EconS 503 - Microeconomic Theory II
Midterm Exam #2 - Answer key

1. **Temporary punishments in Bertrand competition**. Consider an industry with two firms competing in prices a la Bertrand, facing a linear inverse demand function $p(Q) = 100 - Q$, where Q denotes aggregate output. Firms face a common marginal cost $c = 10$. For simplicity, assume that both firms have the same discount factor $\delta \in (0, 1)$.

   (a) *Bertrand equilibrium.* Find equilibrium prices in Nash equilibrium of the Bertrand game when firms interact only once.

   - When firms i and j, where $i, j \in \{1, 2\}$, interact only once, each firm adopts marginal cost pricing, that is, $p^* = c = 10$, yielding a total sales of $Q = 100 - 10 = 90$ units, which are equally divided between the two firms, that is, $q_i^* = q_j^* = 45$. Equilibrium profits are zero in this setting.

   (b) *Infinitely repeated game - Permanent reversion.* Consider now a grim-trigger strategy (GTS) where firms start setting a collusive price that maximizes their joint profits and continue to do so if both firms chose collusive prices in all previous periods. Otherwise, every firm permanently reverts to the Bertrand equilibrium you found in part (a). Under which conditions on δ this GTS can be sustained as a SPNE of the infinitely repeated game?

   - *Collusion.* Rearrange the inverse demand function $p(Q) = 100 - Q$ to obtain the demand function, $Q(p) = 100 - p$. The cartel comprising firms i and j chooses price p to solve the following joint profit maximization problem:

$$\begin{aligned}\max_{p \geq 0} \pi(p) &= p(Q)Q - cQ \\ &= p(100 - p) - 10(100 - p) \\ &= (p - 10)(100 - p)\end{aligned}$$

     Differentiating with respect to the price p, and assuming interior solutions, that is, $p > 0$, we obtain

$$110 - 2p = 0,$$

     which yields a collusive price of $p^C = \$55$.
     Substituting the collusive price, $p^C = 55$, into the demand function, we obtain that aggregate output becomes $Q^C = 100 - 55 = 45$ units. Since firms are symmetric, each firm would serve half of the market with $q_i^C = q_j^C = \frac{45}{2} = 22.5$ units of output. As a result, every firm i in the cartel earns a collusive profit of

$$\pi_i^C = \frac{(p - 10)(100 - p)}{2} = \frac{(55 - 10)(100 - 55)}{2} = \$1,012.5.$$

- *Cooperation.* If at any period t after a history of cooperation firm i charges the collusive price $p^C = 45$ (as prescribed by the grim-trigger strategy), every firm i obtains profits of $962\frac{1}{2}$ in every period, as follows

$$1,012.5 + (\delta \times 1,012.5) + \left(\delta^2 \times 1,012.5\right) + \ldots$$
$$= \frac{1,012.5}{1-\delta}$$

- *Optimal deviation.* Let us now analyze the payoff that firm i can obtain if it deviates from cooperation. If at any period t after a history of coopa-ration firm i deviates from the collusive price of $p^C = 55$, its optimal deviation is to undercut firm j's price by $\varepsilon > 0$, such that it captures the entire market, selling 55 units of output, and earning a deviating profit of $55(45) - 10(45) = \$2,025$. Firms i and j detect this deviation immediately, and revert to marginal cost pricing thereafter, which entails zero profits for all subsequent periods. In this context, the payoff that firm i obtains from deviating at any period t is

$$\underbrace{2,025}_{\pi_i^{Dev}} + \underbrace{(\delta \times 0) + \left(\delta^2 \times 0\right) + \ldots}_{\text{Infinite punishment}}$$
$$= 2,025$$

- *Comparison.* Therefore, to sustain cooperation, it must be that the cooperation profit must be weakly higher than the deviation profit, that is,

$$\frac{1,012.5}{1-\delta} \geq 2,025$$

cross multiplying by $1-\delta$, and solving for discount factor δ, we find that cooperation can be sustained as long as $\delta \geq \frac{1}{2}$.

c. *Repeated game, Temporary reversion.* Consider again the grim-trigger strategy of part (c), but assume that, upon observing a deviation, firms revert to the Bertrand equilibrium of part (a) during T periods, returning to the collusive price if both firms chose the Bertrand equilibrium price during the last T periods (that is, both firms return to cooperation if they observe that both implemented the punishment during the prescribed T periods). Under which conditions on firms' common discount factor this grim-trigger strategy can be sustained as a SPNE of the infinitely repeated game? [*Hint*: Rather than solving for the minimal discount factor sustaining cooperation, δ, solve for the lenght of the temporary punishment T.]

- *Cooperation.* If at any period t after a history of cooperation firm i charges the collusive price $p^C = 55$ (as prescribed by the grim-trigger strategy), firm i obtains

$$1,012.5 + (\delta \times 1,012.5) + \left(\delta^2 \times 1,012.5\right) + \ldots + \left(\delta^T \times 1,012.5\right) + \left(\delta^{T+1} \times 1,012.5\right) + \ldots$$
$$= \left(1,012.5 \times \frac{1-\delta^{T+1}}{1-\delta}\right) + \left(1,012.5 \times \frac{\delta^{T+1}}{1-\delta}\right)$$

- *Optimal deviation.* Let us now analyze the payoff that firm i can obtain if it deviates from cooperation. By deviating, firm i earns all of the cartel profit in this period, nothing in the following T periods, and half of the cartel profits beginning the $T+1$ period, yielding a deviation payoff of

$$\underbrace{2,025}_{\pi_i^{Dev}} + \underbrace{(\delta \times 0) + \left(\delta^2 \times 0\right) + \ldots + \left(\delta^T \times 0\right)}_{\text{Punishment for } T \text{ periods}} + \underbrace{\left(\delta^{T+1} \times 1,012.5\right) + \delta^{T+2} \times 1,012.5 + \ldots}_{\text{Return to cooperation}}$$

$$= 2,025 + \left(1,012.5 \times \frac{\delta^{T+1}}{1-\delta}\right)$$

- *Comparison.* Therefore, to sustain cooperation, it must be that the cooperation profit must be weakly higher than the deviation profit, that is,

$$\left(1,012.5 \times \frac{1-\delta^{T+1}}{1-\delta}\right) + \left(1,012.5 \times \frac{\delta^{T+1}}{1-\delta}\right) \geq 2,025 + \left(1,012.5 \times \frac{\delta^{T+1}}{1-\delta}\right)$$

which can be rearranged as follows

$$\frac{1-\delta^{T+1}}{1-\delta} \geq 2$$
$$\iff 2\delta - 1 \geq \delta^{T+1}$$

