# EconS 503 - Microeconomic Theory II
# Homework #10 - Answer key

1. **Exercises from MWG:**

   (a) Chapter 21 (social choice theory): Exercise 21.D.5.

   (b) Chapter 23 (mechanism design): Exercise 23.C.10.

   - See answer keys to these exercises in the scanned pages at the end of the handout.

2. **Public Good Provision - Different mechanisms.** Imagine that you and your colleagues want to buy a coffee machine for your office. Suppose that some of you may be heavily addicted to coffee and are willing to pay more for the machine than the others. However, you do not know your colleagues' willingness to pay for the machine. The cost of the machine is $C$. We would like to find a decision rule in which (i) each individual reports a valuation (i.e., direct mechanism), and (ii) the coffee maker is purchased if and only if it is efficient to do so. Let us next analyze if it is possible to find a cost-sharing rule which gives incentive for everyone to report his valuation truthfully.

   In particular, assume $n$ individuals, each of them with private valuation $\theta_i \sim U(0,1)$. The allocation function is binary $y \in \{0,1\}$, i.e., the coffee machine is purchased or not. Let $t_i$ be the transfer from individual $i$, implying a utility of

   $$u_i(y, \theta_i, t_i) = y\theta_i - t_i$$

   Let $i \in \{1, ..., n\}$ denote the individuals, and let $i = 0$ denote the original owner of the good.

   (a) What is the efficient assignment rule, $y^*(\theta_1, ..., \theta_n)$?

   - The efficient assignment rule is

   $$y^*(\theta_1, ..., \theta_n) = \begin{cases} 1 & \text{if } \sum_{j=1}^{n} \theta_j \geq C \\ 0 & \text{otherwise} \end{cases}$$

     In words, the coffee machine is purchased if and only if the sum of all valuations exceeds its total cost.

   (b) *Equal-share rule.* Consider the following equal-share rule: When the public good is provided, the cost is equally divided by all $n$ individuals.

      1. Before starting any computation, what would you expect - whether each individual would overstate or understate their valuation?

         - Because of free-rider incentives, each individual may have an incentive to understate his valuation. The equal-share payment rule, however, makes transfers independent of his report.

2. Confirm that the transfer rule is written by:

$$t_i(\theta) = \frac{C}{n} y^*(\theta)$$

   - By the equal-share rule, each individual will pay $\frac{C}{n}$ if the project happens, and 0 otherwise. Hence, the transfer rule is

$$t_i(\theta) = \frac{C}{n} y^*(\theta)$$

3. Let $V_i(\tilde{\theta}_i|\theta_i, \theta_{-i})$ be individual $i$'s payoff when $i$ reports $\tilde{\theta}_i$ instead of his true valuation $\theta_i$, while the others truthfully report their valuations $\theta_{-i}$. Show that

$$V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = \left(\theta_i - \frac{C}{n}\right) y^*(\tilde{\theta}_i, \theta_{-1})$$

   - Using the definition of player $i$'s utility function, we can intert in the above equal-share transfer rule to obtain

$$\begin{aligned} V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) &= \theta_i y^*(\tilde{\theta}_i, \theta_{-1}) - t_i^*\left(\tilde{\theta}_i, \theta_{-i}\right) \\ &= \theta_i y^*(\tilde{\theta}_i, \theta_{-1}) - \frac{C}{n} y^*(\tilde{\theta}_i, \theta_{-1}) \\ &= \left(\theta_i - \frac{C}{n}\right) y^*(\tilde{\theta}_i, \theta_{-1}) \end{aligned}$$

4. Let $U_i\left(\tilde{\theta}_i|\theta_i\right)$ be individual $i$'s expected payof when he reports $\tilde{\theta}_i$ instead of the true valuation $\theta_i$. Show that

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = \left(\theta_i - \frac{C}{n}\right) E_{\theta_{-i}}\left[y^*(\tilde{\theta}_i, \theta_{-1})\right]$$

   - Player $i$'s expected payoff for misreporting $\tilde{\theta}_i \neq \theta_i$ is just the expected value of the utility found above, that is

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = E_{\theta_{-i}}\left[V_i(\tilde{\theta}_i|\theta_i, \theta_{-i})\right] = \left(\theta_i - \frac{C}{n}\right) E_{\theta_{-i}}\left[y^*(\tilde{\theta}_i, \theta_{-1})\right]$$

5. Suppose that $i$'s private valuation $\theta_i$ satisfies $\theta_i > \frac{C}{n}$. Assuming that the others are telling the truth, what is the best response for $i$? What if $\theta_i < \frac{C}{n}$? Is this mechanism strategy-proof? Is this mechanism Bayesian incentive compatible?

   - If player $i$'s valuation $\theta_i$ satisfies $\theta_i > \frac{C}{n}$, $U_i\left(\tilde{\theta}_i|\theta_i\right)$ is maximized when $E_{\theta_{-i}}\left[y^*(\tilde{\theta}_i, \theta_{-1})\right]$ is maximized. Hence, individual $i$ would report $\tilde{\theta}_i$ as large as possible, i.e., $\tilde{\theta}_i = 1$. In contrast, if $\theta_i$ satisfies $\theta_i < \frac{C}{n}$, individual $i$ would report $\tilde{\theta}_i$ as small as possible, i.e., $\tilde{\theta}_i = 0$. The mechanism is neither strategy-proof, nor Bayesian incentive compatible.

