EconS 503 - Advanced Microeconomics II
Homework #9 - Answer key

1. **Moral hazard with multiple tasks, based on Holmström-Milgrom (1991).**[1] Let us consider a moral hazard problem between a principal and an agent. However, let us now allow the agent to take two effort levels e_1 and e_2. This represents, for instance, a salesman choosing how much effort to exert visiting potential customers, how much time to spend creating a more attractive website for online sales, investigating new sales strategies, etc. In this exercise we seek to understand how the multidimensionality in the agent's effort affects our results in the standard moral hazard problem analyzed in this chapter.

   Assume that the cost of exerting effort levels e_1 and e_2 is

$$c(e_1, e_2) = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2$$

   These effort levels produce output y with output function

$$y = f_1e_1 + f_2e_2 + \varepsilon$$

   with performance $p = g_1e_1 + g_2e_2 + \phi$. Random shocks in output, ε, and performance, ϕ, follow distributions of $G(\phi)$ and $H(\varepsilon)$, respectively, with zero expectations, that is, $E(\varepsilon) = E(\phi) = 0$.

   For simplicity, assume that both principal and agent are risk neutral with payoff functions of $\pi = y - w$ for the principal (e.g., firm), where w denotes the salary she pays to the agent; and $U = w - c(e_1, e_2)$ for the agent (e.g., worker). Consider that the principal offers a salary $w = F + bp$ where F is fixed component of the contract and b is the bonus which provides a higher salary to the agent as his performance p increases. In particular, the timing of the game is as follows:

   - The principal and agent sign a contract $w = F + bp$.
   - The agent takes effort levels e_1 and e_2 which are unobservable to the principal.
   - Random shocks ε and ϕ, are realized, affecting the agent's output and performance, respectively.
   - Output y and performance p are observed by the principal and agent.
   - The agent receives wage $w = F + bp$.

   (a) Find the agent's optimal efforts and indirect utility as a function of the bonus parameter b.

[1] Holmström-Milgrom (1991) "Multitask Principal-Agent Analyses: Incentive Contracts, Asset Ownership, and Job Design," *Journal of Law, Economics, & Organization*, vol. 7, pp. 24-52. For a more readable presentation, see Bolton and Dewatripont (2005), pp. 216-28.

- The agent solves the following expected utility maximization problem:

$$\begin{aligned}
\max_{e_1,e_2\geq 0} E_A\left[U\left(e_1,e_2\right)\right] &= E_A\left[w-c\left(e_1,e_2\right)\right] \\
&= \int\left[\underbrace{F+bp}_{w}-\underbrace{\left(\frac{1}{2}e_1^2+\frac{1}{2}e_2^2\right)}_{c(e_1,e_2)}\right]dG\left(\phi\right) \\
&= \int\left[F+b\underbrace{\left(g_1e_1+g_2e_2+\phi\right)}_{p}-\left(\frac{1}{2}e_1^2+\frac{1}{2}e_2^2\right)\right]dG\left(\phi\right) \\
&= F+b\left(g_1e_1+g_2e_2\right)-\frac{1}{2}e_1^2-\frac{1}{2}e_2^2+b\underbrace{\int\phi dG\left(\phi\right)}_{=0\text{ since }E(\phi)=0} \\
&= F+bg_1e_1+bg_2e_2-\frac{1}{2}e_1^2-\frac{1}{2}e_2^2
\end{aligned}$$

Taking first-order conditions with respect to e_1 and e_2, we obtain

$$\begin{aligned}
\frac{\partial E_A\left[U\left(e_1,e_2\right)\right]}{\partial e_1} &= bg_1-e_1=0 \\
\frac{\partial E_A\left[U\left(e_1,e_2\right)\right]}{\partial e_2} &= bg_2-e_2=0
\end{aligned}$$

Assuming interior solutions ($e_1>0$ and $e_2>0$), the agent's optimal efforts are

$$\begin{aligned}
e_1\left(b\right) &= bg_1 \\
e_2\left(b\right) &= bg_2
\end{aligned}$$

