EconS 503 - Microeconomic Theory II
Homework #7 - Answer Key

1. **Exercise from Bolton and Dewatripont:**

   (a) Exercise 6 (see page 650 since all exercises are at the end of the book).

   - For the answer key, see scanned pages at the end of this handout.

2. **PBEs in bargaining.** A buyer and a seller are bargaining. The seller owns an object for which the buyer has a value v; the seller's value is zero. The buyer knows v but the seller does not. The seller's beliefs about v, which are common knowledge, are that $v = 30$ with probability $\frac{1}{2}$ and $v = 10$ with probability $\frac{1}{2}$. There are two periods of bargaining; there is no discounting (i.e., $\delta = 1$).

   - In the first period, the seller makes an offer p_1 that represents a price that the buyer will need to pay to buy the object. The buyer can accept or reject the offer. If the buyer accepts, the offer is implemented and the game ends. If the buyer rejects, the game continues to the second period.
   - In the second period, the seller (again) makes an offer p_2, which is the price the buyer will need to pay to buy the object. The buyer can accept or reject the offer. If the buyer accepts, the offer is implemented and the game ends. If the buyer rejects, then the seller keeps the object and the game ends.

   If the buyer buys the object in the first period, then the payoffs are p_1 for the seller and $v - p_1$ for the buyer. Similarly, if the buyer buys the object in the second period, then the payoffs are p_2 for the seller and $v - p_2$ for the buyer. If the buyer does not buy the object, then the payoffs are zero for each player.

   (a) Provide an extensive-form representation of this game.

   (b) Find a Perfect Bayesian equilibrium in which the seller believes that any buyer that rejects a first-period offer is the type with valuation $v = 10$ with probability 1. (Justify your answer, and remember to fully specify the Perfect Bayesian equilibrium.)

   - For the answer key, see scanned pages at the end of this handout.

3. **Cheap talk with three types.** Consider the cheap talk model with three types discussed in class (Investing recommendations game). Let us focus on the partially separating strategy profile where the Analyst (sender) recommends Buy both when the stock outperforms the market and when its neutral, but recommends Hold when the stock underperforms the market. In class, we made a simplifying assumption on off-the-equilibrium beliefs (after the Investor receives a Sell recommendation), denoted by γ_1, γ_2, and $1 - \gamma_1 - \gamma_2$.

(a) Without restricting off-the-equilibrium beliefs, find under which conditions the above partially separating strategy profile can be sustained as a PBE of this game.

- Upon observing the off-the-equilibrium message of Sell, the investor assigns the following beliefs to the stock's performance:
  $prob(Outperform|Sell) = \gamma_1$

  $prob(Neutral|Sell) = \gamma_2$

  $prob(Underperform|Sell) = 1 - \gamma_1 - \gamma_2$

- Given these beliefs, the expected payoffs for the investor (after observing Sell) are:
  $E(\pi_2(Buy|Sell)) = 2\gamma_1 + \gamma_2 - 1$

  $E(\pi_2(Hold|Sell)) = \gamma_2$

  $E(\pi_2(Sell|Sell)) = 1 - 2\gamma_1 - \gamma_2$

- The investor prefers to Buy than Hold if the expected payoffs from buying are greater than expected payoffs from holding. Formally;

  $E(\pi_2(Buy|Sell)) > E(\pi_2(hold|Sell))$ which boils down to $\gamma_1 > \frac{1}{2}$

  Hold than Sell if:
  $E(\pi_2(Hold|Sell)) > E(\pi_2(Sell|Sell)) = \gamma_1 + \gamma_2 > \frac{1}{2}$

  Buy than Sell if:
  $E(\pi_2(Buy|Sell)) > E(\pi_2(Sell|Sell)) = 2\gamma_1 + \gamma_2 = 1$

- The investor's best response given off-the equilibrium cases is then:
  The investor sells the stock if he believes that the probability of the stock underperforming is high or: $\gamma_1 + \gamma_2 \leq \frac{1}{2}$

  The investor buys if : $\gamma_1 > \frac{1}{2}$

  The investor holds if :$\gamma_1 + \gamma_2 > \frac{1}{2}$ and $\gamma_1 \leq \frac{1}{2}$

(b) Consider now the pooling strategy profile where the Analyst recommends Buy regardless of the stock's type. Under which conditions can this strategy profile be supported as a PBE?