(c) *Proportional payment rule.* Consider now the proportional payment rule:

$$t_i(\theta) = \frac{\theta_i C}{\sum_j \theta_j} y^*(\theta)$$

where every individual $i$ pays a share of the total cost equal to the proportion that his reported valuation signifies out of the total reported valuations.

1. Under this rule, what would you expect - whether each individual would overstate or understate the valuation?
   - Now the payment is a function of the report. Notice that this cost-sharing rule is balanced-budget. Hence, you may expect that the agents have incentive to free-ride.
2. Show that the utility of reporting $\tilde{\theta}_i$ is now

$$V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = \left(\theta_i - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-1})$$

   - The payoff to each individual will be their actual valuation, less the amount the have to pay based on what they report if the project happens, and 0 otherwise. That is,

$$\begin{aligned} V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) &= \theta_i y^*(\tilde{\theta}_i, \theta_{-1}) - t_i^*\left(\tilde{\theta}_i, \theta_{-i}\right) \\ &= \theta_i y^*(\tilde{\theta}_i, \theta_{-1}) - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j} y^*(\tilde{\theta}_i, \theta_{-1}) \\ &= \left(\theta_i - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-1}) \end{aligned}$$

3. For simplicity, suppose two individuals, $n = 2$ and a total cost of $C = 1$. Show that

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = \tilde{\theta}_i\left(\theta_i - \log\left(\tilde{\theta}_i + 1\right)\right)$$

   - Again, by definition, the expected utility of misreporting $\tilde{\theta}_i$ is

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = E_{\theta_{-i}}\left[\left(\theta_i - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-1})\right]$$

   Suppose now that $n = 2$ and $C = 1$. Then the above expression becomes

$$\begin{aligned} U_i\left(\tilde{\theta}_i|\theta_i\right) &= E_{\theta_{-i}}\left[\left(\theta_i - \frac{\tilde{\theta}_i}{\tilde{\theta}_i + \theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-1})\right] \\ &= \int_{1-\tilde{\theta}_i}^{1}\left(\theta_i - \frac{\tilde{\theta}_i}{\tilde{\theta}_i + \theta_j}\right) d\theta_j \\ &= \left[\theta_i\theta_j - \tilde{\theta}_i \log(\tilde{\theta}_i + \theta_j)\right]_{1-\tilde{\theta}_i}^{1} \\ &= \theta_i - \tilde{\theta}_i \log(\tilde{\theta}_i + 1) - \left[\theta_i(1-\tilde{\theta}_i) - \tilde{\theta}_i \log(\tilde{\theta}_i + (1-\tilde{\theta}_i))\right] \\ &= \tilde{\theta}_i\left(\theta_i - \log(\tilde{\theta}_i + 1)\right) \end{aligned}$$

4. Is this mechanism strategy-proof? Is it Bayesian incentive compatible?
   - It is straightforward to show that the expected utility of reporting $\tilde{\theta}_i$ is decreasing in player $i$'s report $\tilde{\theta}_i$, since

$$\frac{\partial}{\partial \tilde{\theta}_i} U_i\left(\tilde{\theta}_i|\theta_i\right)\Big|_{\tilde{\theta}_i=\theta_i} = \frac{\theta_i^2}{1+\theta_i} - \log(\theta_1+1) < 0 \text{ for all } \theta_i \in (0,1]$$

   implying that every player $i$ has incentives to underreport his true valuation $\theta_i$ as much as possible, i.e., $\tilde{\theta}_i = 0$. Hence, this mechanism is neither strategy-proof nor Bayesian incentive compatible.

5. Which way is everyone biased, overstate or understate? What is the intuition?
   - The negative sign in part (iv) suggests that $U_i\left(\tilde{\theta}_i|\theta_i\right)$ is maximized at $\tilde{\theta}_i$ smaller than $\theta_i$. Each individual has an incentive to understate the valuation.

(d) *VCG mechanism.* Let us consider now the VCG mechanism. Recall that the efficient rule $y^*(\theta)$ determines that the coffee machine is bought if and only if total valuations satisfy $\sum_i \theta_i \geq C$. Remember that we need to include the original owner of the public good; $i = 0$. Then, the total surplus when the valuation of individual $i$ is considered in $\theta = (\theta_1, \theta_2, ..., \theta_n)$ is

$$\sum_{j\neq i} v_j\left(y^*(\theta), \theta_j\right) = \begin{cases} \sum_{j\neq i} \theta_j & \text{if } \sum_j \theta_j \geq C \\ C & \text{if } \sum_j \theta_j < C \end{cases}$$

while total surplus when the valuation of individual $i$ is ignored, $\theta_{-i}$, is

$$\sum_{j\neq i} v_j\left(y^*(\theta_{-i}), \theta_j\right) = \begin{cases} \sum_{j\neq i} \theta_j & \text{if } \sum_{j\neq i} \theta_j \geq C \\ C & \text{if } \sum_{j\neq i} \theta_j < C \end{cases}$$