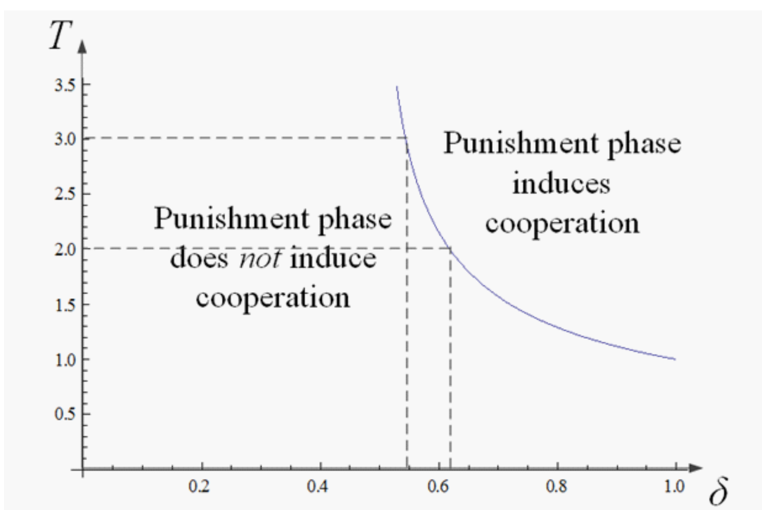
applying logs on both sides, we obtain

$$\ln(2\delta - 1) \geq (T+1)\ln\delta$$

and solving for the lenght of the temporary punishment, T, we find that

$$T \equiv \hat{T} \geq \frac{\ln(2\delta-1)}{\ln\delta} - 1$$

and the last inequality stems from the fact that $\ln\delta < 0$ that reverses the inequality sign. The following figure depicts cutoff $\hat{T}$ as a function of the discount factor δ. Intuitively, the punishment phase must be long enough and firms must care enough about their future profits (as indicated by T and δ pairs on the northwest of the figure) for the GTS with temporary punishment to be sustained as a SPNE of the infinitely repeated game.



When the punishment phase lasts two periods, as indicated by the dotted line at a height of $T = 2$, cooperation can be sustained for discount factors satisfying $\delta \geq 0.61$, graphically represented by the range of δ to the right-hand side of $\delta = 0.61$ in the figure. A similar argument applies when the punishment phase lasts $T = 3$ periods, where we obtain that cooperation can be supported as long as $\delta \geq 0.54$.

- After a history in which at least one firm deviated from cooperation, the GTS prescribes that every firm i implements the punishment during T rounds. This is firm i's best response to firm j implementing the punishment, so there are no further conditions on the discount factor, δ, or the lenght of the punishment phase, T, that we need to impose.

2. **Cournot competition when all firms are uninformed - Allowing for cost correlation.** Consider an industry with two firms competing a la Cournot and inverse demand function $p(Q) = 1 - Q$ where $Q = q_1 + q_2$ denotes aggregate output. Every firm i privately observes its marginal cost of production, $MC_i = 1/4$ or $MC_i = 0$, both equally likely, but it does not observes its rival's marginal costs, MC_j. The probability distribution is, however, common knowledge among firms. Assume that firms' costs are *positive correlated*, that is, when firm i observes that its costs are high ($MC_i = 1/4$), it assigns a probability $p^H > 1/2$ to its rival's costs being high and $1 - p^H < 1/2$ to its rival's costs being low. Similarly, after observing that its own costs are low, $MC_i = 0$, firm i assigns a probability $p^L < 1/2$ to its rival's costs being high and $1 - p^L > 1/2$ to its rival's costs being low. Intuitively, observing that its own costs are high (low) increases the probability that its rival's costs are high (low) as well.

   (a) Find the best response function for every firm i, $q_i^k(q_j^H, q_j^L)$, where $k = \{H, L\}$ denotes firm i's marginal cost (high or low).

   - *Low costs.* When firm i has low costs, it chooses $q_i^L \geq 0$ that solves the following expected profit maximization problem:

$$\max_{q_i^L \geq 0} \ \pi_i^L\left(q_i^L\right) = \overbrace{(1-p^L)\left(1 - q_i^L - q_j^L\right) q_i^L}^{\text{Profits if } j \text{ is low cost}} + \overbrace{p^L\left(1 - q_i^L - q_j^H\right) q_i^L}^{\text{Profits if } j \text{ is high cost}}$$

   Assuming interior solutions, that is, $q_i^L > 0$, the first order condition satisfies

$$\frac{\partial \pi_i^L\left(q_i^L\right)}{\partial q_i^L} = 1 - 2q_i^L - q_j^L - p^L(q_j^H - q_j^L) = 0$$

   such that the best response function of firm i when its costs are low becomes

$$q_i^L\left(q_j^L, q_j^H\right) = \frac{1}{2} - \frac{p^L q_j^H + (1-p^L)q_j^L}{2}$$

   which originates at $1/2$, and decreases in its rival's expected output, $p^L q_j^H + (1-p^L)q_j^L$, at the rate of $\frac{1}{2}$.

   - *High costs.* When firm i has high costs, it chooses $q_i^H \geq 0$ that solves the following expected profit maximization problem:

$$\begin{aligned}\max_{q_i^H \geq 0} \ \pi_i^H\left(q_i^H\right) &= \overbrace{(1-p^H)\left(1 - q_i^H - q_j^L\right) q_i^H}^{\text{Profit if } j \text{ is low cost}} + \overbrace{p^H\left(1 - q_i^H - q_j^H\right) q_i^H}^{\text{Profit if } j \text{ is high cost}} - \frac{1}{4}q_i^H \\ &= \left(\frac{3}{4} - q_i^H - q_j^L - p^H(q_j^H - q_j^L)\right) q_i^H\end{aligned}$$

   Assuming interior solutions, that is, $q_i^H > 0$, the first order condition satisfies

$$\frac{\partial \pi_i^H(q_i^H)}{\partial q_i^H} = \frac{3}{4} - 2q_i^H - q_j^L - p^H(q_j^H - q_j^L)$$

such that the best response function of firm i when its costs are high becomes

$$q_i^H\left(q_j^L, q_j^H\right) = \frac{3}{8} - \frac{p^H q_j^H + (1-p^H)q_j^L}{2}$$

which originates at 3/8, but decreases in its rival's expected output, $p^H q_j^H + (1-p^H)q_j^L$. Comparing it with firm i's best response function when its costs are low, $q_i^L\left(q_j^L, q_j^H\right)$, we can see that, for a given profile of firm j's output, $\left(q_j^L, q_j^H\right)$, firm i responds producing a larger output when its own costs are low than when they are high.

(b) Use your results from part (a) to find the Bayesian Nash Equilibrium (BNE) of the game.

- Since firms i and j are symmetric, we impose symmetry on the equilibrium output that

$$q^L = q_i^L = q_j^L$$
$$q^H = q_i^H = q_j^H$$

Substituting the above results into the best response functions we found in part (a), yields

$$q^L = \frac{1}{2} - \frac{p^L q^H + (1-p^L)q^L}{2}$$
$$q^H = \frac{3}{8} - \frac{p^H q^H + (1-p^H)q^L}{2}$$

Solving for q^L and q^H in the simultaneous equations above, the equilibrium output satisfies

$$q^{L*} = \frac{8+4p^H-3p^L}{12(2+p^H-p^L)} \quad \text{and} \quad q^{H*} = \frac{5+4p^H-3p^L}{12(2+p^H-p^L)}.$$

(c) Evaluate your results in the special cases of perfect positive (negative) cost correlation, where $p^H = 1$ and $p^L = 0$ (where $p^H = 0$ and $p^L = 1$, respectively).