- If the analyst recommends Buy regardless of the stock's type, then the investor's believes for each type of stock coincide with the prior probability distribution over types:
  $prob(Outperform|Sell) = \frac{1}{3}$

$prob(Neutral|Sell) = \frac{1}{3}$

$prob(Underperform|Sell) = \frac{1}{3}$

- Given these beliefs, the expected payoffs of the investor are:
  $E(\pi_2(Buy|buy)) = \frac{1}{3}(1) + \frac{1}{3}(0) + \frac{1}{3}(-1) = 0$

  $E(\pi_2(Hold|buy)) = \frac{1}{3}(0) + \frac{1}{3}(1) + \frac{1}{3}(0)\frac{1}{3} = \frac{1}{3}$

  $E(\pi_2(Sell|buy)) = \frac{1}{3}(-1) + \frac{1}{3}(0) + \frac{1}{3}(1) = 0$

- Given the above payoffs, the analyst will recommend Buy regardless of the stock's type. The investor's best response is then to Hold because it gives him the highest payoff.
- Upon receiving the off-equilibrium message of Hold, the investors expected payoff is:
  $E(\pi_2(Buy|hold)) = 2\mu_1 + \mu_2 - 1$

  $E(\pi_2(Hold|hold)) = \mu_2$

  $E(\pi_2(Sell|hold)) = 1 - 2\mu_1 - \mu_2$

- Therefore, investor prefers to buy than hold if:
  $2\mu_1 + \mu_2 - 1 > \mu_2 = \mu_1 > \frac{1}{2}$

  Hold than sell if:
  $\mu_2 > 1 - 2\mu_1 - \mu_2 = \mu_1 + \mu_2 > \frac{1}{2}$

  Buy than sell if:
  $2\mu_1 + \mu_2 - 1 > 1 - 2\mu_1 - \mu_2 = 2\mu_1 + \mu_2 > \frac{1}{2}$
- As a consequence, the above conditions this pooling strategy profile can be sustained as a PBE.

4. **Cheap talk when the expert receives imprecise signals.** Consider the following cheap talk model between an expert (E), such a special interest group, and a decision maker (DM), such as a politician. For simplicity, assume that the state of the world is discrete, either $\theta = 1$ or $\theta = 0$ with prior probability $p \in (0, 1)$ and $1 - p$, respectively. The expert privately observes an informative but noisy signal s, which also takes two discrete values $s \in \{0, 1\}$. The precision of the signal is given by the conditional probability

$$\text{prob}(s = k|\theta = k) = q,$$

where $k = \{0, 1\}$, and $q > \frac{1}{2}$. In words, the probability that the signal s coincides with the true state of the world θ is q (precise signal), while the probability of an imprecise signal where $s \neq \theta$ is $1 - q$. The time structure of the game is as follows:

1) Nature chooses θ according to the prior p.

2) Expert observes signal s and reports a message $m \in \{0,1\}$

3) Decision maker observes m and responds with $x \in \{0,1\}$

4) θ is observed and payoffs are realized

The payoff function for the decision maker is

$$u(x,\theta) = \left(\theta - \frac{1}{2}\right) x$$

while that of the expert is

$$v(m,\theta) = \begin{cases} 1, & \theta = m \\ 0, & \theta \neq m \end{cases}$$

which, in words, indicates that the expert's payoff is 1 when the message she sends coincides with the true realization of the state of the world, but becomes zero otherwise. Importantly, her payoff is unaffected by the signal, which she only uses to infer the actual realization of parameter θ. Intuitively, $v(m,\theta)$ is often understood as a "reputation function" since it provides the expert with a payoff of 1 only when his message was an accurate representation of the true state of the world (which in this model he does not precisely observe).

(a) Is there a Perfect Bayesian equilibrium (PBE) in which the expert reports his signal truthfully?