The only difference in total surplus arises from the allocation rule which specifies that, when $\theta_i$ is considered, the good is purchased if and only if $\sum_j \theta_j \geq C$, whereas when $\theta_i$ is ignored, the good is bought if and only if $\sum_{j\neq i} \theta_j \geq C$. Hence, the VCG transfer is

$$\begin{aligned} t_i^*(\theta) &= -\left(\sum_{j\neq i} v_j(y^*(\theta), \theta_j) - \sum_{j\neq i} v_j(y^*(\theta_{-i}), \theta_j)\right) \\ &= \begin{cases} C - \sum_{j\neq i} \theta_j & \text{if } \sum_{j\neq i} \theta_j < C \leq \sum_j \theta_j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Intuitively, player $i$ pays the difference between everyone else's valuations, $\sum_{j\neq i} \theta_j$, and the total cost of the good, $C$. Such a payment, however, only occurs when aggregate valuations exceed the total cost, $\sum_j \theta_j \geq C$, and thus the good is purchased, and when the valuations of all other players do not yet exceed the total cost of the good, $\sum_{j\neq i} \theta_j < C$, so the difference $C - \sum_{j\neq i} \theta_j$ is paid by player $i$ in his transfer.

1. Show that in this mechanism player $i$'s utility from reporting a valuation $\tilde{\theta}_i \neq \theta_i$ is

$$\begin{aligned} V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) &= v_i\left(y^*\left(\tilde{\theta}_i, \theta_{-i}\right), \theta_i\right) - t_i^*\left(\tilde{\theta}_i, \theta_{-i}\right) \\ &= \begin{cases} 0 & \text{if } \tilde{\theta}_i + \sum_{j\neq i}\theta_j < C \\ \sum_j \theta_j - C & \text{if } \sum_{j\neq i}\theta_j < C \leq \tilde{\theta}_i + \sum_{j\neq i}\theta_i \\ \theta_i & \text{if } C \leq \sum_{j\neq i}\theta_j \end{cases} \end{aligned}$$

   - This is just the definition of the payoff function for the VCG.

2. Is this mechanism strategy-proof? Is this Bayesian incentive compatible?
   - In order to test if this direct revelation mechanism is strategy-proof,
   - Suppose that $C \leq \sum_{j\neq i}\theta_j$, i.e., the public good will be purchased regardless of individual $i$'s reported valuation. Then $V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = \theta_i$, which is independent of player $i$'s reported valuation, $\tilde{\theta}_i$. Hence, telling the truth is player $i$'s best response.
   - Now suppose that $\sum_{j\neq i}\theta_j < C \leq \sum_j \theta_j$, i.e., individual $i$'s valuation is pivotal. Then by reporting a valutation $\tilde{\theta}_i$ such that $\tilde{\theta}_i \geq C - \sum_{j\neq i}\theta_j$, his utility becomes $V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = \sum_j \theta_j - C \geq 0$. This includes the case of telling the truth; $\tilde{\theta}_i = \theta_i \geq C - \sum_{j\neq i}\theta_j$. If, instead, individual $i$ lies by reporting $\tilde{\theta}_i < C - \sum_{j\neq i}\theta_j$, then his utility becomes $V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = 0$ since the good is not purchased given that $\tilde{\theta}_i < C - \sum_{j\neq i}\theta_j$ entails $\tilde{\theta}_i + \sum_{i\neq j}\theta_j < C$. Hence, misreporting his valuation cannot be profitable.
   - Finally, suppose that $\sum_j \theta_j < C$, i.e., the public good will not be purchased regardless of individual $i$'s valuation. Then, by honestly revealing his valuation, $\tilde{\theta}_i = \theta_i < C - \sum_{j\neq i}\theta_j$, his payoff is $V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = 0$ since the good is not purchased. By lying, $\tilde{\theta}_i \geq C - \sum_{j\neq i}\theta_j$, his payoff is $V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = \sum_j \theta_j - C < 0$. Telling a lie is then not profitable. Hence, truth-telling is the best strategy for $i$, regardless of the values of $\theta_{-i}$. The VCG mechanism is thus strategy-proof, and also Bayesian incentive compatible.

3. For simplicity, suppose two individuals, $n = 2$, and a total cost of $C = 0.5$. Compute $y^*$, $t_1^*$ and $t_2^*$ for the following $(\theta_1, \theta_2)$ pairs.

| $\theta_1$ | $\theta_2$ |
|---|---|
| 0.1 | 0.3 |
| 0.3 | 0.3 |
| 0.3 | 0.8 |
| 0.8 | 0.8 |

   - For the case of $\theta_1 = 0.1$ and $\theta_2 = 0.3$, we have that VCG transfers become

$$t_i^*(\theta) = \begin{cases} -\sum_{j\neq i}\theta_j + C & \text{if } \sum_{j\neq i}\theta_j < C \leq \sum_j \theta_j \\ 0 & \text{otherwise} \end{cases}$$

implying that the transfer player 1 pays is

$$t_1(\theta) = \begin{cases} -0.3 + 0.5 & \text{if } 0.3 < 0.5 \leq 0.4 \\ 0 & \text{otherwise} \end{cases}$$

and the transfer that player 2 pays is

$$t_2(\theta) = \begin{cases} -0.1 + 0.5 & \text{if } 0.1 < 0.5 \leq 0.4 \\ 0 & \text{otherwise} \end{cases}$$