- *Indirect utility function.* Substituting the agent's optimal efforts back into his utility function, yields

$$\begin{aligned}
U\left(b,F\right) &= F+bg_1\cdot\underbrace{bg_1}_{e_1(b)}+bg_2\cdot\underbrace{bg_2}_{e_2(b)}-\frac{1}{2}\underbrace{\left[bg_1\right]}_{e_1(b)}^2-\frac{1}{2}\underbrace{\left[bg_2\right]}_{e_2(b)}^2 \\
&= F+\left(bg_1\right)^2+\left(bg_2\right)^2-\frac{1}{2}\left(bg_1\right)^2-\frac{1}{2}\left(bg_2\right)^2 \\
&= F+\frac{1}{2}\left(bg_1\right)^2+\frac{1}{2}\left(bg_2\right)^2
\end{aligned}$$

(b) Find the principal's optimal contract w^* and his equilibrium profits.

- Operating by backwards induction, the principal anticipates the equilibrium effort levels that the agent chooses in the second stage of the game. Then, the principal solves the following expected profit maximization problem:

$$\begin{aligned}
\max_{b\geq 0} E_P\left[\pi\left(b\right)\right] &= E_P\left[y-w\right] \\
&= E_P\left[\underbrace{f_1 e_1\left(b\right)+f_2 e_2\left(b\right)+\varepsilon}_{y}-\underbrace{\left(F+bp\right)}_{w}\right] \\
&= \left(f_1-bg_1\right)\underbrace{e_1\left(b\right)}_{bg_1}+\left(f_2-bg_2\right)\underbrace{e_2\left(b\right)}_{bg_2}-F+\underbrace{\int \varepsilon dH\left(\varepsilon\right)}_{=0 \text{ since } E(\varepsilon)=0} \\
&= bg_1\left(f_1-bg_1\right)+bg_2\left(f_2-bg_2\right)-F
\end{aligned}$$

Taking first-order condition with respect to the bonus b, we find

$$\begin{aligned}
\frac{\partial E_P\left[\pi\left(b\right),F\right]}{\partial b} &= g_1\left(f_1-bg_1\right)-bg_1^2+g_2\left(f_2-bg_2\right)-bg_2^2 \\
&= g_1\left(f_1-2bg_1\right)+g_2\left(f_2-2bg_2\right)
\end{aligned}$$

Assuming interior solutions, that is, $b>0$, the bonus b^* satisfies

$$b^*=\frac{f_1g_1+f_2g_2}{2\left(g_1^2+g_2^2\right)}$$

- *Equilibrium profits.* Substituting the principal's optimal contract back into the profit function, we obtain

$$\begin{aligned}
\pi\left(b^*,F\right) &= b^*g_1\left(f_1-b^*g_1\right)+b^*g_2\left(f_2-b^*g_2\right)-F \\
&= \frac{f_1g_1+f_2g_2}{2\left(g_1^2+g_2^2\right)}\left(f_1g_1+f_2g_2\right)-\left(\frac{f_1g_1+f_2g_2}{2\left(g_1^2+g_2^2\right)}\right)^2\left(g_1^2+g_2^2\right)-F \\
&= \frac{\left(f_1g_1+f_2g_2\right)^2}{2\left(g_1^2+g_2^2\right)}-\frac{\left(f_1g_1+f_2g_2\right)^2}{4\left(g_1^2+g_2^2\right)}-F \\
&= \frac{\left(f_1g_1+f_2g_2\right)^2}{4\left(g_1^2+g_2^2\right)}-F
\end{aligned}$$

  Inspecting the principal's equilibrium profit above, we notice that any fixed wage $F>0$ reduces her profit, without affecting the agent's optimal efforts; which do not depend on F, as shown in part (a) of the exercise. Therefore, the principal should set the optimal fixed wage at $F^*=0$.