- Under perfect positive cost correlation, where $p^H = 1$ and $p^L = 0$, equilibrium output levels we found in part (b) become

$$q^{L*} = \frac{1}{3} \quad \text{and} \quad q^{H*} = \frac{1}{4}$$

In this case, firm j's costs coincide with those of firm i. In contrast, under perfect negative cost correlation, $p^H = 0$ and $p^L = 1$, equilibrium output levels in part (b) simplify to

$$q^{L*} = \frac{5}{12} \quad \text{and} \quad q^{H*} = \frac{1}{6}$$

Comparing the above output levels, note that, when firm i observes that its own costs are low, its output level q^{L*} is higher when it knows that its rival's costs are high (under negatively correlated costs, where it produces 5/12 units) than when it knows that its rival's costs are low (under positively correlated costs, where it only produces 1/3 units). When firm i observes that its costs are high, its output level q^{H*} is higher when it knows that its rival's costs are also high (under positive cost correlation, where it produces 1/4 units) than when its rival's costs are low (under negatively correlated costs, where it only produces 1/6 units).

3. **First-price auction with entry fees.** Consider a first-price auction with N bidders. Every bidder i's valuation, v_i, is distributed according to a uniform distribution function, that is, $F(v_i) = v_i$ for all $v_i \sim U[0, \overline{v}]$. Consider the following two-stage game: in the first stage, the seller sets an entry fee $E \geq 0$ that every participating bidder must pay, otherwise his bid is ignored; in the second stage, every bidder i independently and simultaneously submit his bid for the object.

   (a) *Second stage.* In this part of the exercise, let us focus on the second stage of the game. For a given entry fee E, find the optimal bidding function that bidder i chooses in the second stage, $b_i(v_i, E)$. [*Hint*: Assume that there exists a critical bidder whose valuation v_e makes him indifferent between participation or not, given a positive entry fee E].

   - Every bidder i's expected utility maximization problem is

   $$\max_{b_i \geq 0} EU_i(b_i) = prob\{win\}(v_i - b_i) - E$$

   where the entry fee, E, is a constant that bidder i must pay when he participates in the auction, whether he wins the object or not.

   - The probability of bidder i winning the object is analogous to the standard First Price Auction without entry fee, which is given by

   $$prob\{win\} = [F(v_i)]^{N-1}$$

   when his valuation exceeds other $N-1$ bidders, $v_i \geq v_j$ for $j \neq i$, $j \in \{1, \ldots, N\}$. Note that for a given bidding strategy, $b : [0, \overline{v}] \rightarrow \mathbb{R}_+$, that is, $b_i(v_i) = b_i$, we can define its inverse, $b_i^{-1}(b_i) = v_i$, implying that the cumulative distribution function, which represents the probability mass of valuation below his, can be rewritten as

   $$F(v_i) = F\left(b_i^{-1}(b_i)\right)$$

   such that bidder i's expected utility maximization problem becomes

   $$\max_{b_i \geq 0} EU_i(b_i) = \left[F\left(b_i^{-1}(b_i)\right)\right]^{N-1}(v_i - b_i) - E$$

   - We assume that the bidding function $b_i(v_i)$ is monotonically increasing in v_i; which we will demonstrate later. In addition, let us define a "critical bidder" with valuation v_e and bid $b_e(v_e)$, where v_e solves[1]

   $$[F(v_e)]^{N-1}(v_e - b_e) - E = 0$$

   which means that his expected utility from participating in the auction (given the entry fee E) is zero.

[1] We need this condition to ensure a one-to-one mapping between the entry fee and the critical bidder's valuation. The monotonically increasing bidding function $b_i(v_i)$ ensures that bidders with valuations above v_e obtain a positive utility from participating in the auction (despite the entry fee E) and thus submit a positive bid for the object.

- Differentiating bidder i's expected utility with respect to his bid b_i yields

$$(N-1)\left[F\left(b_i^{-1}(b_i)\right)\right]^{N-2} f\left(b_i^{-1}(b_i)\right)\frac{\partial b_i^{-1}(b_i)}{\partial b_i}(v_i-b_i)-\left[F\left(b_i^{-1}(b_i)\right)\right]^{N-1}=0$$

Since the inverse of the bidding function yields the bidder's valuation, $b_i^{-1}(b_i)=v_i$, and the derivative of this inverse can be written as $\frac{db_i^{-1}(b_i)}{db_i}=\frac{1}{b_i'\left(b_i^{-1}(b_i)\right)}$, the above expression becomes

$$(N-1)\left[F(v_i)\right]^{N-2} f(v_i)(v_i-b_i)=b_i'(v_i)\left[F(v_i)\right]^{N-1}$$

Further rearranging, we obtain

$$(N-1)\left[F(v_i)\right]^{N-2} f(v_i)\, b_i(v_i)+\left[F(v_i)\right]^{N-1} b_i'(v_i)=(N-1)\left[F(v_i)\right]^{N-2} f(v_i)\, v_i$$

The left-hand side is $\frac{d\left[\left[F(v_i)\right]^{N-1}b_i(v_i)\right]}{dv_i}$. Hence,

$$\frac{\partial\left[\left[F(v_i)\right]^{N-1}b_i(v_i)\right]}{\partial v_i}=(N-1)\left[F(v_i)\right]^{N-2} f(v_i)\, v_i$$

- Integrating the right-hand side of the above expression with respect to v_i (we use integration by parts), and taking the indifferent bidder's valuation v_e as the lower bound of integration, yields

$$\int_{v_e}^{v_i}\frac{d\left[\left[F(x)\right]^{N-1}b_i(x)\right]}{dv_i}dx=\int_{v_e}^{v_i}(N-1)\left[F(x)\right]^{N-2} f(x)\, x\,dx$$

$$\Rightarrow\left[\left[F(x)\right]^{N-1}b_i(x)\right]_{v_e}^{v_i}=\left[\left[F(x)\right]^{N-1}x\right]_{v_e}^{v_i}-\int_{v_e}^{v_i}\left[F(x)\right]^{N-1}dx$$

We can then reorder the terms in the above expression as follows:

$$\left[F(v_i)\right]^{N-1}b_i(v_i)=\left[F(v_i)\right]^{N-1}v_i-\left[F(v_e)\right]^{N-1}\left[v_e-b_e(v_e)\right]-\int_{v_e}^{v_i}\left[F(x)\right]^{N-1}dx$$

Substituting the indifferent bidder condition, $\left[F(v_e)\right]^{N-1}(v_e-b_e)-E=0$, into the above expression, and rearranging, we obtain

$$b_i(v_i)=v_i-\underbrace{\frac{E+\int_{v_e}^{v_i}\left[F(x)\right]^{N-1}dx}{\left[F(v_i)\right]^{N-1}}}_{\text{bid shading}}$$

Note that when entry fees are absent, $E=0$, the equilibrium bidding function collapses to the standard expression found in previous exercises. Since valuations are uniformly distributed, this bidding function simplifies to

$$b_i(v_i)=v_i-\underbrace{\frac{E+\int_{v_e}^{v_i}x^{N-1}dx}{v_i^{N-1}}}_{\text{bid shading}}$$

and, solving the integral, we obtain

$$b_i(v_i) = v_i - \frac{NE + \left[x^N\right]_{v_e}^{v_i}}{Nv_i^{N-1}}$$
$$= v_i - \underbrace{\frac{NE + v_i^N - v_e^N}{Nv_i^{N-1}}}_{\text{Bid shading}}$$

and the indifferent bidder's bid, $b_e(v_e)$, solves

$$v_e^{N-1}(v_e - b_e(v_e)) = E$$

- Lastly, we show that the equilibrium bidding function, $b_i(v_i)$, is monotonically increasing in the bidder's valuation, v_i, since

$$\frac{db_i(v_i)}{dv_i} = 1 - \frac{[F(v_i)]^{2N-2} - (N-1)[F(v_i)]^{N-2}\left(E + \int_{v_e}^{v_i}[F(x)]^{N-1}dx\right)}{[F(v_i)]^{2N-2}}$$
$$= \frac{N-1}{[F(v_i)]^N}\left(E + \int_{v_e}^{v_i}[F(x)]^{N-1}dx\right) > 0.$$

(b) How are equilibrium bids affected by an increase in the entry fee E? Does a higher E limit participation in the auction?