As we can see, the upper inequality does not hold, and thus the good is not purchased, $y^*(\theta) = 0$, and transfers are zero, $t_1^*(\theta) = t_2^*(\theta) = 0$. Following the same steps, the results for valuation pairs $(0.3, 0.3)$, $(0.3, 0.8)$, and $(0.8, 0.8)$ are presented in the following table

| $\theta_1$ | $\theta_2$ | $y^*(\theta)$ | $t_1^*(\theta)$ | $t_2^*(\theta)$ |
|---|---|---|---|---|
| 0.1 | 0.3 | 0 | 0 | 0 |
| 0.3 | 0.3 | 1 | 0.2 | 0.2 |
| 0.3 | 0.8 | 1 | 0 | 0.2 |
| 0.8 | 0.8 | 1 | 0 | 0 |

4. Show that the expected revenue from this mechanism is $E\left[t_1^*(\theta_1, \theta_2) + t_2^*(\theta_1, \theta_2)\right] = \frac{1}{6} \simeq 0.167$. Based on what you calculated in part (iii), is this problematic?

   - If $\theta_2 \geq C$, then player 1 doesn't need to pay anything $t_1^* = 0$. If $\theta_2 < C$, then player 1's transfer is $t_1^* = -\theta_2 + C$ if and only if $\theta_1 + \theta_2 \geq C$. Hence, player 1's expected transfer is

$$\begin{aligned} E_\theta\left[t_1^*(\theta_1, \theta_2)\right] &= \int_{\{(\theta_1,\theta_2)|\theta_1+\theta_2 \geq C\}} (-\theta_2 + C) d\theta_1 d\theta_2 \\ &= \int_0^C \int_{-\theta_2+C}^1 (-\theta_2 + C) d\theta_1 d\theta_2 = \frac{1}{12} \end{aligned}$$

   By symmetry, $E_\theta\left[t_2^*(\theta_1, \theta_2)\right] = \frac{1}{12}$, entailing that expected revenue becomes

$$E_\theta\left[t_1^*(\theta_1, \theta_2) + t_2^*(\theta_1, \theta_2)\right] = \frac{1}{6} \simeq 0.167$$

   This is problematic, because the expected revenue, 0.167, is smaller than the total cost, 0.5, implying a budget deficit. The VCG mechanism has two nice properties: efficiency and incentive compatibility. However, balanced budget condition and participation constraint are not necessarily satisfied.

3. **Lexicographic social welfare functional.** In this exercise, we consider a setting with two alternatives $x$ and $y$, and discuss the following social welfare functional.

$$F(\alpha_1, .., \alpha_N) \begin{cases} \alpha_1 \text{ if } \alpha_1 \neq 0 \\ \alpha_2 \text{ if } \alpha_1 = 0 \text{ and } \alpha_2 \neq 0 \\ \alpha_3 \text{ if } \alpha_1 = \alpha_2 = 0 \text{ and } \alpha_3 \neq 0 \\ \cdots \end{cases}$$

Intuitively, society selects the alternative that individual 1 strictly prefers. However, if he is indifferent between alternatives $x$ and $y$, society follows the strict preferences of individual 2 (if he has a strict preference over $x$ or $y$). If both individuals 1 and 2 are indifferent between $x$ and $y$, the strict preferences of individual 3 are considered, and so on.

Determine whether or not it satisfies the three properties of majority voting (symmetry among agents, neutrality between alternatives, and positive responsiveness).

- *Symmetry among agents*: We can easily find settings in which this property does not hold. In particular, consider a preference profile in which $\alpha_1 > 0 > \alpha_j$, where $j \neq 1$ represents any individual different from 1. In this case, since $\alpha_1 > 0$, the lexicographic swf yields $F(\alpha_1, \alpha_j, \alpha_{-1,j}) > 0$, where $\alpha_{-1,j} = (\alpha_2, \alpha_3, ..., \alpha_{j-1}, \alpha_{j+1}, ..., \alpha_N)$ denotes the preference profile of all individuals other than 1 and $j$. In this context, if we reorder the identities of the individuals so that individual $j$ now becomes 1, and individual 1 becomes $j$, the lexicographic swf would select the alternative that individual $j$ (who is now the first) strictly prefers, that is, $F(\alpha_j, \alpha_1, \alpha_{-1,j}) < 0$. Hence, symmetry among agents is violated.

