the expected welfare found in part (c), see expression (4), by choice of taxes t^H and t^L. In addition, since $q^k = \theta^k e^k$ we can solve for e^k obtaining $e^k = \frac{q^k}{\theta^k}$, which helps us express the government's problem as a function of output and taxes alone (without including effort levels explicitly).

$$\max_{q^H, q^L, t^H, t^L} \quad p\left[q^H - g\left(\frac{q^H}{\theta^H}\right)\right] + (1-p)\left[q^L - g\left(\frac{q^L}{\theta^L}\right)\right] - S$$

subject to

$$q^L - t^L - g\left(\frac{q^L}{\theta^L}\right) \geq q^H - t^H - g\left(\frac{q^H}{\theta^L}\right) \tag{IC$_L$}$$

$$q^H - t^H - g\left(\frac{q^H}{\theta^H}\right) \geq q^L - t^L - g\left(\frac{q^L}{\theta^H}\right) \tag{IC$_H$}$$

$$q^L - t^L - g\left(\frac{q^L}{\theta^L}\right) \geq 0 \tag{PC$_L$}$$

$$q^H - t^H - g\left(\frac{q^H}{\theta^H}\right) \geq 0 \tag{PC$_H$}$$

For compactness, we denote $y^k \equiv q^k - t^k$ in the IC and PC constraints, which helps us simplify the problem as follows (note that the choice variables now changed, as taxes are implicitly in y^k).

$$\max_{q^H, q^L, y^H, y^L} \quad p\left[q^H - g\left(\frac{q^H}{\theta^H}\right)\right] + (1-p)\left[q^L - g\left(\frac{q^L}{\theta^L}\right)\right] - S$$

subject to

$$y^L - g\left(\frac{q^L}{\theta^L}\right) \geq y^H - g\left(\frac{q^H}{\theta^L}\right) \tag{IC$_L$}$$

$$y^H - g\left(\frac{q^H}{\theta^H}\right) \geq y^L - g\left(\frac{q^L}{\theta^H}\right) \tag{IC$_H$}$$

$$y^L - g\left(\frac{q^L}{\theta^L}\right) \geq 0 \tag{PC$_L$}$$

$$y^H - g\left(\frac{q^H}{\theta^H}\right) \geq 0 \tag{PC$_H$}$$

- As in similar screening models, note that PC_L must hold with equality (no information rents for the low-type worker), entailing that

$$y^L = g\left(\frac{q^L}{\theta^L}\right)$$

and, similarly, IC_H holds with equality, implying that

$$y^H - g\left(\frac{q^H}{\theta^H}\right) = \underbrace{g\left(\frac{q^L}{\theta^L}\right)}_{y^L} - g\left(\frac{q^L}{\theta^H}\right).$$

Rearranging this expression, we obtain

$$y^H = g\left(\frac{q^H}{\theta^H}\right) + g\left(\frac{q^L}{\theta^L}\right) - g\left(\frac{q^L}{\theta^H}\right).$$

Inserting the expressions of y^L and y^H that we found above into the maximization problem, we obtain the following program (where we already removed PC_L and IC_H, leaving us with only two constraints: PC_H and IC_L):

$$\max_{q^H, q^L} \quad p\left[q^H - g\left(\frac{q^H}{\theta^H}\right)\right] + (1-p)\left[q^L - g\left(\frac{q^L}{\theta^L}\right)\right] - S$$

subject to

$$\underbrace{g\left(\frac{q^L}{\theta^L}\right)}_{y^L} - g\left(\frac{q^L}{\theta^L}\right) \geq \underbrace{g\left(\frac{q^H}{\theta^H}\right) + g\left(\frac{q^L}{\theta^L}\right) - g\left(\frac{q^L}{\theta^H}\right)}_{y^H} - g\left(\frac{q^H}{\theta^L}\right) \qquad (\text{IC}_L)$$

$$\underbrace{g\left(\frac{q^H}{\theta^H}\right) + g\left(\frac{q^L}{\theta^L}\right) - g\left(\frac{q^L}{\theta^H}\right)}_{y^H} - g\left(\frac{q^H}{\theta^H}\right) \geq 0 \qquad (\text{PC}_H)$$

which simplifies to

$$\max_{q^H, q^L} \quad p\left[q^H - g\left(\frac{q^H}{\theta^H}\right)\right] + (1-p)\left[q^L - g\left(\frac{q^L}{\theta^L}\right)\right] - S$$

subject to

$$g\left(\frac{q^L}{\theta^H}\right) + g\left(\frac{q^H}{\theta^L}\right) \geq g\left(\frac{q^H}{\theta^H}\right) + g\left(\frac{q^L}{\theta^L}\right) \qquad (\text{IC}_L)$$

$$g\left(\frac{q^L}{\theta^L}\right) \geq g\left(\frac{q^L}{\theta^H}\right). \qquad (\text{PC}_H)$$