- As in other games on first-price auctions, the second term represents bidder i's bid shading, which is increasing in the entry fee E. Intuitively, the entry fee affects all bidders uniformly, inducing those with valuations below v_e to not participate in the auction and, in addition, reducing the bid of those who participate. Intuitively, every participating bidder expects a lower expected utility in equilibrium, both if he wins and if he loses the auction, leading him to decrease his bid (larger bid shading) to compensate for his lower expected utility.

(c) *First stage.* Anticipating the optimal bidding function $b_i(v_i)$ you found in part (a), what is the optimal entry fee E^* that the seller sets in the first stage to maximize his expected revenue from the auction? For simplicity, assume that the critical bidder, who is indifferent between participation or not, submits a bid, $b_e(v_e) = 0$.

- We first find the seller's expected revenue from the auction, and then differentiate it with respect to the entry fee E, to identify the revenue maximizing entry fee E^*.
- *Finding the seller's revenue from the auction.* For compactness, let us define $G(x) = (F[x])^{N-1}$ to be the joint cumulative probability density function for $N-1$ bidders, where valuation x satisfies $x \in [0, \overline{v}]$. Then the above optimal

bidding function can be rewritten as

$$\begin{aligned} b_i(v_i) &= v_i - \frac{E + \int_{v_e}^{v_i} G(x)\,dx}{G(v_i)} \\ &= \frac{1}{G(v_i)}\left[G(v_i)\,v_i - E + G(v_e)\,v_e - G(v_e)\,v_e - \int_{v_e}^{v_i} G(x)\,dx\right] \\ &= \frac{1}{G(v_i)}\int_{v_e}^{v_i} xg(x)\,dx \end{aligned}$$

by the fact that $G(v_e)\,v_e = E$ and the opposite of integration by parts.
From an ex-ante point of view (before observing his own valuation for the object), bidder i's expected payment to the seller is given by the probability of winning the auction times the bid he pays for the object upon winning, that is,

$$\begin{aligned} \pi_i(v_i|v_i \geq v_e) &= prob(win) \times b_i(v_i) + E \\ &= G(v_i) \times \frac{1}{G(v_i)}\int_{v_e}^{v_i} xg(x)\,dx + E \\ &= \int_{v_e}^{v_i} xg(x)\,dx + E \end{aligned}$$

where the second line indicates that, as discussed in previous parts of the exercise, bidder i wins the auction if his valuation v_i is above everyone else's, that is, $v_i \geq v_j$ for every bidder $j \neq i$; and in addition, he participates in the auction and pays a participation fee E. The probability of his valuation exceeding that of every other bidder is given by $(F[v_i])^{N-1}$, and we represent it more compactly as $G(v_i) = (F[v_i])^{N-1}$, and the last line inserts the equilibrium bidding function found above, $b_i(v_i)$.

- Since the seller cannot observe bidders' values, he finds the expected payment from each bidder i, $E[\pi_i(v_i|v_i \geq v_e)]$, and then sums up for all N bidders, $\sum_{i=1}^{N} E[\pi_i(v_i|v_i \geq v_e)]$, which gives us the seller's revenue from the auction (this is, of course, understood from an *ex-ante* perspective since the seller does not observe bidders' valuations). We find the seller's revenue as follows

$$\begin{aligned} E[\pi(v_e)] &= \sum_{i=1}^{N} E[\pi_i(v_i|v_i \geq v_e)] \\ &= N\int_{v_e}^{\overline{v}}\left[E + \int_{v_e}^{v_i} xg(x)\,dx\right] f(z)\,dz \end{aligned}$$

Since the participation fee, E, is a constant, it is unaffected by the integration, helping us to rewrite the seller's revenue as

$$\begin{aligned} E\left[\pi\left(v_e\right)\right] &= NE\int_{v_e}^{\overline{v}} f\left(z\right)dz + N\int_{v_e}^{\overline{v}}\left(\int_{v_i}^{\overline{v}} f\left(z\right)dz\right)xg\left(x\right)dx \\ &= NE\left[1-F\left(v_e\right)\right] + N\int_{v_e}^{\overline{v}}\left[1-F\left(v_i\right)\right]xg\left(x\right)dx \\ &= Nv_eG\left(v_e\right)\left[1-F\left(v_e\right)\right] + N\int_{v_e}^{\overline{v}}\left[1-F\left(v_i\right)\right]xg\left(x\right)dx \end{aligned}$$

where the first line expands the integral into two parts; and the second line integrates the probability density function into the cumulative distribution function from the critical type's valuation v_e to the upper bound $\overline{v}$, with the second part exchanges the order of integration that considers the density mass of bidders whose valuation are above v_e. Finally, the third line stems from the fact that $E = \left[F\left(v_e\right)\right]^{N-1}\left(v_e - b_e\right) = G\left(v_e\right)v_e$ when we assume that the indifferent bidder submits a bid of $b_e\left(v_e\right) = 0$.

- *Revenue-maximizing reservation price.* We can now differentiate the seller's revenue with respect to the critical valuation v_e,

$$\begin{aligned} \frac{dE\left[\pi\left(v_e\right)\right]}{dv_e} &= N\left[G\left(v_e\right)\left(1-F\left(v_e\right)-v_ef\left(v_e\right)\right)+\left(1-F\left(v_e\right)\right)v_eg\left(v_e\right)-\left(1-F\left(v_e\right)\right)v_eg\left(v_e\right)\right] \\ &= NG\left(v_e\right)\left(1-F\left(v_e\right)\right)\left[1-v_e\frac{f\left(v_e\right)}{1-F\left(v_e\right)}\right] \end{aligned}$$

Assuming interior solutions, we set the above first order condition equal to zero.

$$v_e = \frac{1-F\left(v_e\right)}{f\left(v_e\right)}$$

The right-hand side of the above inequality is the inverse hazard rate, $\frac{1-F(v_e)}{f(v_e)}$, which measures how sensitive is the distribution of the bidders' valuation $F(\cdot)$ to a change in the critical valuation v_e. That is, if the density mass is concentrated in the region above the critical valuation v_e, then $1-F\left(v_e\right)$ would be large relative to $f\left(v_e\right)$ so that the seller can further increase the entry fee E to raise his expected revenue by selling the object to those bidders with higher valuations. Next, we will elaborate on the relationship between the entry fee E and the critical valuation v_e.