- *Neutrality between alternatives*: Let us first recall that the lexicographic social welfare functional $F$ is equal to $\alpha_1$ if $\alpha_1 \neq 0$; is equal to $\alpha_2$ if $\alpha_1 = 0$ but $\alpha_2 \neq 0$; and, similarly, is equal to $\alpha_k$ when $\alpha_1, \alpha_2, ..., \alpha_{k-1} = 0$ but $\alpha_k \neq 0$. [Formally, $F(\alpha_1, ..., \alpha_k, ..., \alpha_N) = \alpha_k$, where $\alpha_k$ is the first non-zero element in the preference profile $(\alpha_1, \alpha_2, ..., \alpha_N)$. That is, $(0, ..., 0, \alpha_k, ..., \alpha_N)$] Hence, the negative of $F(\alpha_1, ..., \alpha_k, ..., \alpha_N)$ is

$$-F(\alpha_1, ..., \alpha_k, ..., \alpha_N) = -\alpha_k,$$

  In addition, if we create the profile of individual preferences in which every individual's $\alpha_i$ has been "reversed" to $-\alpha_i$, i.e., $(-\alpha_1, ..., -\alpha_k, ..., -\alpha_N)$, its social welfare functional becomes

$$F(-\alpha_1, ..., -\alpha_k, ..., -\alpha_N) = -\alpha_k$$

  Intuitively, since the original preference profile is $(\alpha_1, \alpha_2, ..., \alpha_N) = (0, ..., 0, \alpha_k, ..., \alpha_N)$, the new preference profile is $(-\alpha_1, ..., -\alpha_k, ..., -\alpha_N) = (0, ..., 0, -\alpha_k, ..., -\alpha_N)$, implying that the lexicographic social welfare functional produces $-\alpha_k$. Hence, $F(\alpha_1, ..., \alpha_k, ..., \alpha_N) = -F(-\alpha_1, ..., -\alpha_k, ..., -\alpha_N) = \alpha_k$; as required by neutrality between alternatives.

- *Positive responsiveness.* Again, let $\alpha_k$ be the first non-zero element in the preference profile $(\alpha_1, ..., \alpha_N)$. Consider a preference profile $(\alpha_1, ..., \alpha_k, ..., \alpha_N)$ producing a social welfare functional $F(\alpha_1, ..., \alpha_k, ..., \alpha_N) \geq 0$. Let us now specify another preference profile $(\alpha'_1, ..., \alpha'_k, ..., \alpha'_N)$ in which alternative $x$ increased its importance relative to $y$ for at least one individual, i.e., $(\alpha'_1, ..., \alpha'_k, ..., \alpha'_N) \geq (\alpha_1, ..., \alpha_k, ..., \alpha_N)$ where $(\alpha'_1, ..., \alpha'_k, ..., \alpha'_N) \neq (\alpha_1, ..., \alpha_k, ..., \alpha_N)$. In this new preference profile, let $\alpha'_j$ represent the first non-zero element (being analogous to $\alpha_k$ in the original preference profile). Two cases then arise:

(a)
- If $j < k$, then $F(\alpha'_1, ..., \alpha'_j, ..., \alpha'_k, ..., \alpha'_N) = \alpha'_j > 0$ since $(\alpha'_1, ..., \alpha'_k, ..., \alpha'_N) \geq (\alpha_1, ..., \alpha_k, ..., \alpha_N)$.
- If $j \geq k$, then $F(\alpha'_1, ..., \alpha'_k, ..., \alpha'_j, ..., \alpha'_N) = \alpha'_k > 0$.

- Hence, for any relation between $j$ and $k$, the lexicographic social welfare functional is strictly positive, $F(\alpha'_1, ..., \alpha'_k, ..., \alpha'_N) > 0$; as required by positive responsiveness. Intuitively, alternative $x$ is chosen over $y$ by the social welfare functional under the new preference profile. The only difference between the two cases analyzed above is that in the first (second) case alternative $x$ is socially chosen over $y$ because individual $j$ ($k$, respectively) was the first individual with a strict preference for $x$ over $y$ (as all other previous elements in the profile are zero).

4. **Social theory, short proofs.** Provide a short proof of the following claims. In some cases, a counterexample may suffice.

(a) The Borda count satisfies monotonicity.

- Assume that we use the Borda count and consider an arbitrary sequence of individual preference relations yielding alternative $x$ as a social choice. Now assume that someone exchanges $x$'s position with that of the alternative above $x$ on his or her list. We want to show that $x$ is still a social choice. But the change in the single list described above simply adds one point to $x$'s total, subtracts one point from that of the other alternative involved, and leaves the scores of all the other alternatives unchanged. Thus, $x$ is still a social choice; as we wanted to show.

(b) The Borda count satisfies the Pareto condition.

- Assume that we use the Borda count and consider an arbitrary sequence of individual preference relations where everyone prefers alternative $x$ to alternative $y$. We must show that $y$ is not a social choice. Since $x$ is ranked higher than $y$ by every individual, $x$ receives more points from each preference list than does $y$. Thus, when we add up the points awarded from each list we clearly have a strictly higher total for $x$ than for $y$. This does not guarantee that $x$ is the social choice, but it certainly guarantees that $y$ is not; as we wanted to show.
  In summary, swapping $x$'s position with the alternative above $x$ on some preference list adds one point to $x$'s score and subtracts one point from that of the other alternative; the scores of all other alternatives remain unchanged.

(c) The Hare procedure violates IIA.

