EconS 503 – Microeconomic Theory II[1]
Spring 2020

1 Signaling games with continuous action spaces – A step-by-step approach

1.1 Introduction

The labor market signaling game in Munoz-Garcia (2017, Example 8.6) assumes, for simplicity, that the worker's strategy space was binary (either acquire eduction or not) and that the firm's response was also binary (hire the worker as a manager or as a cashier). In reality, however, workers can choose among a wider range of education levels, $e \geq 0$, and firms can respond offering a range of salaries, $w \geq 0$, implying that both players face continuous, rather than discrete, action spaces. We analyze this setting in this section, following Spence (1974). We start describing the worker's and firm's payoff functions.

1.2 Payoff functions

A worker with productivity θ_K, where $K = \{H, L\}$, has utility function is

$$u(w, e|\theta_K) = w - c(e, \theta_K)$$

where the first term represents the salary he receives from the firm, w, while the second term denotes his cost of acquiring e units (e.g., years) of education given his productivity being θ_K. The cost of education function $c(e, \theta_K)$ satisfies the following assumptions:

1. Cost of education is zero when the worker acquires no education, that is, $c(0, \theta_K) = 0$.

2. Cost of education is strictly increasing and convex in education (i.e., $c_e > 0$ and $c_{ee} > 0$), indicating that additional years of education become progressively more costly.

3. Cost of education $c(e, \theta_K)$ is decreasing in the worker's productivity, θ_K; that is,

$$c(e, \theta_H) < c(e, \theta_L),$$

which implies that a given education e (e.g., a college degree) is easier to acquire for the high-productivity than for the low-productivity worker. A similar argument applies to the marginal cost of education, $c_e(e, \theta_K)$, which is also lower for the high-productivity than the low-productivity worker; that is, $c_e(e, \theta_H) < c_e(e, \theta_L)$.

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For instance, cost functions such as $c(e, \theta_K) = \frac{Ae^2}{\theta_K}$ or $c(e, \theta_K) = \frac{Ae^3}{\theta_K}$, where $A > 0$ denotes a constant, satisfy the above three assumptions.

Graphically, we can depict the worker's indifference curve in the $(e, w)-$quadrant by, first, solving for w, which yields $w = u + c(e, \theta_K)$. Since the cost of education is strictly increasing and convex in e, indifference curves are also increasing and convex in e; as depicted in figure 8.22. Intuitively, a higher education level must be accompanied by an increase in his salary for the worker's utility to remain unchanged. Additionally, indifference curves to the northwest are associated to a higher utility level since, for a given education e, the worker receives a higher wage or, for a given wage w, the worker acquires less education.

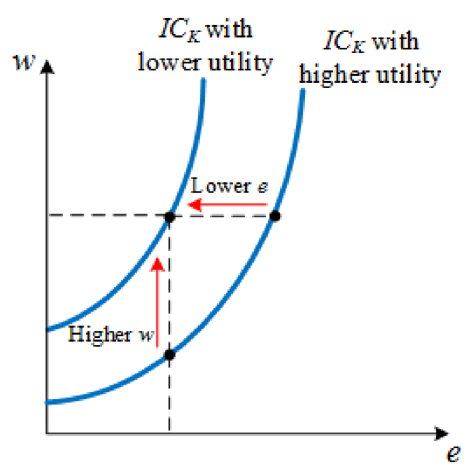


Figure 8.22. Indifference curves for a representative worker.

We assume that the labor market is competitive. In a complete information setting where the firm observes the worker's type, this assumption implies that every firm pays a salary equal to the worker's productivity $w = \theta_K$. If a firm paid a lower salary, $w < \theta_K$, other firms could offer a slightly higher salary w' where $w < w' \leq \theta_K$, making a weakly positive profit. If a firm paid a higher salary $w > \theta_K$, it would attract the worker but make a negative profit. A similar argument applies when the firm does not observe the worker's type and pays a salary equal to his expected productivity, $w = E[\theta_K]$, where this expectation is based on the firm's beliefs about the worker's type.

1.3 Complete information

As a benchmark, we consider a complete information context where the firm observes the worker's productivity, θ_K. In this setting, the only subgame perfect equilibrium has the worker acquiring zero education regardless of his productivity, $e_H = e_L = 0$. The firm responds with salary $w(\theta_H) = \theta_H$

when the worker's productivity is high and $w(\theta_L) = \theta_L$ when his productivity is low. Intuitively, the worker cannot use education as a signal about his type since the firm observes his productivity.