Evaluating $v_e = \frac{1-F(v_e)}{f(v_e)}$ for a uniformly distributed valuation $v_i \sim U\left[0,1\right]$, we obtain

$$v_e = 1 - v_e, \text{ or } v_e^* = \frac{1}{2}$$

so that half of the bidders would participate in the auction in equilibrium.

- Substituting $v_e = \frac{1-F(v_e)}{f(v_e)}$ into the indifferent bidder's valuation function, the optimal entry fee E^* solves

$$\begin{aligned} E^* &= \left[F\left(v_e\right)\right]^{N-1}v_e \\ &= \left[F\left(v_e\right)\right]^{N-1}\frac{1-F\left(v_e\right)}{f\left(v_e\right)} \end{aligned} \tag{4}$$

Substituting $v_e^* = \frac{1}{2}$ into the expression of E^*, we find

$$E^* = v_e^{N-1}(1 - v_e) = \frac{1}{2^N}$$

so that the entry fee E^* decreases in the number of bidders N at a decreasing rate. Note that when N becomes infinitely large, that is, $N \to \infty$, the profit-maximizing entry fee that the seller sets approaches zero asymptotically; as depicted in the next figure.

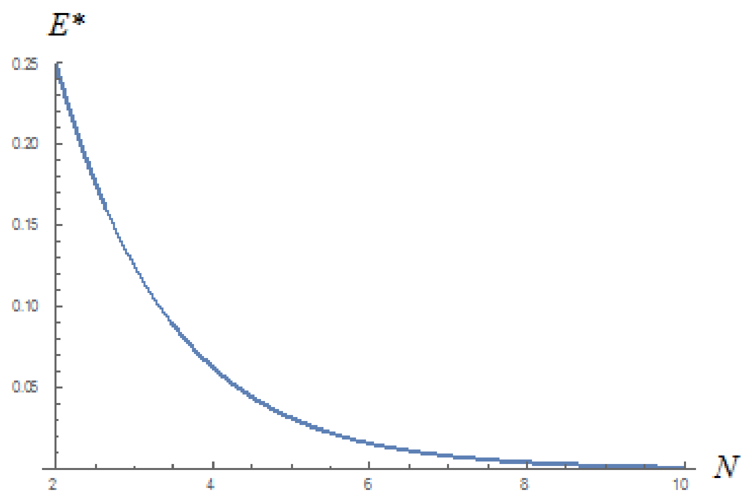


Figure 1. Optimal entry fee E^* as a function of the number of bidders, N.

For instance, when only $N = 2$ bidders compete for the object, the optimal entry fee becomes $E^* = \frac{1}{4}$, while when $N = 3$ bidders compete the entry fee decreases to $E^* = \frac{1}{8}$.

- Therefore, the optimal bidding function, $b_i^*(v_i)$, becomes

$$b_i^*(v_i) = v_i - \frac{NE^* + v_i^N - (v_e^*)^N}{Nv_i^{N-1}} = v_i - \frac{N\frac{1}{2^N} + v_i^N - \frac{1}{2^N}}{Nv_i^{N-1}} = \frac{N-1}{N}\left[1 - (2v_i)^{-N}\right] v_i$$

For instance, when only $N = 2$ bidders compete for the object, this optimal bidding function simplifies to

$$b_i^*(v_i) = \frac{v_i}{2} - \frac{1}{8v_i},$$

when $N = 3$ bidders compete, their optimal bidding function becomes

$$b_i^*(v_i) = \frac{2v_i}{3} - \frac{1}{12v_i^2},$$

while when $N = 10$ bidders compete in the auction, it becomes

$$b_i^*(v_i) = \frac{9v_i}{10} - \frac{9}{10240v_i^9}.$$

The next figure depicts these bidding functions, where valuations are restricted in $v_i \in \left[\frac{1}{2}, 1\right]$ since the bidder indifferent between participating and not participating in the auction when valuations are uniformly distributed is $v_e^* = \frac{1}{2}$, as shown above.

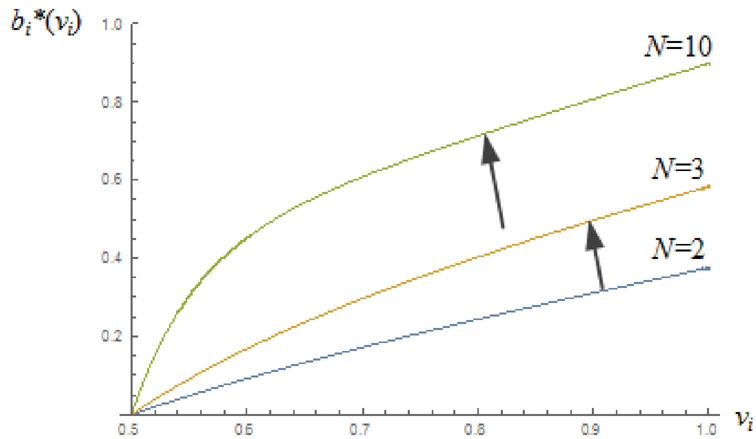


Figure 2. Optimal bidding function shifts up in N.

Every player's bids are increasing in his valuation for the object, v_i, and shift upwards when he competes against a larger number of bidders, N. We next confirm these two points more formally. First, let us differentiate the optimal bidding function, $b_i^*(v_i)$, with respect to valuation v_i,

$$\begin{aligned}\frac{db_i^*(v_i)}{dv_i} &= \frac{N-1}{N}\left[1 - (2v_i)^{-N} + N(2v_i)^{-N}\right] \\ &= \frac{N-1}{N}\left[1 + (N-1)(2v_i)^{-N}\right] > 0\end{aligned}$$

so that bidder i's bid is increasing in his valuation for the object, v_i. Second, let us now differentiate the optimal bidding function, $b_i^*(v_i)$, with respect to the number of bidders, N,

$$\begin{aligned}\frac{\partial b_i^*(v_i)}{\partial N} &= \frac{1}{N^2}\left[1 - (2v_i)^{-N}\right]v_i + \frac{N-1}{N}\left[N\log(2v_i)\right]v_i \\ &= v_i\left[\frac{1-(2v_i)^{-N}}{N^2} + (N-1)\log(2v_i)\right]\end{aligned}$$

and a sufficient condition for $\frac{\partial b_i^*(v_i)}{\partial N} \geq 0$ is $v_i \geq \frac{1}{2}$, which entails that $\log(2v_i) \geq 0$. However, we already showed that bidders who participate in this auction have a private value of $v_i \geq v_e^* = \frac{1}{2}$, such that for those bidders who participate, their equilibrium bids increase when facing competition from more bidders.

4. **Selten's horse.** Consider the "Selten's Horse" game depicted in Figure 1. Player 1 is the first mover in the game, choosing between C and D. If he chooses C, player 2 is called on to move between C' and D'. If player 2 selects C' the game is over. If player 1 chooses D or player 2 chooses D', then player 3 is called on to move without being informed whether player 1 chose D before him or whether it was player 2 who chose D'. Player 3 can choose between L and R, and then the game ends.