- Is there a Perfect Bayesian equilibrium (PBE) in which the expert reports his signal truthfully?
- *Updated beliefs.* In a strategy profile where the expert sends a message that coincides with the signal she receives (that is, sending message $m = 1$ after receiving signal $s = 1$, but sending message $m = 0$ after receiving signal $s = 0$)[1], the decision maker and the expert sustain the same beliefs about θ since $m = s$. Specifically, after receiving a signal of $s = 1$ (a message of $m = 1$), both expert and decision maker use Bayes' rule to update their beliefs yielding

$$\mu_1 = \frac{pq}{pq + (1-p)(1-q)}$$

  while after receiving a signal of $s = 0$ (a message of $m = 0$), both expert and receiver updated their beliefs as follows

$$\mu_0 = \frac{p(1-q)}{p(1-q) + (1-p)q}$$

- *Decision maker's response.* Given the above beliefs, after receiving a message $m = 1$ from the expert, the decision maker responds with $x = 1$ if

$$\mu_1 \left(1 - \frac{1}{2}\right) 1 + (1-\mu_1)\left(0 - \frac{1}{2}\right) 1 \geq \mu_1 \left(1 - \frac{1}{2}\right) 0 + (1-\mu_1)\left(0 - \frac{1}{2}\right) 0$$

[1] This truthful reporting of signals can be described more compactly by saying that the expert's strategy is a message $m(s) = s$ for every signal $s \in \{0,1\}$.

or

$$\mu_1 \frac{1}{2} + (1-\mu_1)\left(-\frac{1}{2}\right) \geq 0$$

or, simplifying, $\mu_1 \geq \frac{1}{2}$. From the above expression of posterior belief μ_1, this condition holds if

$$\frac{pq}{pq + (1-p)(1-q)} \geq \frac{1}{2}$$

or, after rearranging, $p \geq 1-q$, which holds by assumption. That is, the decision maker responds with $x = 1$ after receiving message $m = 1$ for all admissible parameter values. Similarly, after receiving a message $m = 0$, the decision maker responds with $x = 1$ if

$$\mu_0 \left(1-\frac{1}{2}\right) 1 + (1-\mu_0)\left(0-\frac{1}{2}\right) 1 \geq \mu_0 \left(1-\frac{1}{2}\right) 0 + (1-\mu_0)\left(0-\frac{1}{2}\right) 0$$

or, after simplying, $\mu_0 \geq \frac{1}{2}$. From the above expression of posterior belief μ_0, this condition holds if

$$\frac{p(1-q)}{p(1-q) + (1-p)q} \geq \frac{1}{2}$$

or $p \geq q$. In words, the decision maker responds with $x = 1$ after observing message $m = 0$ when the probability of $\theta = 1$, p, is higher than the probability of the expert receiving precise signals, q. Otherwise (when $p < q$), the decision maker responds with $x = 0$ after observing message $m = 0$. Therefore, when $p < q$ we can say that the decision maker responds with an action $x(m) = m$ to every message $m \in \{0, 1\}$ he receives from the sender.

- *Expert's messages - After receiving signal $s = 1$.* If the expert reports her signal truthfully (sending message $m = 1$), her expected payoff is

$$\mu_1 v(1,1) + (1-\mu_1) v(1,0) = \mu_1$$

Intuitively, the above expression says that the expert sends a message $m = 1$ but does not know if the state of the world is $\theta = 1$, which yields a payoff of 1 since $\theta = m$; or if the state of the world is $\theta = 0$, which yields a payoff of zero for her since $\theta \neq m$. If, instead, she misreports her signal (sending message $m = 0$), her expected payoff becomes

$$\mu_1 v(0,1) + (1-\mu_1) v(0,0) = 1 - \mu_1$$

Therefore, the expert truthfully reports her signal if $\mu_1 \geq 1 - \mu_1$, or $\mu_1 \geq \frac{1}{2}$. Using the expression of posterior belief μ_1, we obtain that

$$\frac{pq}{pq + (1-p)(1-q)} \geq \frac{1}{2}$$

which collapses to $p \geq 1-q$. In words, after receiving a signal of $s = 1$, the expert truthfully conveys her signal if the probability of receiving such a signal is higher than the probability of an imprecise signal, $1-q$.