In addition, the program now only has two choice variables (q^H and q^L) rather than four. Letting λ and μ be the Lagrange multipliers for constraints IC_L and PC_H, respectively, we obtain that

$$\begin{aligned}\mathcal{L} \;=\; & p\left[q^H - g\left(\frac{q^H}{\theta^H}\right)\right] + (1-p)\left[q^L - g\left(\frac{q^L}{\theta^L}\right)\right] - S \\ & +\lambda\left[g\left(\frac{q^L}{\theta^H}\right) + g\left(\frac{q^H}{\theta^L}\right) - g\left(\frac{q^H}{\theta^H}\right) - g\left(\frac{q^L}{\theta^L}\right)\right] \\ & +\mu\left[g\left(\frac{q^L}{\theta^L}\right) - g\left(\frac{q^L}{\theta^H}\right)\right]\end{aligned}$$

(f) Find the solution to the government's problem in part (d). Compare your answer to the complete information solution found in part (b).

- Taking FOCs with respect to q^H and q^L, we find that

$$p\left[1-\frac{1}{\theta^H}g'\left(\frac{q^H}{\theta^H}\right)\right]+\lambda\left[\frac{1}{\theta^L}g'\left(\frac{q^H}{\theta^L}\right)-\frac{1}{\theta^H}g'\left(\frac{q^H}{\theta^H}\right)\right]=0$$

$$(1-p)\left[1-\frac{1}{\theta^L}g'\left(\frac{q^L}{\theta^L}\right)\right]+\lambda\left[\frac{1}{\theta^H}g'\left(\frac{q^L}{\theta^H}\right)-\frac{1}{\theta^L}g'\left(\frac{q^H}{\theta^L}\right)\right]+\mu\left[\frac{1}{\theta^L}g'\left(\frac{q^L}{\theta^L}\right)-\frac{1}{\theta^H}g'\left(\frac{q^L}{\theta^H}\right)\right]=$$

- At this point, we can do some algebra to show that Lagrange multipliers λ and μ are both zero, entailing that its associated constraints hold with strict inequality. Alternatively, remember that PC_H must hold with strict inequality as, otherwise, the high-type worker would be making no information rents in the incomplete information game (earning the same utility as in the complete information version). Similarly, IC_L must hold with strict inequality as, otherwise, the low-type worker would be indifferent between the contract meant for him and that meant for the high type. As a consequence, $\lambda = \mu = 0$, which simplifies the above first-order conditions to

$$p\left[1-\frac{1}{\theta^H}g'\left(\frac{q^H}{\theta^H}\right)\right]=0$$

$$(1-p)\left[1-\frac{1}{\theta^L}g'\left(\frac{q^L}{\theta^L}\right)\right]=0$$

In other words, we get the complete-information solution

$$\theta^H = g'\left(\frac{q^H}{\theta^H}\right) \quad \text{and} \quad \theta^L = g'\left(\frac{q^L}{\theta^L}\right)$$

(g) *Parametric example (Incomplete information).* Assume the same cost of effort function as in the parametric example developed in part (c), $g(e^k) = (e^k)^2$, and the same set of productivity parameters $\theta^H = 1$ and $\theta^L = \frac{1}{2}$. In addition, consider that both types of workers are equally likely, i.e., $p = \frac{1}{2}$. Find the optimal values of q^H and q^L in the incomplete information setting. Then, find the optimal y^H and y^L, where $y^k \equiv q^k - t^k$.