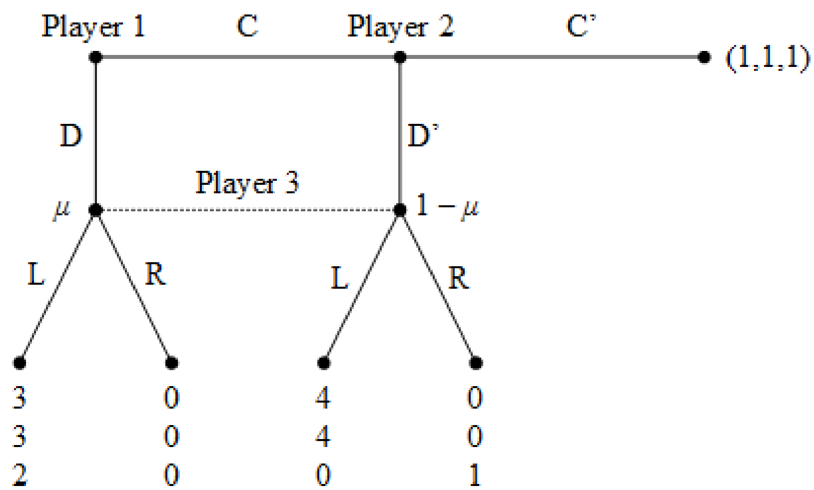


Figure 1. Selten's horse.

(a) Define the strategy spaces for each player. Then find all pure strategy Nash equilibria (psNE) of the game. [*Hint*: This is a three-player game, so you can consider that player 1 chooses rows, player 2 columns, and player 3 chooses matrices.]

- The strategy spaces of the players are as follows:

$$S_1 = \{C, D\}$$
$$S_2 = \{C', D'\}$$
$$S_3 = \{L, R\}$$

In Figure 2, we represent the strategies and payoffs of the three players in the following normal form representation of the game, where Player 1 chooses between the rows, Player 2 chooses between the columns, and Player 3 chooses between the matrixes.

Player 3 choosing L

Player 1 \ Player 2	C'	D'
C	1, 1, 1	4, 4, 0
D	3, 3, 2	3, 3, 2

Player 3 choosing R

Player 1 \ Player 2	C'	D'
C	1, 1, 1	0, 0, 1
D	0, 0, 0	0, 0, 0

Figure 2. Selten's horse - Matrix representation.

- We next underline the best responses of the three players in Figure 3, and identify that (C, C', R) and (D, C', L) are the pure strategy Nash equilibria of this game.

Player 3 choosing L

Player 1 \ Player 2	C'	D'
C	1, 1, $\underline{1}$	$\underline{4}$, $\underline{4}$, 0
D	$\underline{3}$, $\underline{3}$, $\underline{2}$	3, $\underline{3}$, $\underline{2}$

Player 3 choosing R

Player 1 \ Player 2	C'	D'
C	$\underline{1}$, $\underline{1}$, $\underline{1}$	$\underline{0}$, 0, $\underline{1}$
D	0, $\underline{0}$, 0	$\underline{0}$, $\underline{0}$, 0

Figure 3. Selten's horse - Underlining best response payoffs.

(b) Argue that one of the two psNEs you found in part (a) is not sequentially rational. A short verbal explanation suffices.

- (D, C', L) is not sequentially rational. If Player 1 chooses D, then Player 3's belief is $\mu = 1$, responding with L (see left-hand side at the bottom of the tree). Anticipating that Player 3 choosing L, Player 2 compares his payoff from C', 1, against that from D' (which is followed by Player 3 responding with L), 4, and thus chooses D'. Therefore, Player 2 choosing C' is not sequentially rational.

(c) Show that strategy profile $\{C, C', R\}$ can be sustained as a PBE of the game. (You don't need to show that this is actually the unique PBE we can sustain in this game.) Discuss that this strategy profile is based on credible beliefs by player 3.

- We check the pooling strategy profile, C, C', where Player 1 chooses C and Player 2 selects C'.
  As depicted in Figure 4, since player 1 chooses C (as illustrated by the blue horizontal arrow) and player 2 chooses C' (as illustrated by the green horizontal arrow), messages D and D' are off-the-equilibrium path, leaving the

beliefs of Player 3 unrestricted, that is, $\mu \in [0,1]$. In other words, Player 3's information set should never be reached in this strategy profile.

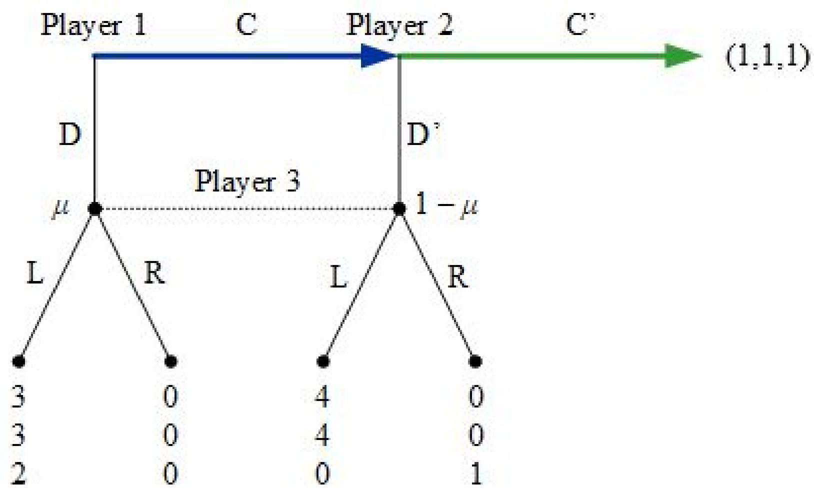


Figure 4. Pooling Strategy Profile C, C'

Therefore, if Player 3 is ever called out to move, he compares the expected payoff from responding with L and R, as follows:

$$EU_3(L) = 2 \times \mu + 0 \times (1-\mu) = 2\mu$$
$$EU_3(R) = 0 \times \mu + 1 \times (1-\mu) = 1-\mu$$

Player 3 then responds with L if $2\mu > 1-\mu$, which simplifies to $\mu > \frac{1}{3}$. Otherwise, he responds with R. This gives rise to two cases (one in which $\mu > \frac{1}{3}$, and Player 3 responds with L; and another in which $\mu \leq \frac{1}{3}$ and Player 3 responds with R), which we separately analyze below.
- *Case 1,* $\mu > \frac{1}{3}$. As depicted in Figure 5a, Player 3 responds with L (as illustrated by the red arrows) since $\mu > \frac{1}{3}$. In this context, Player 2 can improve his payoff by deviating from C', which yields a payoff of 1, to D', which yields a payoff of 4. Therefore, the pooling strategy profile C, C' cannot be supported as a PBE of this game when Player 3's beliefs satisfy $\mu > \frac{1}{3}$.