- *Expert's messages - After receiving signal $s = 0$.* If the expert reports his signal truthfully (that is, sending message $m = 0$), her expected payoff is

$$\mu_0 v(0,1) + (1-\mu_0)v(0,0) = \mu_0 0 + (1-\mu_0)1 = 1-\mu_0$$

  Intuitively, the above expression says that the expert sends a message $m = 0$ but does not know if the state of the world is $\theta = 1$, which yields a payoff of zero for her since $\theta \neq m$; or if the state of the world is $\theta = 0$, which yields a payoff of 1 since $\theta = m$. If, instead, the expert sends message $m = 1$ (lying about her message), her expected payoff becomes

$$\mu_0 v(1,1) + (1-\mu_0)v(1,0) = \mu_0 v(1,1) + (1-\mu_0)v(1,0) = \mu_0$$

  Therefore, the expert truthfully reports her signal if $1-\mu_0 \geq \mu_0$, or $\frac{1}{2} \geq \mu_0$. Examining the expression of posterior belief μ_0, we find that

$$\frac{p(1-q)}{p(1-q)+(1-p)q} \leq \frac{1}{2}$$

  simplifies to $p \leq q$. In words, after receiving a signal of $s = 0$, the expert truthfully conveys her signal if the probability of an accurate signal, q, is higher than the probability of receiving a signal of $s = 1$. Combining the above conditions $p \geq 1-q$ and $p \leq q$, we obtain $1-q \leq p \leq q$.

- *Summary*:
  - When $p \geq q$, a PBE where the expert truthfully reports her signal can be sustained if $1-q \leq p \leq q$ (from the expert) and $p \geq q$ (from the decision maker), which are only compatible when $p = q$. In words, the prior probability of the state of the world being $\theta = 1$, p, must coincide with the probability with which the expert receiving precise signals, q. While the expert truthfully reports her signals to the decision maker, the decision maker does not follow the expert's advise when observing a message of $m = 0$.
  - When $p < q$, a PBE where the expert truthfully reportsher signal can be sustained if $1-q \leq p \leq q$ (from the expert) and $p < q$ (from the decision maker), where the expert sends a message $m(s) = s$ for every signal $s \in \{0,1\}$ she received, while the decision maker responds with an action $x(m) = m$ for every message $m \in \{0,1\}$ he receives. In this PBE, the expert truthfully reports her signals to the decision maker, and the decision maker follows the expert's advise after every message.

a) Standard figure.

b) At period 1, the buyer buys if $v - p_1 \geq 0$, or $v \geq p_1$. Hence, if $p_1 < 10$ the buyer buys regardless of his valuation; if p_1 lies between 10 and 30, he buys only when his valuation is v=30; and if p_1p1 is above 30 the buyer rejects regardless of his valuation. Hence, the seller expected profit is $\frac{1}{2}p_1$ since both types are equally likely. (In addition, if a first-period price of 30 is rejected the game proceeds to period 2, where the seller infers that the buyer's valuation must be lower than 30)

At period 2, let $\mu = Prob(v = 30|History\ at\ t = 2)$, and the buyer buys if his valuation v satisfies $v \geq p_2$

In addition, when beliefs satisfy

- $\mu > \frac{1}{2}$, the seller's optimal second-period price is $p_2 = \$30$,
- when $\mu < \frac{1}{2}$ the seller's optimal second-period price is $p_2 = \$10$, and
- when $\mu = \frac{1}{2}$ the seller is indifferent between setting a second-period price of $p_2 = \$10$ and $p_2 = \$30$.

We can now apply Bayes' rule to update the seller's beliefs about the seller's valuation upon observing that a price p_1 was rejected in the first period. In particular,

$$\mu(p_1) = \frac{\frac{1}{2}\alpha}{\frac{1}{2}\alpha + \left(1 - \frac{1}{2}\right)} = \frac{1}{2}$$

where α is the probability that the low-value buyer does not buy the product at a price p_1. Rearranging, we obtain

$$\frac{\alpha}{1 + \alpha} = \frac{1}{2}$$

And solving for probability α yields $\alpha = 1$, implying that in the second period the seller believes that the buyer who rejected price p_1 in the first-period game must be a high-value buyer with certainty.