- Since in both information contexts optimal outputs are given by the same FOC, we have that

$$q^H = \frac{1}{2} \quad \text{and} \quad q^L = \frac{1}{8}$$

(as shown in the parametric example of part c). However, taxes under complete and incomplete information do not coincide. In order to show this, let us first recall that, under complete information, we found that

$$t^H_{CI} = \frac{1}{4}, \quad \text{and} \quad t^L_{CI} = \frac{1}{16}$$

Let us now find optimal taxes under incomplete information. Using condition IC_H, which holds with equality, we obtain

$$\begin{aligned} y_{II}^H - g\left(\frac{q^H}{\theta^H}\right) &= y_{II}^L + g\left(\frac{q^L}{\theta^H}\right) \\ &\Longleftrightarrow\ y_{II}^H - \frac{1}{4} = \frac{1}{16} - \left(\frac{1}{8}\right)^2 \\ &\Longleftrightarrow\ y_{II}^H = \frac{19}{64} > \frac{1}{4} \end{aligned}$$

for the high-type. Similarly, using PC_L, which also holds with equality, we find that

$$y_{II}^L = g(e^L) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

for the low-type. We can next operate as in the case of complete information, that is, using $t^k \equiv q^k - y^k$ to find the optimal taxes under incomplete information, as follows

$$\begin{aligned} t_{II}^H &= q^H - y_{II}^H = \frac{1}{2} - \frac{19}{64} = \frac{13}{64}, \text{ and} \\ t_{II}^L &= q^L - y_{II}^L = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}. \end{aligned}$$

Hence, under incomplete information, the high-type worker pays lower taxes and obtains a higher utility, ultimately generating an information rent of

$$\Delta v^H = v_{II}^H - v_{CI}^H = \frac{3}{64} > 0.$$

However, the low-type worker's information rent is zero. That is, he produces the same first-best output and tax under both information contexts. In summary, while output levels suffer no information distortion (i.e., they coincide under both information contexts), taxes differ, thus giving rise to a lower tax for the high-productivity worker and the same tax for the low-productivity worker.

5. **Gibbard-Satterthwaite theorem.** In this chapter, we analyzed the aggregation of individual preferences into a social preference relation satisfying a set of desirable properties. However, we assumed individual preferences were truthfully reported by each individual. In this exercise, we examine a setting in which individuals do not necessarily truthfully reveal their preferences. In particular, we are interested in social choice functions that are "strategy proof."

   First, note that a *social choice function* $c\left(\succsim^1, \succsim^2, ..., \succsim^N\right) \in X$ maps the profile of individual preferences $\left(\succsim^1, \succsim^2, ..., \succsim^N\right)$ into an alternative $x \in X$. That is, society uses the social choice function (scf) to "select" an alternative $x \in X$, using the information in the profile of individual preferences $\left(\succsim^1, \succsim^2, ..., \succsim^N\right)$. Hence, we say that a scf $c(\cdot)$ is *strategy-proof* if every individual i prefers the alternative that the scf selects when he reports his true preferences, $c\left(\succsim^i, \succsim^{-i}\right) = x$, than that arising when he misreports his preferences, $c(\succsim'^i, \succsim^{-i}) = y$, i.e., $x \succsim^i y$, where $\succsim^{-i}$ denotes the profile of individual preferences by all other agents $\left(\succsim^1, ..., \succsim^{i-1}, \succsim^{i+1}, ..., \succsim^N\right)$. In words, if a scf is strategy proof, individuals have no strict incentives to misreport their preferences, regardless of the preferences other individuals report, $\succsim^{-i}$; which holds true even if the other individuals misreport their preferences. We seek to show, in several steps, Gibbard-Satterthwaite's theorem, which says that: If there are three or more alternatives in X, then every strategy-proof scf is dictatorial.[3] In the next questions of this exercise, we will start showing that (1) a strategy-proof scf must exhibit two properties: Pareto efficiency and monotonicity; and (2) every Pareto efficient and monotonic scf must be dictatorial.

We need to define what we mean by Pareto efficient scf: A scf is *Pareto efficient* when every individual i's strict preference for x over y, $x \succ^i y$, where $x, y \in X$, yields the scf to select x, i.e., $c\left(\succsim^1, \succsim^2, ..., \succsim^N\right) = x$. We also define what we mean by monotonic scfs: Consider a initial profile of individual preferences, $\left(\succsim^1, \succsim^2, ..., \succsim^N\right)$, yielding that alternative x is chosen by the scf, i.e., $c\left(\succsim^1, \succsim^2, ..., \succsim^N\right) = x$. Assume that the preferences of at least individual i change from $x \succsim^i y$ to $x \succ'^i y$, for every $y \in X$, i.e., alternative x rises to the only spot at the top of his ranking of alternatives, and the preference for x is not lowered for any individual, i.e., $x \not\prec y$. We then say that a scf is *monotonic* if the scf still selects x under the new profile of individual preferences, $c\left(\succ'^1, \succ'^2, ..., \succ'^N\right) = x$. Hence, loosely speaking, a scf is monotonic if it keeps selecting x as socially preferred when x becomes the top alternative for at least one individual.