- As a result, the optimal contract becomes

$$\begin{aligned}
w^* &= F^*+b^*p \\
&= \frac{f_1g_1+f_2g_2}{2\left(g_1^2+g_2^2\right)}p
\end{aligned}$$

  which depends on the random shock ϕ. Specifically, when a favorable shock $\phi>0$ is realized, the agent outperforms so that the principal pays a higher wage to him. On the other hand, when an unfavorable shock $\phi<0$ is realized, the agent underperforms so that the principal pays a lower wage to him. Therefore, in expectation (that is, at stage 1 of the game, before the shocks are realized), the expected wage is

$$
\begin{aligned}
E[w^*] &= b\frac{f_1 g_1 + f_2 g_2}{2(g_1^2+g_2^2)}\left(g_1^2+g_2^2\right) + \frac{f_1 g_1 + f_2 g_2}{2(g_1^2+g_2^2)}\underbrace{E[p]}_{=0} \\
&= \frac{(f_1 g_1 + f_2 g_2)^2}{4(g_1^2+g_2^2)}
\end{aligned}
$$

(c) *Comparative Statics.* How is the optimal contract you found in part (b) affected by the output rates f_1 and f_2? How is it affected by the performance rates g_1 and g_2? Explain.

- Differentiating the optimal wage with respect to f_1 and f_2, we find

$$
\begin{aligned}
\frac{\partial w^*}{\partial f_1} &= \frac{g_1(f_1 g_1 + f_2 g_2)}{2(g_1^2+g_2^2)} > 0 \\
\frac{\partial w^*}{\partial f_2} &= \frac{g_2(f_1 g_1 + f_2 g_2)}{2(g_1^2+g_2^2)} > 0
\end{aligned}
$$

  which means that as either effort level becomes more effective in producing output, wage payment increases.

- Differentiating the wage contract with respect to g_1 and g_2, yields

$$
\begin{aligned}
\frac{\partial w^*}{\partial g_1} &= \frac{f_1(f_1 g_1 + f_2 g_2)(g_1^2+g_2^2) - g_1(f_1 g_1 + f_2 g_2)^2}{2(g_1^2+g_2^2)^2} \\
&= \frac{g_2(f_1 g_1 + f_2 g_2)(f_1 g_2 - f_2 g_1)}{2(g_1^2+g_2^2)^2} \\
\frac{\partial w^*}{\partial g_2} &= \frac{f_2(f_1 g_1 + f_2 g_2)(g_1^2+g_2^2) - g_2(f_1 g_1 + f_2 g_2)^2}{2(g_1^2+g_2^2)^2} \\
&= \frac{g_1(f_1 g_1 + f_2 g_2)(f_2 g_1 - f_1 g_2)}{2(g_1^2+g_2^2)^2}
\end{aligned}
$$

  Therefore, if $\frac{f_1}{f_2} > \frac{g_1}{g_2}$, we obtain that $\frac{\partial w^*}{\partial g_1} > 0$ and $\frac{\partial w^*}{\partial g_2} < 0$. Intuitively, if effort 1 is more effective in generating output than in delivering performance, relatively to effort 2, the optimal wage increases in the performance rate of effort 1, g_1, but decrease in that of effort 2, g_2. The opposite holds if $\frac{f_1}{f_2} < \frac{g_1}{g_2}$ such that $\frac{\partial w^*}{\partial g_1} < 0$ and $\frac{\partial w^*}{\partial g_2} > 0$. In this case, the optimal wage increases in the performance rate of effort 2, g_2, but decreases in that of effort 1, g_1.

(d) Given the optimal contract found above, what are the principal's expected payoff, the agent's expected utility, and the expected social welfare in equilibrium?

- Substituting the optimal contract w^* into the agent's indirect utility function, we find

$$\begin{aligned}
U(w^*) &= F^* + \frac{1}{2}(b^* g_1)^2 + \frac{1}{2}(b^* g_2)^2 \\
&= 0 + \frac{1}{2}\left(\frac{f_1 g_1 + f_2 g_2}{2(g_1^2 + g_2^2)} g_1\right)^2 + \frac{1}{2}\left(\frac{f_1 g_1 + f_2 g_2}{2(g_1^2 + g_2^2)} g_2\right)^2 \\
&= \frac{1}{8}\left(\frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}\right)^2 (g_1^2 + g_2^2) \\
&= \frac{(f_1 g_1 + f_2 g_2)^2}{8(g_1^2 + g_2^2)}
\end{aligned}$$