Chapter 3

Hidden Information, Signaling

3.1 Question 6

Consider a firm that can invest an amount I in a project generating high observable cash flow $C > 0$ with probability θ and 0 otherwise: $\theta \in \{\theta_L, \theta_H\}$ with $\theta_H - \theta_L \equiv \Delta > 0$ and $\Pr[\theta = \theta_L] = \beta$. The firm needs to raise I from external investors who do not observe the value of θ. Assume that $\theta_L C - I > 0$. Everybody is risk neutral and there is no discounting.

1. Suppose that the firms can only promise to repay an amount R chosen by the firm (with $0 \leq R \leq C$) when cash flow is C and 0 otherwise. Can a good firm signal its type?

2. Suppose now that the firm also has the possibility of pledging some assets as collateral for the loan: Should a "default" occur (the firm being unable to repay R), an asset of value K to the firm is transferred to the creditor whose valuation is xK with $0 < x < 1$. The size of the collateral K is a choice variable. Give a necessary and sufficient condition for the "best" Perfect Bayesian Equilibrium to be separating. How does it depend on β and x? Explain.

3.1.1 No Collateral

Both firms would want to undergo the project since $\theta_L C > I$. A good firm cannot signal its type, since for a separating equilibrium to exist we need $R_H \neq R_L$. However, this cannot be an equilibrium. This can be seen clearly from the incentive compatibility condition for a firm of type i

$$\theta_i (C - R_i) \geq \theta_i (C - R_j).$$

Whenever $R_i \neq R_j$ at least one type of firm will want to deviate.

Intuitively speaking, since both firms receive C when the project is successful and 0 when it fails, and we only have one repayment instrument, the bad firm can perfectly mimic the good firm.

3.1.2 Collateral

Separating Equilibrium

The best separating equilibrium is clearly the one with the least amount of K so $K_L = 0$. The loss $(1-x)K$ is higher for a low-type firm since it has a higher probability of being in default.

A separating equilibrium can be supported by the following beliefs:

$$\begin{aligned} \Pr(\theta = \theta_L | K > K^*) &= 0 \\ \Pr(\theta = \theta_L | K \le K^*) &= 1 \end{aligned}$$

and we have

$$\begin{aligned} K_L &= 0 \\ R_L &= \frac{I}{\theta_L}. \end{aligned}$$

This holds since otherwise the low type could offer a higher payment and still be better off–there is no benefit from a positive K.

Thus we shall have the following incentive compatibility and individual rationality constraints:

$$\begin{aligned} \theta_L R_L &= I && \text{(IRL)} \\ xK^H(1-\theta_H) + \theta_H R_H &= I && \text{(IRH)} \\ \theta_L(C - R_L) &\ge \theta_L(C - R_H) - (1-\theta_L)K_H && \text{(ICL)} \\ \theta_H(C - R_H) - (1-\theta_H)K_H &\ge \theta_H(C - R_L) && \text{(ICH)} \end{aligned}$$

Rewriting (ICL) we obtain

$$R_L - R_H \le \frac{1-\theta_L}{\theta_L} K_H.$$

Similarly, re-arranging (ICH) gives

$$R_L - R_H \ge \frac{1-\theta_H}{\theta_H} K_H.$$

Putting these expressions together we obtain

$$\frac{1-\theta_H}{\theta_H} K_H \le R_L - R_H \le \frac{1-\theta_L}{\theta_L} K_H.$$

This works even if x is very small. The intuition is that the high type benefits from a lower R more often and suffers from the loss of K less often, since

$\theta_H > \theta_L$. Thus the best separating equilibrium minimizes the use of (socially) wasteful collateral, that is K_H is set as low as possible. Hence, in equilibrium, only (ICL) is binding and (ICH) is slack. Thus, in what follows we can ignore (ICH). Solving the following equalities which we obtained from the constraints using the fact that (ICL) is binding in equilibrium and combining (IRL) and (ICL), we find

$$xK^H(1-\theta_H) + \theta_H R_H = I \qquad \text{(IRH)}$$
$$K_H(1-\theta_L) + \theta_L R_H = I. \qquad \text{(IRL, ICL)}$$