(a) Show that strategy-proofness implies monotonicity on the scf.

- Consider an arbitrary profile of individual preferences, $(\succsim^i, \succsim^{-i})$, where alternative x is chosen by the scf, i.e., $c(\succsim^i, \succsim^{-i}) = x$. In addition, consider an arbitrary indidividual i, whose preferences change from $x \succsim^i y$ to $x \succ'^i y$ for every $y \in X$, so alternative x rises to the only spot at the top of i's ranking of alternatives. We then need to show that the scf still selects x, i.e., $c\left(\succsim'^i, \succsim^{-i}\right) = x$.
- *Proof.* By contradiction, suppose that the scf doesn't select x under the new preferences, i.e.,

$$c\left(\succsim'^i, \succsim^{-i}\right) = y \neq x$$

  Then, the social choice that arises when individual i truthfully reports his preferences, y, is less preferred than the alternative that would emerge when he misreports his preferences, x, i.e., $x \succ'^i y$. Therefore, under the new preference relation $\succsim'^i$, individual i has incentives to misreport his preferences, thus violating strategy-proofness. (Q.E.D.)

(b) Use monotonicity to show that the scf must be Pareto efficient.

[3] The definition of a dictatorial scf is similar to , in the definition in swf. In particular, we say that a scf $c(\cdot)$ is *dictatorial* if there is an individual d (the dictator) such that, if $x \succsim^d y$ for every two alternatives $x, y \in X$, then the scf selects x, i.e., $c\left(\succsim^1, \succsim^2, ..., \succsim^N\right) = x$. That is, a scf is dictatorial if there is an individual d such that $c(\cdot)$ chooses d's top choices, regardless of the preferences of all other individuals.

- We need to show that in a profile of individual preference relations $\succsim$ where $x \succ^i y$ for every $y \neq x$ and for every individual i, the scf selects $c(\succsim) = x$.
- *Proof:* Consider a profile of individual preferences $\succsim'$ which yields $c(\succsim') = x$, where $x \succ'^i y$ does not necessarily hold for all individuals. Construct now a new profile of individual preferences $\succsim''$ where $x \succ''^i y$ holds for one individual i, i.e., alternative x has been moved to the top of i's ranking, but leaving the remaining ranking unaffected. By monotonicity, $c(\succsim'') = x$. We can now repeat the process for all individuals (moving x to the top of their rankings). Applying monotonicity again yields $c(\succsim) = x$, as required. (Q.E.D.)

After demonstrating that strategy-proofness implies monotonicity and Pareto efficiency, we are ready to show the main result of Gibbard-Satterthwaite's theorem (namely, that in a context where the set of alternatives has more than three elements, and where the scf satisfies monotonicity and Pareto efficiency, then such scf must be dictatorial). We will demonstrate that using five steps.

(c) *Step 1.* Consider a profile of strict rankings in which alternative x is ranked highest and y lowest for every individual i; as illustrated in the next table. In this setting, Pareto efficiency implies that the scf must select x.

$\succsim^1$	...	$\succsim^{n-1}$	$\succsim^n$	$\succsim^{n+1}$	...	$\succsim^N$	Social choice
x	...	x	x	x	...	x	x
.		.	.	.		.	
.		.	.	.		.	
.		.	.	.		.	
y	...	y	y	y	...	y	

Table 12.1

Consider now that we change individual 1's ranking by raising y in it one position at a time. Show that there must exist an individual n for which the social ranking changes when y is raised above x in individual n's ranking.

- By monotonicity, the social choice must remain alternative x as long as $x \succ^i y$. But when y is raised above x, the social choice can change to y, or remain at x. If the social choice is still x, we then begin raising y for individual 2 one position at a time. Eventually, the social ranking will change when y is raised above x in individual n's ranking.

(d) *Step 2.* Consider now a different profile of individual preferences in which: x is moved to the bottom of individual i's ranking, for all $i < n$, and x is moved to the second last position in individual i's ranking, for all $i > n$. Show that this change in individual preferences does not change the selection of the scf.

