EconS 503 - Microeconomic Theory II
Homework #10 - Due date: Friday, May 1st, via email.

1. **Exercises from MWG:**

   (a) Chapter 21 (social choice theory): Exercise 21.D.5.

   (b) Chapter 23 (mechanism design): Exercise 23.C.10.

2. **Public Good Provision - Different mechanisms** Imagine that you and your colleagues want to buy a coffee machine for your office. Suppose that some of you may be heavily addicted to coffee and are willing to pay more for the machine than the others. However, you do not know your colleagues' willingness to pay for the machine. The cost of the machine is C. We would like to find a decision rule in which (i) each individual reports a valuation (i.e., direct mechanism), and (ii) the coffee maker is purchased if and only if it is efficient to do so. Let us next analyze if it is possible to find a cost-sharing rule which gives incentive for everyone to report his valuation truthfully.

   In particular, assume n individuals, each of them with private valuation $\theta_i \sim U(0,1)$. The allocation function is binary $y \in \{0,1\}$, i.e., the coffee machine is purchased or not. Let t_i be the transfer from individual i, implying a utility of

$$u_i(y, \theta_i, t_i) = y\theta_i - t_i$$

   Let $i \in \{1, ..., n\}$ denote the individuals, and let $i = 0$ denote the original owner of the good.

   (a) What is the efficient assignment rule, $y^*(\theta_1, ..., \theta_n)$?

   (b) *Equal-share rule.* Consider the following equal-share rule: When the public good is provided, the cost is equally divided by all n individuals.

      1. Before starting any computation, what would you expect - whether each individual would overstate or understate their valuation?
      2. Confirm that the transfer rule is written by:

$$t_i(\theta) = \frac{C}{n} y^*(\theta)$$

      3. Let $V_i(\tilde{\theta}_i|\theta_i, \theta_{-i})$ be individual i's payoff when i reports $\tilde{\theta}_i$ instead of his true valuation θ_i, while the others truthfully report their valuations θ_{-i}. Show that

$$V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = \left(\theta_i - \frac{C}{n}\right) y^*(\tilde{\theta}_i, \theta_{-1})$$

4. Let $U_i\left(\tilde{\theta}_i|\theta_i\right)$ be individual i's expected payof when he reports $\tilde{\theta}_i$ instead of the true valuation θ_i. Show that

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = \left(\theta_i - \frac{C}{n}\right) E_{\theta_{-i}}\left[y^*(\tilde{\theta}_i, \theta_{-1})\right]$$

5. Suppose that i's private valuation θ_i satisfies $\theta_i > \frac{C}{n}$. Assuming that the others are telling the truth, what is the best response for i? What if $\theta_i < \frac{C}{n}$? Is this mechanism strategy-proof? Is this mechanism Bayesian incentive compatible?

(c) *Proportional payment rule.* Consider now the proportional payment rule:

$$t_i(\theta) = \frac{\theta_i C}{\sum_j \theta_j} y^*(\theta)$$

where every individual i pays a share of the total cost equal to the proportion that his reported valuation signifies out of the total reported valuations.

1. Under this rule, what would you expect - whether each individual would overstate or understate the valuation?
2. Show that the utility of reporting $\tilde{\theta}_i$ is now

$$V_i(\tilde{\theta}_i|\theta_i, \theta_{-i}) = \left(\theta_i - \frac{\tilde{\theta}_i C}{\tilde{\theta}_i + \sum_{j\neq i}\theta_j}\right) y^*(\tilde{\theta}_i, \theta_{-1})$$

3. For simplicity, suppose two individuals, $n = 2$ and a total cost of $C = 1$. Show that

$$U_i\left(\tilde{\theta}_i|\theta_i\right) = \tilde{\theta}_i\left(\theta_i - \log\left(\tilde{\theta}_i + 1\right)\right)$$

4. Is this mechanism strategy-proof? Is it Bayesian incentive compatible?
5. Which way is everyone biased, overstate or understate? What is the intuition?