1.4 Incomplete information

We next discuss that, under incomplete information, education can become an informative (although costly) signal. Assume that the worker privately observes his type θ_K before choosing his education level e. The firm observes education e but does not know the worker's type θ_K. However, it assigns a prior probability $p \in [0, 1]$ to the worker's productivity being high and $1 - p$ to his productivity being low. This probability distribution is common knowledge among players.

1.4.1 Separating PBE

We next check if a separating strategy profile where each type of worker acquires a different education level can be sustained as a Perfect Bayesian Equilibrium (PBE). We follow the same four steps as in the discrete version of the game.

First step. In a separating strategy profile, the high-productivity worker chooses education level e_H while the low-productivity worker chooses e_L, where $e_L \neq e_H$.

Second step. Upon observing education level e_H, the firm assigns full probability to facing a high-productivity worker, that is, $\mu(\theta_H|e_H) = 1$. In contrast, after observing education level e_L, the firm assigns no probability to facing the high-productivity worker, i.e., $\mu(\theta_H|e_L) = 0$, as it is convinced of dealing with a low-productivity worker. If, instead, the firm observes the worker acquiring an off-the-equilibrium education level, i.e., an education e different than e_H and e_L, it cannot update its beliefs using Bayes' rule, leaving them unrestricted, that is, $\mu(\theta_H|e) \in [0, 1]$ for all $e \neq e_H \neq e_L$. (We impose some conditions on off-the-equilibrium beliefs below, but at this point we leave these beliefs unconstrained.)

Third step. Given the above beliefs, we must now find the firm's optimal responses. Upon observing education e_H, the firm pays a salary $w(e_H) = \theta_H$ since it is convinced of facing a high-productivity worker. Similarly, upon observing education e_L, the firm pays a salary $w(e_L) = \theta_L$ since it puts full probability at dealing with a low type. Intuitively, salaries coincide with the worker's productivity, thus being the same as under complete information. Yet, education levels in the separating PBE do not coincide with those under complete information, as we show below.

Upon observing off-the-equilibrium education $e \neq e_H \neq e_L$, the firm beliefs are $\mu(\theta_H|e) \in [0, 1]$, as discussed in the second step above. Therefore, the firm pays a salary between $w(e) = \theta_L$, which occurs when its beliefs are $\mu(\theta_H|e) = 0$, and $w(e) = \theta_H$, which happens when $\mu(\theta_H|e) = 1$, that is, $w(e) \in [\theta_L, \theta_H]$. As illustrated in figure 8.23, this wage schedule means that $w(e_L) = \theta_L$ at point A, $w(e_H) = \theta_H$ at point B, but for all other education levels $e \neq e_H \neq e_L$, the wage can lie weakly above θ_L and below θ_H.

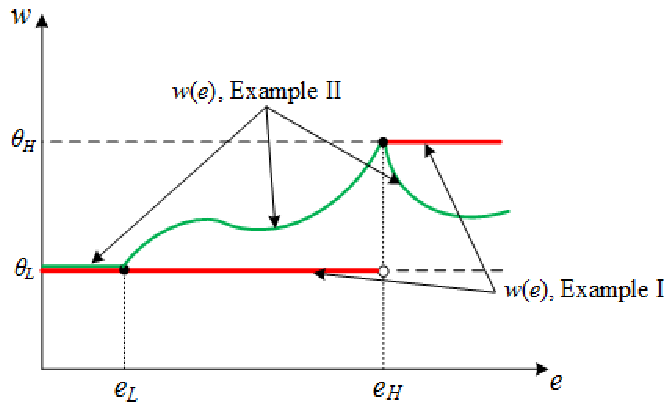


Figure 8.23. Separating PBE - Two wage schedules.