- After raising alternative y to the top position for the first $n-1$ individuals, we have the following table of individual preferences.

$\succsim^1$	...	$\succsim^{n-1}$	$\succsim^n$	$\succsim^{n+1}$	...	$\succsim^N$	Social choice
y	...	y	x	x	...	x	x
x	...	x	y	.		.	
.		.	.	.		.	
.		.	.	.		.	
.		.	.	y	...	y	

Table 12.2

After raising y to the top position for one more individual (individual n), we obtain the following table. Note that raising y to the top position of individual n now changes the social choice, from x to y.

$\succsim^1$	...	$\succsim^{n-1}$	$\succsim^n$	$\succsim^{n+1}$	...	$\succsim^N$	Social choice
y	...	y	y	x	...	x	y
x	...	x	x	.		.	
.		.	.	.		.	
.		.	.	.		.	
.		.	.	y	...	y	

Table 12.3

$\succsim^1$	...	$\succsim^{n-1}$	$\succsim^n$	$\succsim^{n+1}$	...	$\succsim^N$	Social choice
y	...	y	x	.	...	.	x
.	...	.	y	.		.	
.		.	.	.		.	
.		.	.	x		x	
x	...	x	.	y	...	y	

Table 12.4

$\succsim^1$	...	$\succsim^{n-1}$	$\succsim^n$	$\succsim^{n+1}$	...	$\succsim^N$	Social choice
y	...	y	y	.	...	.	y
.	...	.	x	.		.	
.		.	.	.		.	
.		.	.	x		x	
x	...	x	.	y	...	y	

Table 12.5

Comparing tables 12.3 and 12.5, note that, by monotonicity, the scf must still select y: indeed, the social choice in Table 12.3 was y, and no individuals' ranking of y versus any other alternative has changed when we moved from Table 12.3 to 12.5.

- Let us now compare tables 12.4 and 12.5. They only differ in the ranking of individual n: he prefers x to y in Table 12.4, but prefers y to x in Table 12.5. Hence, if the scf selects y in Table 12.5, it must still select either y or x in Table 12.3 (by the same logic as in Step 1). But, can the scf select y in Table 12.3? No! By monotonicity, the scf should select alternative y in Table 12.1 as well, which is a contradiction. Therefore, the social choice in Table 12.3 must be x.

(e) *Step 3.* In this step, we use the assumption that the number of elements in the set of alternatives X is equal or larger than 3. For that, we only need to consider an alternative $z \neq x, y$ in our above steps.

- Consider the next table. Note that the social choice is still x: indeed, we have not changed the ranking of x against any other alternative in any individual's ranking.

$\succsim^1$	...	$\succsim^{n-1}$	$\succsim^n$	$\succsim^{n+1}$	...	$\succsim^N$	Social choice
.		.	x	.	...	.	x
.		.	z	.		.	
.		.	y	.		.	
z	...	z		z	...	z	
y	...	y	.	x	...	x	
x	...	x	.	y	...	y	

Table 12.6

(f) *Step 4.* Consider a profile of individual preferences compatible with those in Step 3. Switch the ranking of alternatives x and y for all individuals $i > n$; as depicted in the next table.

$\succsim^1$	...	$\succsim^{n-1}$	$\succsim^n$	$\succsim^{n+1}$	...	$\succsim^N$	Social choice
.		.	x	.	...	.	x
.		.	z	.		.	
.		.	y	.		.	
z	...	z		z	...	z	
y	...	y	.	y	...	y	
x	...	x	.	x	...	x	

Table 12.7

Show that alternative x must be socially selected.

- When moving from the profile of preferences in part (e) to that in part (f), we made alternative y preferred for $N - n$ individuals (see last columns). Hence, either x is still selected by the scf (as under the table in part f), or y becomes selected. But, can alternative y be selected? No! By Pareto efficiency, $z \succ^i y$ for every individual i. Thus, x must be socially selected.

(g) *Step 5.* Argue that the scf must be dictatorial.

- In the previous step, we showed that individual n's top choice (any arbitrary alternative x) becomes selected by the scf, regardless of the profile of preferences by all other individuals. Hence, the scf is dictatorial.
- A natural question is how to avoid the unfortunate result in the Gibbard-Satterthwaite theorem. That is, under which conditions truthtelling becomes incentive compatible for all individuals (i.e., the scf is strategy-proof) and the scf is not dictatorial? As discussed in the chapter on Mechanism Design, this occurs when utility functions are quasilinear and we use mechanisms such as the Vickrey-Clarks-Groves (VCG) mechanism. Indeed, the VCG mechanism is strategy-proof, e.g., inducing truthful report of each individual's benefit of a public project, and it is also non-dictatorial (no individual imposes his/her own preferences on the group).