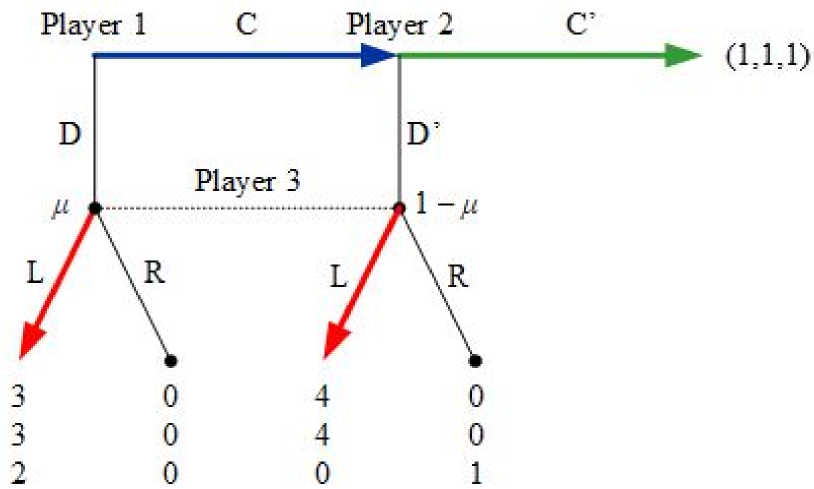


Figure 5a. Pooling Strategy Profile C, C' when $\mu > \frac{1}{3}$.

- *Case 2,* $\mu \leq \frac{1}{3}$. As depicted in Figure 5b, Player 3 responds with R (as illustrated by the red arrows) given that his beliefs are $\mu \leq \frac{1}{3}$. In this context, Player 2 does not deviate because his prescribed strategy, C', gives him a payoff of 1, while deviating to D' would give him a payoff of 0. Similarly, Player 1 does not deviate because his prescribed strategy, C, gives him a payoff of 1, exceeds his payoff from deviating to D, zero. Therefore, strategy profile C, C' can be supported as a PBE of this game when Player 3's beliefs satisfy $\mu \leq \frac{1}{3}$.

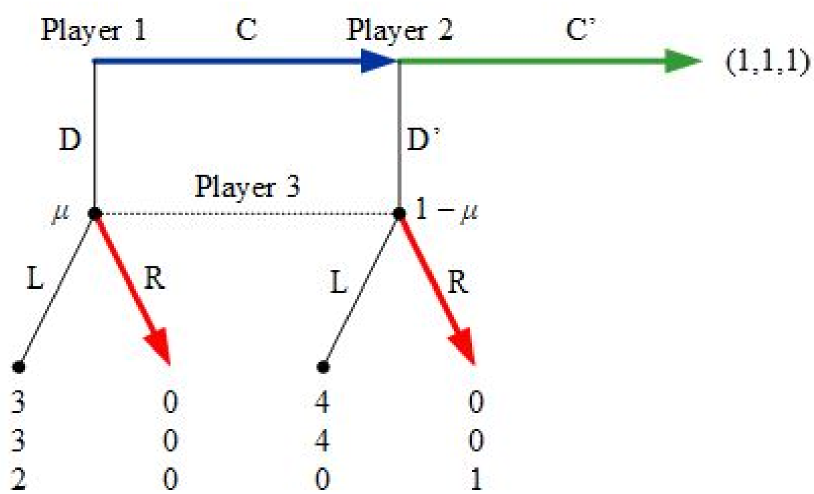


Figure 5b. Pooling Strategy Profile C, C' when $\mu \leq \frac{1}{3}$.

5. **BONUS EXERCISE - A model of sales, based on Varian (1980).**[2] Firms offer sales at different times. In this exercise, we show that offering sales (or, more generally, randomizing over prices) is a strategy that helps firms maximize their expected profits. This exercise belongs to the literature on "price dispersion" where firms face a share of consumers who are uninformed about prices, and offer different prices, either at different locations (spatial price dispersion) or at different points in time (temporal price dispersion, as we analyze in this exercise). Price discrimination models, in contrast, assume that consumers can perfectly observe prices.

   Consider an industry with N firms and free entry, so firms enter until the profits from doing so are zero. Consumers have a reservation price r for an homogeneous good and purchase at most one unit. A share α^I of consumers is informed about prices, buying from the cheapest firm, and a share $1-\alpha^I$ are uninformed, who purchase from any firm. Therefore, there are $\alpha^U = \frac{1-\alpha^I}{N}$ uninformed consumers per firm. Firms face a symmetric cost function $C(q) = F + cq$, where $F > 0$ denotes fixed costs and c represents its marginal cost. Every firm can only charge one price for its product.

   As a reference, note that $C(\alpha^I+\alpha^U) = F+c(\alpha^I+\alpha^U)$ denotes the cost from serving the maximum amount of customers (both informed and uninformed consumers). Therefore, the ratio

$$p_L \equiv \frac{F + c(\alpha^I + \alpha^U)}{\alpha^I + \alpha^U}$$

   represents the average cost in this setting.

   We next show that, in the above context, every firm has incentives to randomize its pricing over a certain interval. The following questions should help you find the specific cumulative distribution function $F(p)$ that every firm uses in the mixed-strategy Nash equilibrium of the game.

   (a) Show that $F(p) = 0$ for all $p < p_L$, and that $F(p) = 1$ for all $p > r$.

   - This question essentially asks us to "trim" the support of price randomization in $F(p)$ and characterize its lower and upper bounds.
   - *Lower bound.* When charging prices below p_L, a firm must be making losses, since its price lies below its cost in the most favorable scenario (when all types of consumers purchase the good). Therefore, the firm does not assign a probability weight on prices below p_L.
   - *Upper bound.* If a firm charges a price above the reservation price r, no customer buys from it, regardless of whether he is informed or uninformed. The firm then has no incentives to assign a probability weight on prices above r. Combining our above results, the price p in $F(p)$ must lie in the interval $[p_L, r]$.

   (b) Show that the cumulative distribution function $F(p)$ is non-degenerated, that is, there is no pure strategy Nash equilibrium.

   - If firm i uses a pure strategy, charging price $p_i = p_L$, it makes a loss, thus having incentives to exit the industry. (Recall that, in equilibrium, firms

[2] Varian, H. (1980) "A model of sales," American Economic Review, 70, pp. 651–59.

make zero profits.) If, instead, the firm sets a higher price p_i that satisfies $r \geq p_i > p_L$, other firms would have incentives to undercut firm i's price by a small ε. Therefore, firm i does not use a pure strategy.

(c) For simplicity, assume that $F(p)$ is continuous.[3] Find expected profits from the pricing strategy $F(p)$.

- If a firm sets the lowest price, it attract all consumers, and its profit is

$$\pi_s(p) = p(\alpha^I + \alpha^U) - F - c(\alpha^I + \alpha^U)$$

where the subscript s denotes that the firm is successful at attracting all consumers.

If, instead, the firm is unsuccessful, it only sells its product to uninformed consumers, earning

$$\pi_f(p) = p\alpha^U - F - c\alpha^U$$

where the subscript f denotes "failure".

The probability that firm i sets a price p higher than its rival $j \neq i$ is

$$F(p) = \text{Prob}\{p \geq p_j\}$$

so the probability that $p < p_j$ is the converse, $1 - F(p)$. As a result, the probability that p is lower than the prices of all its $N-1$ rivals is

$$[1 - F(p)]^{N-1},$$

which represents the probability that firm i sells to informed consumers. Finally, the probability that firm i does not sell to informed consumers is

$$1 - [1 - F(p)]^{N-1}.$$

- We are now ready to write firm i's expected profit

$$\int_{p_L}^{r} \left[\underbrace{\pi_s(p)\,[1 - F(p)]^{N-1}}_{\text{Success}} + \underbrace{\pi_f(p)\left[1 - [1 - F(p)]^{N-1}\right]}_{\text{Failure}} \right] f(p)dp$$

(d) Using the no entry condition, find the cumulative distribution function $F(p)$ with which every firm randomizes.