Fourth step. Given the above wage schedule from the firm, we must identify under which conditions the low-productivity worker has incentives to choose education e_L while the high-productivity worker chooses e_H. Starting with the low-type, let us find when e_L satisfies $e_L = 0$ (the lowest possible education). Figure 8.24 depicts the indifference curve of the low-productivity worker, IC_L, that passes through point $(w, e) = (\theta_L, 0)$. At education level $e_L = 0$, the firm is convinced to deal with a low-productivity worker, paying a wage $w(e_L) = \theta_L$. Figure 8.24 also depicts a wage schedule $w(e)$ which guarantees that this type of worker has no incentives to deviate from education level $e_L = 0$. For instance, at point A, the worker acquires education level e_1, receiving a salary above θ_L, but the indifference curve passing through point A, IC_L^A, lies to the southeast of IC_L, thus associated to a lower utility level. Intuitively, the worker finds that the additional cost of education he suffers when deviating from $e_L = 0$ to $e_1 > 0$ offsets the extra salary he receives. A similar argument applies for any other education levels $e > e_L$, since mapping them into the firm's wage schedule $w(e)$ we obtain (w, e) pairs lying to the sotheast of IC_L. More generally, for the low-productivity worker to stick to $e_L = 0$, we need that the firm's wage schedule lies below IC_L.[2]

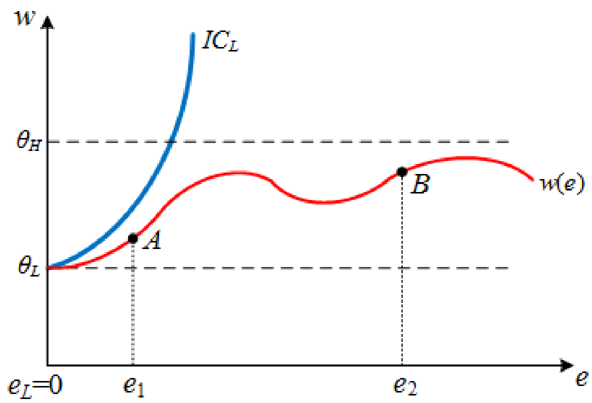


Figure 8.24. Separating PBE - Low-productivity worker.

The high-productivity worker chooses the education level prescribed in this separating strategy profile, e_H, rather than deviating to the low-type's education, $e_L = 0$, if

$$\theta_H - c(e_H, \theta_H) \geq \theta_L - c(0, \theta_H)$$

since he can anticipate that acquiring e_H identifies him as a high-productivity worker, yielding a salary $w(e_H) = \theta_H$ while acquiring $e_L = 0$ identifies him as a low-productivity worker, where he receives $w(e_L) = \theta_L$ (see third step). Since cost $c(0, \theta_H) = 0$ by assumption, we can rearrange the above inequality as

$$\theta_H - \theta_L \geq c(e_H, \theta_H).$$

Intuitively, the high-productivity worker chooses education level e_H when the wage differential he enjoys, relative to his salary when deviating to $e_L = 0$, as captured by $\theta_H - \theta_L$, compensates him for the additional cost of education he suffers, $c(e_H, \theta_H) - c(0, \theta_H) = c(e_H, \theta_H)$. We can write a similar condition to express his incentives to not deviate towards off-the-equilibrium education levels $e \neq e_H \neq e_L$, as follows

$$\theta_H - c(e_H, \theta_H) \geq w(e) - c(e, \theta_H).$$

Figure 8.25 depicts a wage schedule $w(e)$ that provides the high-productivity worker with incentives to choose education level e_H, where he reaches indifference curve IC_H, rather than deviating twoards any other education $e \neq e_H$. For instance, at a lower education e_1, his salary is represented

[2] To see this point, depict a different wage schedule $w(e)$ in figure 8.24, starting at $(w, e) = (\theta_L, 0)$ but lying above IC_L for at least some (w, e) pairs. The low-productivity worker will now have incentives to deviate from education level $e_L = 0$ to a positive education e' since the additional salary he earns compensates him for the additional cost of education.

by the height of point A. An indifference curve crossing through this point yields a lower utility level than that at IC_H since it lies to the southeast of IC_H. Therefore, the high-productivity worker does not have incentives to deviate from e_H to e_1. A similar argument applies to all other education levels $e < e_H$, inclucing $e_L = 0$; and to all education levels above e_H, such as e_2, where his salary is represented by the height of point B.