6. **Checking properties on a social welfare function - Kaneko and Nakamura (1979)**[4] Consider the following social welfare function

$$SW(x) = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdot \ldots \cdot x_N^{\alpha_N} = \prod_{i=1}^{N} x_i^{\alpha_i}$$

where $x = (x_1, x_2, \ldots x_N)$ denotes an alternative, where $x \in \mathbb{R}^N$, and $\alpha_i > 0$ represents the weight that the social planner assigns to agent i. Alternatively, this social welfare function can be represented in its linear form by applying logs, as follows,

$$\alpha_1 \ln x_1 + \alpha_2 \ln x_2 \cdot \ldots \cdot \alpha_N \ln x_N = \sum_{i=1}^{N} \alpha_i \ln x_i$$

In this exercise, we show that ordinal preferences among the alternatives can be represented by the above (cardinal) social welfare function that satisfies:

- Paretian (P),
- Anonymity (A),
- Neutrality (N), and
- Independence of Irrelevant Alternatives (IIA).

Show that the above social welfare function $SW(x)$ satisfies these five properties.

- *Paretian.* Consider two alternatives, x and y, where $x \geq y$ for all agents, then

$$\begin{aligned} SW(x) &= \prod_{i=1}^{N} x_i^{\alpha_i} \\ &\geq \prod_{i=1}^{N} y_i^{\alpha_i} = SW(y) \end{aligned}$$

  so that, when all agents prefer alternatives x to y, the social planner also prefers alternative x to y; thereby satisfying the Pareto Optimality (P) condition.

- *Anonymity.* Consider a permutation of agents, where agent i is assigned a new identity, $\pi(i) \neq i$.

$$\begin{aligned} SW(\pi(x)) &= \prod_{\pi_i=1}^{N} x_{\pi_i}^{\alpha_{\pi_i}} \\ &= \prod_{i=1}^{N} x_i^{\alpha_i} = SW(x) \end{aligned}$$

  because when the agents are assigned a new identity, both their endowments x and weights α are permuted, so that the social welfare generated from the allocation $\pi(x)$ coincides with that emerges from the original allocation x; thereby satisfying the Anonymity (A) condition.

[4] Kaneko M. and Nakamura K. (1979). The Nash Social Welfare Function. *Econometrica,* 47(2), pp. 423-35.

- *Neutrality.* Consider a permutation of alternatives, where $x_i = \beta_i y_i$, for which $\beta_i > 0$ and $\beta_i \neq \beta_j$ in general (that is, we consider alternative x to be different from alternative y by a factor of β_i that is generally not the same for each agent i), then

$$\begin{aligned} SW(x) &= \prod_{i=1}^{N} x_i^{\alpha_i} \\ &= \prod_{i=1}^{N} (\beta_i y_i)^{\alpha_i} \\ &= \left(\prod_{i=1}^{N} \beta_i^{\alpha_i}\right)\left(\prod_{i=1}^{N} y_i^{\alpha_i}\right) \\ &= \left(\prod_{i=1}^{N} \beta_i^{\alpha_i}\right) SW(y) \end{aligned}$$

  so that when the alternatives are assigned a new identity, social welfare is also scaled by a factor of $\prod_{i=1}^{N} \beta_i^{\alpha_i}$ that preserves the ranking of alternatives, where

$$\left(\prod_{i=1}^{N} \beta_i^{\alpha_i}\right) SW(y) = \left(\prod_{i=1}^{N} \beta_i^{\alpha_i}\right) SW\left(\frac{x}{\beta}\right) = \left(\prod_{i=1}^{N} \beta_i^{\alpha_i}\right) \prod_{i=1}^{N} \left(\frac{x_i}{\beta_i}\right)^{\alpha_i} = SW(x);$$

  thereby satisfying the Neutrality (N) condition.

- *Independence of Irrelevant Alternatives.* Consider an alternative z. Since $SW(x)$ only considers alternative x and $SW(y)$ only considers alternative y, the ranking of x and y does not depend on z; thereby satisfying the Independence of Irrelevant Alternatives (IIA) condition.