- Since firms make no profits in equilibrium (otherwise entry or exit would still be profitable), the above expected profit must be equal to zero, which entails

$$\pi_s(p)\,[1 - F(p)]^{N-1} + \pi_f(p)\left[1 - [1 - F(p)]^{N-1}\right] = 0$$

Rearranging,

$$F(p) = 1 - \left(\frac{\pi_f(p)}{\pi_f(p) - \pi_s(p)}\right)^{\frac{1}{N-1}}$$

The denominator is negative since $\pi_f(p) < \pi_s(p)$ for any price $p \in [p_L, r]$. Therefore, the numerator must also be negative, $\pi_f(p) < 0$.

[3] That is, there is no "mass point" in the pricing strategy $F(p)$ that every firm uses. Intuitively, the firm chooses all prices in the $[p_L, r]$ interval with positive probability. More compactly, this means that the density function $f(p) > 0$ for all $p \in [p_L, r]$.

(e) Show that the cumulative distribution function $F(p)$ has full support in $p \in [p_L, r]$. That is, $F(p_L + \varepsilon) > 0$ and $F(r - \varepsilon) < 1$ for any $\varepsilon > 0$.

- *Prices slightly above p_L.* If, instead, $F(p_L + \varepsilon) = 0$, firm i is assigning no probability weight to prices slightly higher than the lower bound p_L. Therefore, firm i assigns probability weight to prices strictly above $p_L + \varepsilon$. In that case, another firm j could undercut firm i's price and set for instance a price $p_L + \frac{\varepsilon}{2}$ to make positive profits. Hence, $F(p_L + \varepsilon) > 0$ for any $\varepsilon > 0$.
- *Prices slightly below r.* If, instead, $F(r - \varepsilon) = 1$, firm i assigns no probability to prices slightly below r. At $\widetilde{p} < r$, only uninformed consumers purchase the good and the firm earns $\widetilde{p}\alpha^U - F - c\alpha^U$, yielding zero profits. However, a deviation to price $p = r$ yields $r\alpha^U - F - c\alpha^U$ which is positive, thus making such deviation profitable. Therefore, $F(r - \varepsilon) < 1$ for any $\varepsilon > 0$.

(f) Taking into account that $\pi_f(r) = 0$, find the equilibrium number of firms in the industry, n^*.

- Condition $\pi_f(r) = 0$ entails

$$r\alpha^U - F - c\alpha^U = 0$$

Substituting $\alpha^U = \frac{1-\alpha^I}{N}$ into the above expression, yields

$$F = (r - c)\underbrace{\frac{1 - \alpha^I}{N}}_{\alpha^U}$$

Solving for N, we obtain

$$N^* = \frac{(r - c)\left(1 - \alpha^I\right)}{F}.$$

Therefore, the higher the profit margin $r - c$, the larger share of the uninformed consumers $1 - \alpha^I$, and the lower the entry cost F, the more firms in equilibrium.

(g) Taking into account that $\pi_s(p_L) = 0$, and the equilibrium number of firms N^*, find the lower bound of firms' randomization strategy, p_L.

- Condition $\pi_s(p_L) = 0$ entails

$$p_L(\alpha^I + \alpha^U) - F - c(\alpha^I + \alpha^U) = 0$$

Substituting $\alpha^U = \frac{1-\alpha^I}{N}$ into the above expression, yields

$$F = (p_L - c)\left(\alpha^I + \frac{1 - \alpha^I}{N}\right)$$

Further inserting the equilibrium number of firms, N^*, found in part (f), we have

$$F = (p_L - c)\left(\alpha^I + \frac{1 - \alpha^I}{\frac{(r-c)(1-\alpha^I)}{F}}\right)$$

Rearranging, we obtain

$$F = (p_L - c)\left(\frac{(r-c)\,\alpha^I + F}{r-c}\right)$$

Solving for p_L, we find the lower bound of firms' randomization strategy

$$p_L = \frac{c\,(r-c)\,\alpha^I + rF}{(r-c)\,\alpha^I + F}$$

(h) Evaluate your above results in the special case in which all consumers are uninformed.

- When all consumers are uninformed, $\alpha^I = 0$, the lower bound of firms' randomization strategy, p_L, becomes

$$p_L = \frac{rF}{F} = F$$

which coincides with the upper bound of firms' randomization strategy. In other words, firms put full probability weight on one price, $p = r$, with every firm extracting all surplus from a share $\frac{1}{N}$ of consumers.

(i) *Numerical example.* Evaluate your results in parts (d), (f), and (g) at parameter values $r = 1$, $F = \frac{2}{9}$, $c = 0$, and $\alpha^I = \frac{1}{3}$.

- In this setting, the equilibrium number of firms becomes

$$\begin{aligned} N^* &= \frac{(1-0)\left(1-\frac{1}{3}\right)}{\frac{2}{9}} \\ &= 3. \end{aligned}$$

In addition, the lower bound is

$$\begin{aligned} p_L &= \frac{0\,(1-0)\,\frac{1}{3} + \frac{2}{9}\cdot 1}{(1-0)\,\frac{1}{3} + \frac{2}{9}} \\ &= \frac{2}{5}. \end{aligned}$$

In this context, the share of uninformed consumers for every firm becomes

$$\begin{aligned} \alpha^U &= \frac{1-\alpha^I}{N^*} \\ &= \frac{1-\frac{1}{3}}{3} \\ &= \frac{2}{9}. \end{aligned}$$

- Finally, the cumulative distribution function is

$$F(p) = 1 - \left(\frac{\pi_f(p)}{\pi_f(p) - \pi_s(p)}\right)^{\frac{1}{N-1}}$$

where profits from successfully attracting all customers are

$$\begin{aligned}\pi_s(p) &= (p-c)\left(\alpha^I+\alpha^U\right)-F \\ &= (p-0)\left(\frac{1}{3}+\frac{2}{9}\right)-\frac{2}{9} \\ &= \frac{5p-2}{9}\end{aligned}$$

and profits from only attracting uninformed consumers are

$$\begin{aligned}\pi_f(p) &= (p-c)\,\alpha^U-F \\ &= (p-0)\cdot\frac{2}{9}-\frac{2}{9} \\ &= -\frac{2\,(1-p)}{9}\end{aligned}$$

Therefore, the above function $F\,(p)$ becomes

$$\begin{aligned}F\,(p) &= 1-\left(\frac{2\,(1-p)}{5p-2-2\,(1-p)}\right)^{\frac{1}{2}} \\ &= 1-\sqrt{\frac{2\,(1-p)}{7p-4}}.\end{aligned}$$

which is distributed between the lower bound $p_L=\frac{2}{5}$ and the upper bound $r=1$.
Differentiating $F(p)$ with respect to p, we find its probability density function

$$f\,(p)=\frac{3\,(7p-4)^{-\frac{3}{2}}}{\sqrt{2\,(1-p)}}$$

which is positive so that firms randomize over the full support of the interval $\left[\frac{2}{9},1\right]$.