Rewriting these conditions yields

$$\begin{aligned} R_H &= \frac{I}{\theta_H} - \frac{1-\theta_H}{\theta_H} x K_H \\ R_H &= \frac{I}{\theta_L} - \frac{1-\theta_L}{\theta_L} K_H, \end{aligned}$$

and after some algebraic manipulation we obtain

$$\begin{aligned} K_H &= \frac{\Delta I}{\theta_H (1-\theta_L) - x\theta_L (1-\theta_H)} \\ R_H &= \frac{I}{\theta_H} - \frac{1-\theta_H}{\theta_H} \frac{x\Delta I}{\theta_H (1-\theta_L) - x\theta_L (1-\theta_H)}. \end{aligned}$$

From above, we have

$$\begin{aligned} K_L &= 0 \\ R_L &= \frac{I}{\theta_L}, \end{aligned}$$

since this is the best, or the least cost equilibrium. K is costly and hence there is no reason to use it in the low state. Notice that this separating equilibrium always exists.

Pooling Equilibrium

We can compare this to the best pooling equilibrium, where

$$\begin{aligned} K^P &= 0 \\ R^P &= \frac{I}{\theta_H - \beta\Delta}. \end{aligned}$$

However, this pooling equilibrium may not exist. It will exist if and only if

$$\theta_H \left(C - R^P\right) \geq \theta_H (C - R) - (1-\theta_H) K$$

where

$$\begin{aligned} R &= \frac{I - (1-\theta_L) xK}{\theta_L} \\ I &= \theta_L R + (1-\theta_L) xK, \end{aligned}$$

which follows from the assumption that for any deviation from $K^P = 0$ the investors will believe that the firm is of low type. So the pooling equilibrium is sustainable if there are no deviations given these beliefs. This is the worst belief in the sense that if we cannot find a pooling equilibrium supported by these beliefs then no pooling equilibrium exists (there will always be a profitable deviation from it). Combining the equations, we obtain

$$\theta_H \left(C - \frac{I}{\theta_H - \beta\Delta} \right) \geq \theta_H \left(C - \frac{I - (1-\theta_L)\, xK}{\theta_L} \right) - (1-\theta_H)\, K,$$

which is equivalent to

$$x \leq (\theta_H I (\frac{1}{\theta_L} - \frac{1}{\beta\theta_L + (1-\beta)\theta_H}) + (1-\theta_H)K) \frac{\theta_L}{\theta_H(1-\theta_L)K}.$$

Thus, the smaller x or β the more likely is the existence of a pooling equilibria. This is intuitive. A smaller x means that the signal is more costly, and hence a profitable deviation from the least-cost pooling equilibrium that has no costly collateral, is very difficult. Similarly, with a smaller β the less likely it is that the firm is a bad type (so a smaller cross-subsidy is needed).

Comparison

One way to compare the different equilibria would be to compare ex-ante expected profits of the firm for the separating and pooling equilibrium (compare section 3.1.1). The expected profits for the separating equilibrium are given by

$$\begin{aligned} \pi^S &= (1-\beta)\, \pi_H^S + \beta \pi_L^S \\ &= (1-\beta)\, [\theta_H(C - R_H) - (1-\theta_H)K_H] + \beta\theta_L(C - R_L) \\ &= C\, [\theta_L + (1-\beta)\, \Delta] - I - (1-\beta)\, (1-\theta_H)(1-x)K_H. \end{aligned}$$

where K_H is as defined above.

In contrast, expected profits for the pooling equilibrium are

$$\begin{aligned} \pi^P &= (1-\beta)\, \pi_H^P + \beta \pi_L^P \\ &= (1-\beta)\, \theta_H(C - R^P) + \beta\theta_L(C - R^P) \\ &= [\theta_L + (1-\beta)\, \Delta]\, (C - R^P) \\ &= [\theta_L + (1-\beta)\, \Delta]\, C - I. \end{aligned}$$

Hence, we have

$$\pi^P > \pi^S,$$

whenever the pooling equilibrium exists. The pooling equilibrium leads to higher profits as it avoids the use of (wasteful) collateral. Thus, the best perfect Bayesian equilibrium (when defined in this way) is separating if and only if the pooling equilibrium does not exist. Another way would be look for one equilibria Pareto-dominating the other. Here we would see when both types prefer the pooling equilibrium. This happens when β is very small and thus the tiny gain from signalling (avoiding the infinitesimal cross subsidy) is smaller than the costly signal. See section 3.1.1 for details.