- Similarly, substituting the optimal contract w^* into the principal's indirect utility function, we obtain

$$\pi(w^*) = \frac{(f_1 g_1 + f_2 g_2)^2}{4(g_1^2 + g_2^2)}$$

- Therefore, the expected social welfare is given by the sum of the principal's expected profit and the agent's expected utility in equilibrium.

$$\begin{aligned}
SW^* &= \pi(w^*) + U(w^*) \\
&= \frac{(f_1 g_1 + f_2 g_2)^2}{4(g_1^2 + g_2^2)} + \frac{(f_1 g_1 + f_2 g_2)^2}{8(g_1^2 + g_2^2)} \\
&= \frac{3(f_1 g_1 + f_2 g_2)^2}{8(g_1^2 + g_2^2)}
\end{aligned}$$

(e) What is the socially optimal contract? Compare it against the contract that emerges in the subgame perfect equilibrium of the game you found in part (b).

- The expected social welfare for both the principal and the agent is given by the sum

$$\begin{aligned}
SW &= E[\pi(b, F) + U(b, F)] \\
&= \underbrace{[b g_1 (f_1 - b g_1) + b g_2 (f_2 - b g_2) - F]}_{E[\pi(b,F)]} + \underbrace{\left[F + \frac{1}{2}(b g_1)^2 + \frac{1}{2}(b g_2)^2\right]}_{E[U(b,F)]} \\
&= b f_1 g_1 - b^2 g_1^2 + b f_2 g_2 - b^2 g_2^2 + \frac{1}{2} b^2 g_1^2 + \frac{1}{2} b^2 g_2^2 \\
&= b(f_1 g_1 + f_2 g_2) - \frac{1}{2}(b g_1)^2 - \frac{1}{2}(b g_2)^2
\end{aligned}$$

where the first line is operating by both parties maximizing joint payoffs *as if* efforts are observable; and because the fixed wage, F, which is an action-independent transfer from the principal to the agent, is cancelled out, we can set $F^{**} = 0$ without affecting the expected social welfare.

- Taking first-order condition with respect to b, yields

$$\frac{\partial SW}{\partial b} = f_1 g_1 + f_2 g_2 - b g_1^2 - b g_2^2$$

Assuming interior solutions, that is, $b > 0$, the socially optimal bonus b^{**} becomes

$$b^{**} = \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2} \tag{1}$$

- Comparing with the equilibrium bonus found part (b), $b^* = \frac{f_1 g_1 + f_2 g_2}{2\left(g_1^2 + g_2^2\right)}$, we see that $b^{**} = 2b^*$. In words, this result indicates that effort unobservability (in part b) reduces the bonus b by half. Intuitively, as the principal cannot observe the effort levels chosen by the agent, the principal believes that the observed performance can be a matter of luck (i.e., due to random shocks) other than the efforts exerted by the agent; and therefore, she reduces the variable wage (i.e., bonus) to the agent.

(f) What is the deadweight loss in this contractual setting?

- Substituting the socially optimal bonus found in part (e), b^{**}, into the social welfare function, we obtain

$$\begin{aligned}
SW^{**} &= E\left[\pi\left(b^{**}, F\right) + U\left(b^{**}, F\right)\right] \\
&= \underbrace{\left[b^{**} g_1 \left(f_1 - b g_1\right) + b^{**} g_2 \left(f_2 - b g_2\right) - F\right]}_{E[\pi(b,F)]} + \underbrace{\left[F + \frac{1}{2}\left(b^{**} g_1\right)^2 + \frac{1}{2}\left(b^{**} g_2\right)^2\right]}_{E[U(b,F)]} \\
&= b^{**} f_1 g_1 - \left(b^{**}\right)^2 g_1^2 + b^{**} f_2 g_2 - \left(b^{**}\right)^2 g_2^2 + \frac{1}{2}\left(b^{**}\right)^2 g_1^2 + \frac{1}{2}\left(b^{**}\right)^2 g_2^2 \\
&= b^{**}\left(f_1 g_1 + f_2 g_2\right) - \frac{1}{2}\left(b^{**} g_1\right)^2 - \frac{1}{2}\left(b^{**} g_2\right)^2 \\
&= \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}\left(f_1 g_1 + f_2 g_2\right) - \frac{1}{2}\left(\frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}\right)^2 \left(g_1^2 + g_2^2\right)^2 \\
&= \frac{\left(f_1 g_1 + f_2 g_2\right)^2}{2\left(g_1^2 + g_2^2\right)}
\end{aligned}$$