- Consider the alternatives $a$, $b$, and $c$ and the following individual preferences:

| $\succsim^1$ | $\succsim^2$ | $\succsim^3$ | $\succsim^4$ |
|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ |
| $b$ | $b$ | $c$ | $b$ |
| $c$ | $c$ | $a$ | $a$ |

Alternative $a$ is the social choice when the Hare procedure is used because alternatives $b$ and $c$ have only one first-place vote each. In particular, $a$ is a winner and $b$ is a nonwinner. Now suppose that voter 4 changes his or her list by moving the alternative $c$ down between $b$ and $a$, as illustrated in the next table.

| $\succsim^1$ | $\succsim^2$ | $\succsim^3$ | $\succsim^4$ |
|---|---|---|---|
| $a$ | $a$ | $b$ | $b$ |
| $b$ | $b$ | $c$ | $c$ |
| $c$ | $c$ | $a$ | $a$ |

Notice that we still have $b$ over $a$ in voter 4's list. Under the Hare procedure, we now have that $a$ and $b$ are tied for the win, since each has two first-place votes. Thus, although no one changed his or her mind about whether $a$ is preferred to $b$, or $b$ to $a$, the alternative $b$ went from being a nonwinner to being a winner. This shows that IIA fails for the Hare procedure.

Homework #8 - Answer Key

we know that:

$$\frac{\partial v(k(r,\theta_{-i}),r)}{\partial k} \geq \frac{\partial v(k(r,\theta_{-i}),\theta_i)}{\partial k} \quad \text{for all} \quad r \geq \theta_i. \qquad \text{(vi)}$$

using (v) and (vi) we get:

$$u(\theta_i,\hat{\theta}_i) - u(\theta_i,\theta_i) \leq \int_{\theta}^{\hat{\theta}} \left[\frac{\partial v(k(r,\theta_{-i}),r)}{\partial k}\cdot\frac{\partial k(r,\theta_{-i})}{\partial r} + \frac{\partial t(r,\theta_{-i})}{\partial r}\right] dr = 0$$

because the bracketed term equals zero for all $r$ (see equation (23.C.12)). This, however, contradicts our negation assumption that $u(\theta_i,\hat{\theta}_i)-u(\theta_i,\theta_i)>0$ so $f(\cdot)$ must be truthfully implementable.

<u>Case 2:</u> Suppose $\hat{\theta}_i<\theta_i$. We can proceed as before, however the inequality in (vi) above will be reversed, and we will have a minus sign before the integral, so we will get the same contradiction.

**23.C.10** [<u>First Printing Errata:</u> At the end of the first paragraph insert: "Assume throughout that conditions are such that (23.C.8) holding is a necessary condition for $(k^*(\cdot),t_1(\cdot),\ldots,t(\cdot)_I)$ to be truthfully implementable in dominant strategies." Also, in the second line of part c) insert the word "implementable" before "ex post efficient social choice function".]

a) <u>Sufficiency:</u> Suppose that we can write $V^*(\theta) = \sum_i V_i(\theta_{-i})$. Consider the transfer functions of the form

$$t_i(\theta) = \left[\sum_{j\neq i} v_j(k^*(\theta),\theta_j)\right] + h_i(\theta_{-i}) \ .$$

where for all i,

$$h_i(\theta_{-i}) = -(I - 1)V_i(\theta_{-i}) \quad \text{for all } \theta_{-i} \ .$$

By proposition 23.C.4, $(k^*(\cdot),t_1(\cdot),\ldots,t(\cdot)_I)$ is truthfully implementable in dominant strategies. Moreover, for all $\theta$ we have,

$$\sum_i t_i(\theta) = \sum_i \left[\sum_{j\neq i} v_j(k^*(\theta),\theta_j)\right] + \sum_i h_i(\theta_{-i})$$

$$= (I-1)V^*(\theta) - (I-1)\sum_i V_i(\theta_{-i}) = 0$$

__Necessity:__ Suppose $(k^*(\cdot),t_1(\cdot),\ldots,t(\cdot)_I)$ is ex post efficient and is truthfully implementable in dominant strategies. Since (23.C.8) is necessary (by assumption) for truthful implementation, this means that there exist functions $(h_i(\theta_{-i}))_{i=1}^{I}$ such that

$$(I-1)V^*(\theta) + \sum_i h_i(\theta_{-i}) = \sum_i \left[\sum_{j\neq i} v_j(k^*(\theta),\theta_j)\right] + \sum_i h_i(\theta_{-i})$$

$$= \sum_i t_i(\theta) = 0$$

But this implies that by defining

$$V_i(\theta_{-i}) = \left(\frac{-1}{I-1}\right)h_i(\theta_{-i}) \ ,$$

we can then write $V^*(\theta) = \sum_i V_i(\theta_{-i})$ .