(d) *VCG mechanism.* Let us consider now the VCG mechanism. Recall that the efficient rule $y^*(\theta)$ determines that the coffee machine is bought if and only if total valuations satisfy $\sum_i \theta_i \geq C$. Remember that we need to include the original owner of the public good; $i = 0$. Then, the total surplus when the valuation of individual i is considered in $\theta = (\theta_1, \theta_2, ..., \theta_n)$ is

$$\sum_{j\neq i} v_j\left(y^*(\theta), \theta_j\right) = \begin{cases} \sum_{j\neq i}\theta_j & \text{if } \sum_j \theta_j \geq C \\ C & \text{if } \sum_j \theta_j < C \end{cases}$$

while total surplus when the valuation of individual i is ignored, θ_{-i}, is

$$\sum_{j\neq i} v_j\left(y^*(\theta_{-i}), \theta_j\right) = \begin{cases} \sum_{j\neq i}\theta_j & \text{if } \sum_{j\neq i} \theta_j \geq C \\ C & \text{if } \sum_{j\neq i} \theta_j < C \end{cases}$$

The only difference in total surplus arises from the allocation rule which specifies that, when θ_i is considered, the good is purchased if and only if $\sum_j \theta_j \geq C$, whereas when θ_i is ignored, the good is bought if and only if $\sum_{j\neq i}\theta_j \geq C$. Hence, the VCG transfer is

$$\begin{aligned} t_i^*(\theta) &= -\left(\sum_{j\neq i} v_j(y^*(\theta),\theta_j) - \sum_{j\neq i} v_j(y^*(\theta_{-i}),\theta_j)\right) \\ &= \begin{cases} C-\sum_{j\neq i}\theta_j & \text{if } \sum_{j\neq i}\theta_j < C \leq \sum_j \theta_j \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Intuitively, player i pays the difference between everyone else's valuations, $\sum_{j\neq i}\theta_j$, and the total cost of the good, C. Such a payment, however, only occurs when aggregate valuations exceed the total cost, $\sum_j \theta_j \geq C$, and thus the good is purchased, and when the valuations of all other players do not yet exceed the total cost of the good, $\sum_{j\neq i}\theta_j < C$, so the difference $C - \sum_{j\neq i}\theta_j$ is paid by player i in his transfer.

1. Show that in this mechanism player i's utility from reporting a valuation $\tilde{\theta}_i \neq \theta_i$ is

$$\begin{aligned} V_i(\tilde{\theta}_i|\theta_i,\theta_{-i}) &= v_i\left(y^*\left(\tilde{\theta}_i,\theta_{-i}\right),\theta_i\right) - t_i^*\left(\tilde{\theta}_i,\theta_{-i}\right) \\ &= \begin{cases} 0 & \text{if } \tilde{\theta}_i + \sum_{j\neq i}\theta_j < C \\ \sum_j \theta_j - C & \text{if } \sum_{j\neq i}\theta_j < C \leq \tilde{\theta}_i + \sum_{j\neq i}\theta_i \\ \theta_i & \text{if } C \leq \sum_{j\neq i}\theta_j \end{cases} \end{aligned}$$

2. Is this mechanism strategy-proof? Is this Bayesian incentive compatible?
3. For simplicity, suppose two individuals, $n = 2$, and a total cost of $C = 0.5$. Compute y^*, t_1^* and t_2^* for the following (θ_1, θ_2) pairs, so you complete the table.

θ_1	θ_2	$y^*(\theta)$	$t_1^*(\theta)$	$t_2^*(\theta)$
0.1	0.3			
0.3	0.3			
0.3	0.8			
0.8	0.8			

4. Show that the expected revenue from this mechanism is $E\left[t_1^*(\theta_1,\theta_2) + t_2^*(\theta_1,\theta_2)\right] = \frac{1}{6} \simeq 0.167$. Based on what you calculated in part (iii), is this problematic?

3. **Lexicographic social welfare functional.** In this exercise, we consider a setting with two alternatives x and y, and discuss the following lexicographic social welfare functional.

$$F(\alpha_1,..,\alpha_N)\begin{cases} \alpha_1 & \text{if } \alpha_1 \neq 0 \\ \alpha_2 & \text{if } \alpha_1 = 0 \text{ and } \alpha_2 \neq 0 \\ \alpha_3 & \text{if } \alpha_1 = \alpha_2 = 0 \text{ and } \alpha_3 \neq 0 \\ \ldots \end{cases}$$

Intuitively, society selects the alternative that individual 1 strictly prefers. However, if he is indifferent between alternatives x and y, society follows the strict preferences of individual 2 (if he has a strict preference over x or y). If both individuals 1 and 2 are indifferent between x and y, the strict preferences of individual 3 are considered, and so on.

Determine whether or not it satisfies the three properties of majority voting (symmetry among agents, neutrality between alternatives, and positive responsiveness).

4. **Social theory, short proofs.** Provide a short proof of the following claims. In some cases, a counterexample may suffice.

   (a) The Borda count satisfies monotonicity.

   (b) The Borda count satisfies the Pareto condition.

   (c) The Hare procedure violates IIA.