Therefore, the deadweight loss is

$$\begin{aligned}
DWL &= SW^{**} - SW^{*} \\
&= \frac{\left(f_1 g_1 + f_2 g_2\right)^2}{2\left(g_1^2 + g_2^2\right)} - \frac{3\left(f_1 g_1 + f_2 g_2\right)^2}{8\left(g_1^2 + g_2^2\right)} \\
&= \frac{\left(f_1 g_1 + f_2 g_2\right)^2}{8\left(g_1^2 + g_2^2\right)}.
\end{aligned}$$

(g) *Numerical example.* Consider output rates $f_1 = \frac{1}{2}$ and $f_2 = \frac{1}{3}$, and performance rates $g_1 = \frac{2}{3}$ and $g_2 = \frac{1}{4}$. In this context, evaluate the equilibrium bonus b^*, efforts e_1^* and e_2^*, wage w^*, the agent's expected utility $U\left(w^*\right)$, the principal's expected profit $\pi\left(w^*\right)$, and social welfare SW^*. Then, evaluate the socially optimal bonus b^{**}, social welfare SW^{**} at b^{**}, and deadweight loss due to unobservability of effort.

- *Equilibrium outcomes.* Evaluating the equilibrium bonus, $b^* = \frac{f_1 g_1 + f_2 g_2}{2\left(g_1^2 + g_2^2\right)}$, found in part (b) of the exercise at the above parameter values, we obtain

$$b^* = \frac{\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}}{2\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)}$$
$$= \frac{30}{73}$$

Evaluating equilibrium efforts, $e_1(b^*) = b^* g_1$ and $e_2(b^*) = b^* g_2$, found in same part of the exercise, we obtain

$$e_1^* = \frac{20}{73}$$
$$e_2^* = \frac{15}{146}$$

Evaluating equilibrium wage, $w^* = F^* + b^* p = 0 + b^* p$, at the above parameter values, we obtain

$$w^* = \frac{30}{73} p$$

as a function of the performance p which in turn depends on the random shock ϕ, such that expected wage of the agent, $E[w^*] = \frac{(f_1 g_1 + f_2 g_2)^2}{4\left(g_1^2 + g_2^2\right)}$, at stage 1 of the game (i.e., before the shocks are realized) found in part (b) of the exercise becomes

$$E[w^*] = \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{4\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)}$$
$$= \frac{25}{292} \simeq 0.08$$

Therefore, expected utility of the agent, $U(w^*) = \frac{(f_1 g_1 + f_2 g_2)^2}{8\left(g_1^2 + g_2^2\right)}$, found in part (d) of the exercise becomes

$$U^* = \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{8\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)}$$
$$= \frac{25}{584} \simeq 0.04$$

and expected profit of the principal, $\pi(w^*) = \frac{(f_1 g_1 + f_2 g_2)^2}{4\left(g_1^2 + g_2^2\right)}$, which is also found in part (d) becomes

$$\pi^* = \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{4\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)}$$
$$= \frac{25}{292} \simeq 0.08$$

which implies a social welfare of $SW^* = \pi^* + U^* = \frac{75}{584} \simeq 0.12$.