b) If $v_i(k,\theta_i) = \theta_i k - \frac{1}{2}k^2$ for all i, then, $k^*(\theta) = \text{Argmax}_k(\sum_i\theta_i)k - \frac{3}{2}k^2$ for all $\theta$, and so the FOC implies that $k^*(\theta) = \frac{\sum_i\theta_i}{3}$ . Hence,

$$V^*(\theta) = \sum_{i=1}^{3}\left[\theta_i\left(\frac{\sum_i\theta_i}{3}\right) - \frac{1}{2}\left(\frac{\sum_i\theta_i}{3}\right)^2\right]$$

$$= \left(\frac{\sum_i\theta_i}{3}\right)\sum_i\left[\theta_i - \frac{1}{2}\left(\frac{\sum_i\theta_i}{3}\right)\right]$$

$$= (\theta_1+\theta_2+\theta_3)[\theta_1+\theta_2+\theta_3 - \tfrac{1}{2}(\theta_1+\theta_2+\theta_3)]$$

$$= \tfrac{1}{2}(\textstyle\sum_i\theta_i)^2$$

$$= (\theta_1^2+\theta_2^2+\theta_3^2+2\theta_1\theta_2+2\theta_1\theta_3+2\theta_2\theta_3) \ .$$

We now define,

$$V_1(\theta_2,\theta_3) = \frac{\theta_2^2+\theta_3^2}{2} + 2\theta_2\theta_3 \ ,$$

$$V_2(\theta_1,\theta_3) = \frac{\theta_1^2+\theta_3^2}{2} + 2\theta_1\theta_3 \ ,$$

$$V_3(\theta_1,\theta_2) = \frac{\theta_1^2 + \theta_2^2}{2} + 2\theta_1\theta_2 \ .$$

and the result then follows from part a) above since

$$V^*(\theta) = V_1(\theta_2,\theta_3) + V_2(\theta_1,\theta_3) + V_3(\theta_1,\theta_2) \ .$$

c) If $V^*(\theta) = \sum_i V_i(\theta_{-i})$ then clearly $\dfrac{\partial^I V^*(\theta)}{\partial\theta_1\cdots\partial\theta_I} = 0$.

d) In this case, $V^*(\theta_1,\theta_2) = v_1(k^*(\theta),\theta_1) + v_2(k^*(\theta),\theta_2)$, therefore,

$$\frac{\partial V^*}{\partial\theta_1} = \left(\frac{\partial v_1}{\partial k} + \frac{\partial v_2}{\partial k}\right)\frac{\partial k}{\partial\theta_1} + \frac{\partial v_1}{\partial\theta_1} \ ,$$

$$\frac{\partial^2 V^*}{\partial\theta_1\partial\theta_2} = \left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right)\left(\frac{\partial k}{\partial\theta_1}\right)\left(\frac{\partial k}{\partial\theta_2}\right) + \frac{\partial^2 v_2}{\partial k\partial\theta_2}\frac{\partial k}{\partial\theta_1} + \frac{\partial^2 v_1}{\partial k\partial\theta_1}\frac{\partial k}{\partial\theta_2}.$$

Since,

$$\frac{\partial v_1}{\partial k} + \frac{\partial v_2}{\partial k} = 0 \ ,$$

we have,

$$\frac{\partial^2 v_i}{\partial k\partial\theta_i} = -\frac{\partial k}{\partial\theta_i}\left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right),$$

which in turn implies that

$$\frac{\partial^2 V^*}{\partial\theta_1\partial\theta_2} = -\left(\frac{\partial^2 v_1}{\partial k^2} + \frac{\partial^2 v_2}{\partial k^2}\right)\left(\frac{\partial k}{\partial\theta_1}\right)\left(\frac{\partial k}{\partial\theta_2}\right) \neq 0 \ ,$$

thus proving the statement.

23.C.11 Let agent i's Bernoulli utility function be $u_i(v_i(k,\theta_i) + \bar{m}_i + t_i$ and assume in negation that Proposition 23.C.4 no longer holds. That is, there exists i, $\theta_i$, $\hat{\theta}_i$, and $\theta_{-i}$ such that:

$$u_i(v_i(k^*(\hat{\theta}_i,\theta_{-i}),\theta_i) + \bar{m}_i + t_i(\hat{\theta}_i,\theta_{-i}) > u_i(v_i(k^*(\theta),\theta_i) + \bar{m}_i + t_i(\theta)$$

Substituting from (23.C.8) we get:

$$u_i(v_i(k^*(\hat{\theta}_i,\theta_{-i}),\theta_i)+\bar{m}_i+\sum_{j\neq i} v_j(k^*(\hat{\theta}_i,\theta_{-i}),\theta_j) + h_i(\theta_{-i})) >$$

but when comparing each of these two alternatives to all other alternatives there can be disagreement.

21.D.3 Clearly, the generated social preferences are acyclic: for all possibilities of individual preferences, agent 1's most preferred alternative will be at least as good (but not necessarily <u>better</u>) than any other alternative. To see that quasitransitivity may not hold, let the preferences be $x \succ_1 z \succ_1 y$ and $y \succ_2 x \succ_2 z$. From the definition of social preferences we have that $x\ F_p(\succsim_1,\ldots,\succsim_I)\ z$ and $z\ F_p(\succsim_1,\ldots,\succsim_I)\ y$, but we do not have $x\ F_p(\succsim_1,\ldots,\succsim_I)\ y$ because individual 2 can veto this. Note also, that in spite of the veto power of 2, x is the only maximal element in this case: we have that $x\ F_p(\succsim_1,\ldots,\succsim_I)\ z$ and $x\ F(\succsim_1,\ldots,\succsim_I)\ y$. However, y is not a maximal element since $z\ F_p(\succsim_1,\ldots,\succsim_I)\ y$, which happens because individual 2 cannot veto z preferred to y.