- *Socially optimal outcomes.* Evaluating the socially optimal bonus $b^{**} = \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}$ we found in part (e) of the exercise at the above parameter values, we obtain

$$b^{**} = \frac{\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}}{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2} = \frac{60}{73} \simeq 0.82$$

and social welfare, $SW^{**} = \frac{(f_1 g_1 + f_2 g_2)^2}{2\left(g_1^2 + g_2^2\right)}$, evaluated at bonus b^{**}, as found in part (e) of the exercise, is

$$SW^{**} = \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{2\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)} = \frac{25}{146} \simeq 0.17$$

- *Comparison.* Deadweight loss, $DWL = \frac{(f_1 g_1 + f_2 g_2)^2}{8\left(g_1^2 + g_2^2\right)}$, as found in part (f) of the exercise, is

$$DWL = \frac{\left(\frac{1}{2}\frac{2}{3} + \frac{1}{3}\frac{1}{4}\right)^2}{8\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right)} = \frac{25}{584} \simeq 0.04$$

2. **Principal agent model with two principals (Common agency problem), based on Stole (1990).**[2] Consider two principals, 1 and 2, who assign different tasks to a common agent (e.g., worker). There are 2 output levels in principal j's task, where $j = \{1, 2\}$, denoted as q_j^H and q_j^L, where $q_j^H > q_j^L > 0$. The probability of output q_j^H coincides with the agent's effort on task j, e_j , where $e_j \in [0, 1]$, such that the probability of the opposite output level, q_j^L, becomes $(1 - e_j)$. Intuitively, the more effort the agent puts in task j, the more likely it is to realize a high output level on that task.

   The agent incurs a cost of effort, $c(e_1, e_2)$, which is increasing and convex in both arguments. Additionally, the agent enjoys a utility $v(l)$ from leisure l, which is increasing and concave in leisure. Assume that the amount of leisure he enjoys and the amount of labor he supplies to both principals add up to 1, that is, $e_1 + e_2 + l = 1$. Every principal j observes the output but not the effort the agent exerts, and pays w_j^H and w_j^L to the agent when high and low levels of output are realized, respectively. For simplicity, assume that the principals are risk-neutral but the agent is risk averse, with utility function $u(\cdot)$ where $u(0) = 0$, $u' > 0$, and $u'' < 0$.

   (a) *Independent task assignment.* Write down principal 1's objective function and the constraints that he faces, assuming that the two principals act independently in assigning different tasks to the agent.

[2] Stole, L. (1990) Mechanism design under common agency, Mimeo.

- Principal 1 chooses the salary pair (w_1^H, w_1^L) to maximize her expected profit,

$$\max_{e_1, w_1^H, w_1^L} \quad e_1\left(q_1^H - w_1^H\right) + (1 - e_1)\left(q_1^L - w_1^L\right)$$

subject to the following constraints,

$$\underbrace{e_1 u\left(w_1^H\right) + (1-e_1)\,u\left(w_1^L\right)}_{EU_1} + \underbrace{e_2 u\left(w_2^H\right) + (1-e_2)\,u\left(w_2^L\right)}_{EU_2} + v\left(1 - e_1 - e_2\right) - c\left(e_1, e_2\right) \geq \overline{u} \tag{PC}$$

$$u\left(w_1^H\right) - u\left(w_1^L\right) \geq v_1\left(1 - e_1 - e_2\right) + c_1\left(e_1, e_2\right) \tag{IC}$$

- *Participation constraint (PC).* The first constraint is the participation constraint, where the expected utility from earning wages from principal 1, EU_1, and from principal 2, EU_2, in addition to his utility from leisure, after taking off his cost of effort, should be no less than the reservation utility $\overline{u}$.
- *Incentive compatibility (IC).* The second constraint is the incentive compatibility condition, where the expected payoff from exerting effort at the margin more than compensates for the loss of leisure and the marginal cost of effort.

(b) Setup the Lagrangian function and solve for the optimal effort that the agent exerts on task 1 (you can keep the Lagrangian multipliers).

- The Lagrangian function of principal 1 becomes

$$\begin{aligned} & e_1\left(q_1^H - w_1^H\right) + (1-e_1)\left(q_1^L - w_1^L\right) \\ & + \lambda_1\left[e_1 u\left(w_1^H\right) + (1-e_1)\,u\left(w_1^L\right) + e_2 u\left(w_2^H\right) + (1-e_2)\,u\left(w_2^L\right) + v\left(1-e_1-e_2\right) - c\left(e_1,e_2\right)\right. \\ & + \mu_1\left[u\left(w_1^H\right) - u\left(w_1^L\right) - v_1\left(1-e_1-e_2\right) - c_1\left(e_1,e_2\right)\right] \end{aligned}$$

Differentiating with respect to e_1, and assuming interior solutions, we obtain

$$q_1^H - w_1^H - q_1^L + w_1^L + \mu_1\left[v_{11}\left(1-e_1-e_2\right) - c_{11}\left(e_1,e_2\right)\right] = 0$$

which, after rearranging, yields

$$\mu_1 = \frac{\left(q_1^H - q_1^L\right) - \left(w_1^H - w_1^L\right)}{\underbrace{c_{11}\left(e_1,e_2\right)}_{>0} - \underbrace{v_{11}\left(1-e_1-e_2\right)}_{<0}}$$

To determined the sign this ratio, note that $c_{11}\left(e_1,e_2\right) > 0$ by the convexity in the cost of effort and $v_{11}\left(1-e_1-e_2\right) < 0$ by the concavity in the utility from leisure; thus implying that the denominator is positive. Turning now to the numerator, we can claim that it is positive because the increase in output, $q_1^H - q_1^L$, must be larger than the increase in wages, $w_1^H - w_1^L$, for the principal to induce the agent to exert effort. Otherwise the principal would be losing money.
Differentiating with respect to w_1^H, and assuming interior solutions,

$$-e_1 + \lambda_1 e_1 u'\left(w_1^H\right) + \mu_1 u'\left(w_1^H\right) = 0$$

which, after rearranging, yields

$$e_1^{Indep} = \frac{\mu_1 u'\left(w_1^H\right)}{1-\lambda_1 u'\left(w_1^H\right)}$$

where the superscript *Indep* stands for independent task assignment for the principals.

(c) *Joint task assignment.* Write down the principal's Lagrangian function if they jointly assign tasks to the agent, and solve for the optimal effort that the agent exerts on the two tasks.

- The Lagrangian function is

$$\begin{aligned} & e_1\left(q_1^H - w_1^H\right) + \left(1-e_1\right)\left(q_1^L - w_1^L\right) + e_2\left(q_2^H - w_2^H\right) + \left(1-e_2\right)\left(q_2^L - w_2^L\right) \\ & +\lambda\left[e_1 u\left(w_1^H\right) + \left(1-e_1\right)u\left(w_1^L\right) + e_2 u\left(w_2^H\right) + \left(1-e_2\right)u\left(w_2^L\right) + v\left(1-e_1-e_2\right) - c\left(e_1,e_2\right) - \right. \\ & +\mu\left[u\left(w_1^H\right) - u\left(w_1^L\right) - v_1\left(1-e_1-e_2\right) - c_1\left(e_1,e_2\right)\right] \\ & +\kappa\left[u\left(w_2^H\right) - u\left(w_2^L\right) - v_2\left(1-e_1-e_2\right) - c_2\left(e_1,e_2\right)\right] \end{aligned}$$

In words, the objective function now lists the expected profits for principal 1 and 2, rather than one of the two principals alone as in part (a) of the exercise. Regarding the constraints, they still start with a participation constraint that intuitively says that the agent's expected salary from both principals compensates him to participate in the contract, followed by two incentive compatibility conditions (one for each principal).
Differentiating with respect to effort e_1, and assuming interior solutions, we find

$$q_1^H - w_1^H - q_1^L + w_1^L + \mu\left[v_{11}\left(1-e_1-e_2\right) - c_{11}\left(e_1,e_2\right)\right] + \kappa\left[v_{21}\left(1-e_1-e_2\right) - c_{21}\left(e_1,e_2\right)\right] = 0$$

Differentiating with respect to effort e_2, and assuming interior solutions, we obtain

$$q_2^H - w_2^H - q_2^L + w_2^L + \mu\left[v_{21}\left(1-e_1-e_2\right) - c_{21}\left(e_1,e_2\right)\right] + \kappa\left[v_{22}\left(1-e_1-e_2\right) - c_{22}\left(e_1,e_2\right)\right] = 0$$