21.D.4 Assume $\succsim$ is single peaked with $x^* \in [0,1]$ as the peak alternative. Assume that y and z are two distinct alternatives, neither of them equals $x^*$, such that $y \sim z$. We need to show that for all $\lambda \in (0,1)$, $\lambda y + (1-\lambda)z \succ y$. First, we must have either $y < x^* < z$ or $z < x^* < y$ (or else we could not have $y \sim z$). W.l.o.g. assume that $y < x < z$, and choose some $\lambda \in (0,1)$. Letting $x' = \lambda y + (1-\lambda)z$, we either have that $x' \in (y,x^*]$, in which case $x' \succ y$, or we have that $x' \in (x^*,z)$, in which case $x' \succ z$, which implies that $x' \succ y$.

21.D.5 The six possible orderings are: (1) (x,y,z); (2) (x,z,y); (3) (y,x,z); (4) (y,z,x); (5) (z,x,y); and (6) (z,y,x). Recall that the preferences of the Condorcet paradox are $x \succ_1 y \succ_1 z$ , $z \succ_2 x \succ_2 y$, and $y \succ_3 z \succ_3 x$. Agent 1's preferences are not single peaked for orderings (2) and (4), agent 2's preferences are not single peaked for orderings (1) and (6), and agent 3's preferences are not single peaked for orderings (3) and (5).

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21.D.6 Let $X = \{x,y,z\}$, $I = \{1,2,3,4\}$, and consider the following profile of preferences: $x \succ_1 y \succ_1 z$, $z \succ_2 y \succ_2 x$, $x \succ_3 z \succ_3 y$, and $y \succ_4 x \succ_4 z$. We thus have $\#\{i : x \succ_i y\} = \#\{i : y \succ_i x\} = \#\{i : z \succ_i y\} = \#\{i : y \succ_i z\} = 2$, which implies that x is socially indifferent to y, and y is socially indifferent to z, so we can write $z\ \hat{F}(\succsim_1,\ldots,\succsim_6)\ y$ and $y\ \hat{F}(\succsim_1,\ldots,\succsim_6)\ x$. It is not true, however, that $z\ \hat{F}(\succsim_1,\ldots,\succsim_6)\ x$ since $\#\{i : x \succ_i z\} = 3$ and $\#\{i : z \succ_i x\} = 1$ which implies $x\ \hat{F}_p(\succsim_1,\ldots,\succsim_6)\ z$. Therefore, in this case majority voting fails to generate a fully transitive social welfare functional.

21.D.7 (a) Since the cone spanned by $\{\nabla u_1(0), \nabla u_2(0), \nabla u_3(0)\}$ is the entire space $R^2$, we must have no linear dependence between the $\nabla u_i(0)$'s, and the directions of the gradients must be as drawn in Figure 21.D.7(a1) (they cannot all point in directions that can be "bounded" by some straight line going through the origin). The dashed lines $p_1$, $p_2$, and $p_3$ are the perpendicular lines to the direction of each gradient respectively, and the indifference curve of agent i that passes through the origin is tangent to $p_i$. These indifference curves are shown in Figure 21.D.7(a2).

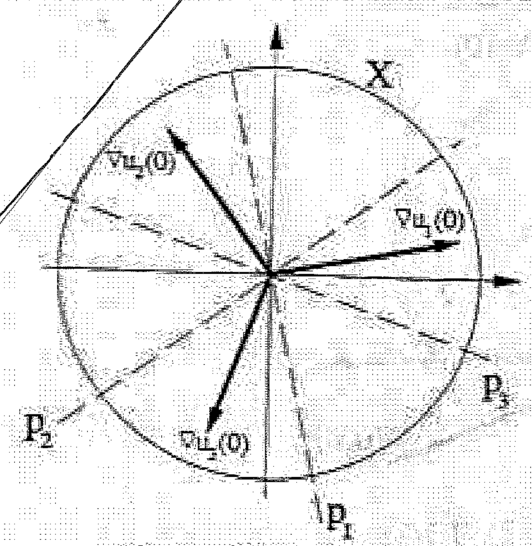


Figure 21.D.7(a1)

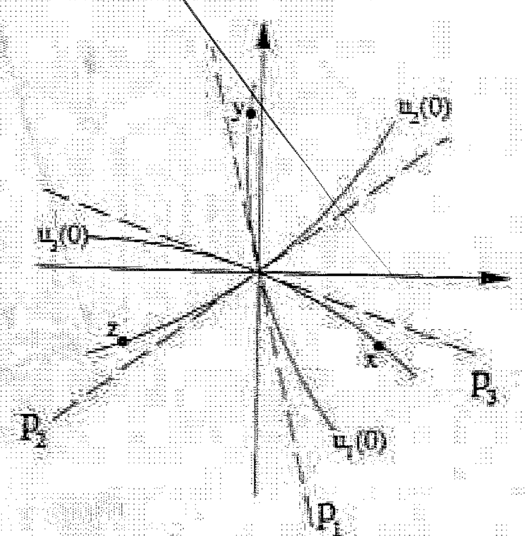


Figure 21.D.7(a2)