Rearranging the above expressions, and solving by Cramer's rule, yields

$$\mu = \frac{\left(c_{22}-v_{22}\right)\left[\left(q_1^H - q_1^L\right) - \left(w_1^H - w_1^L\right)\right]}{\left(c_{11}-v_{11}\right)\left(c_{22}-v_{22}\right) - \left(c_{21}-v_{21}\right)^2} - \frac{\left(c_{21}-v_{21}\right)\left[\left(q_2^H - q_2^L\right) - \left(w_2^H - w_2^L\right)\right]}{\left(c_{11}-v_{11}\right)\left(c_{22}-v_{22}\right) - \left(c_{21}-v_{21}\right)^2}$$

$$\kappa = \frac{\left(c_{11}-v_{11}\right)\left[\left(q_2^H - q_2^L\right) - \left(w_2^H - w_2^L\right)\right]}{\left(c_{11}-v_{11}\right)\left(c_{22}-v_{22}\right) - \left(c_{21}-v_{21}\right)^2} - \frac{\left(c_{21}-v_{21}\right)\left[\left(q_1^H - q_1^L\right) - \left(w_1^H - w_1^L\right)\right]}{\left(c_{11}-v_{11}\right)\left(c_{22}-v_{22}\right)\left(c_{21}-v_{21}\right)^2}$$

Differentiating the Lagrangian function with respect to salaries w_1^H and w_2^H, and assuming interior solutions,

$$\begin{aligned} -e_1 + \lambda e_1 u'\left(w_1^H\right) + \mu u'\left(w_1^H\right) &= 0 \\ -e_2 + \lambda e_2 u'\left(w_2^H\right) + \kappa u'\left(w_2^H\right) &= 0 \end{aligned}$$

which, after rearranging, yields

$$e_1^{Coop} = \frac{\mu u'\left(w_1^H\right)}{1-\lambda u'\left(w_1^H\right)}$$
$$e_2^{Coop} = \frac{\kappa u'\left(w_2^H\right)}{1-\lambda u'\left(w_2^H\right)}$$

where the superscript *Coop* stands for cooperation between the principals.

(d) Compare the level of effort that the agent exerts in parts (b) and (c). Interpret.

- Consider the first part of μ, check if

$$\frac{c_{22}-v_{22}}{\left(c_{11}-v_{11}\right)\left(c_{22}-v_{22}\right)-\left(c_{21}-v_{21}\right)^2} > \frac{1}{c_{11}-v_{11}}$$

which is true because $\left(c_{11}-v_{11}\right)\left(c_{22}-v_{22}\right) \geq \left(c_{11}-v_{11}\right)\left(c_{22}-v_{22}\right)-\left(c_{21}-v_{21}\right)^2$.

- Therefore, considering that $v\left(1-e_1-e_2\right)$ is linear in efforts, $v_{21}=0$ such that:
  - If $c_{21} \leq 0$, then $\mu \geq \mu_1$ such that effort levels satisfy $e_1^{Coop} \geq e_1^{Indep}$. Intuitively, when cost exhibits complementarity in effort, in which exerting effort 1 reduces the marginal cost to exert effort 2, then the common agent will exert more effort on both tasks when the principals jointly assign tasks. In other words, positive externality that exists between efforts enable the principals to induce more efforts when the tasks are jointly assigned, compared to the case in which they independently assign tasks.
  - If $c_{21} > 0$, then it is possible (but not always) that $\mu < \mu_1$ such that effort levels satisfy $e_1^{Coop} < e_1^{Indep}$. Intuitively, when cost exhibits substitutability in effort, exerting effort 1 increases the marginal cost to exert effort 2. In other words, every principal, by increasing the effort level he assigns to the common agent, creates a negative externality on the other principal, who now faces a higher cost of effort and thus needs to offer a higher wage to the common agent. In this setting, when the principals jointly assign effort levels, they internalize this negative externality, reducing the effort levels they assign to the common agent.