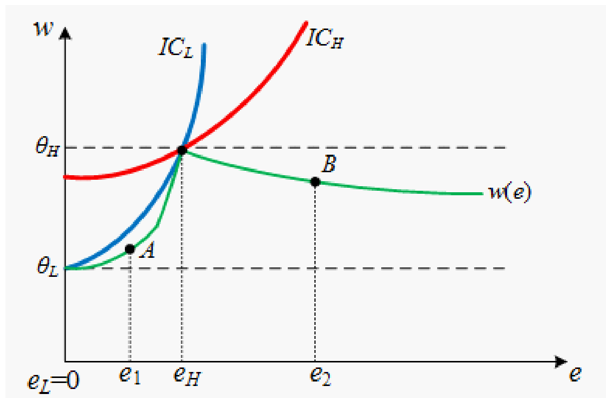
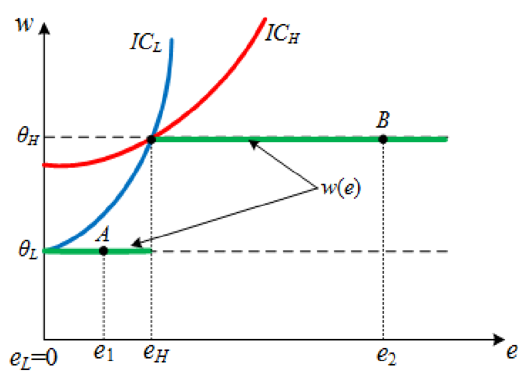


Figure 8.25. Separating PBE - Low and High-productivity worker.

Other wage schedules also induce the high-productivity worker to choose education e_H, such as that of figure 8.26a. Intuitively, this wage schedule indicates that the firm pays the lowest salary $w(e) = \theta_L$ upon observing $e < e_H$, but pays the highest salary otherwise. As a practice, the figure also depicts education levels e_1 and e_2, confirming that the worker does not have incentives to deviate from e_H.



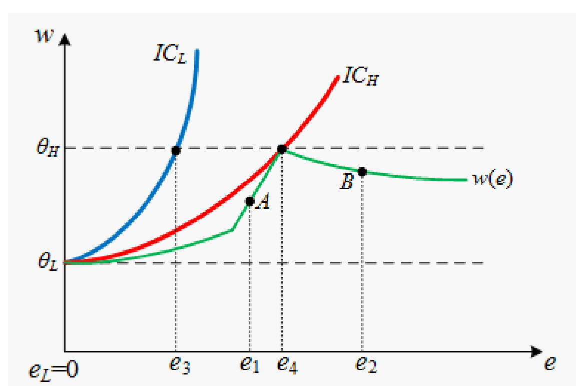


Figure 8.26a. Separating PBE - One extreme. Figure 8.26b. Separating PBE -Another extreme.

Interestingly, different wage schedules help us support different education levels e_H for the high-productivity worker, such as that in figure 8.26b, where the firm only pays the high salary θ_H when

e_H is extremely high, that is, $e_H = e_4$. In this case, the high-productivity worker, despite receiving salary $w(e_4) = \theta_H$, is indifferent between acquiring education e_4 or deviating to $e_L = 0$. Specifically, e_4 solves $\theta_H - \theta_L = c(e_4, \theta_H)$.

Example 8.7. For instance, if $\theta_H = 2$, $\theta_L = 1$, and the cost of education function is given by $c(e, \theta_K) = \frac{e^2}{\theta_K}$, we can solve for e_4 in the above equation, $\theta_H - \theta_L = \frac{e^2}{\theta_H}$, to obtain $e_4 = \sqrt{2}$.

More generally, we can claim that, in separating PBEs of the labor market signaling game, while the low-productivity worker chooses $e_L = 0$, the high-productivity worker selects an education level e_H in the range $[e_3, e_4]$, where $e_H = e_3$ represents the "least-costly separating PBE," since the high type conveys his type to the firm acquiring the lowest education level, whereas $e_H = e_4$ represents the "most-costly separating PBE." Importantly, in all education levels $e_H \in [e_3, e_4]$, the low-productivity worker has no incentives to choose e_H (i.e., to mimic the high type) as that would yield a lower utility level than that at IC_L.

The least-costly separating education level $e_H = e_3$ solves

$$\theta_H - c(e_3, \theta_L) = \theta_L - c(0, \theta_L).$$

Intuitively, the low-productivity worker is indifferent between his equilibrium strategy $e_L = 0$, receiving a wage of θ_L, and deviating to education level e_3 which provides him with wage θ_H. Since $c(0, \theta_L) = 0$ by assumption, the above equation simplifies to

$$\theta_H - \theta_L = c(e_3, \theta_L).$$

Example 8.8. In the setting of Example 8.7, education level e_3 solves $\theta_H - \theta_L = \frac{e^2}{\theta_L}$. Since $\theta_H = 2$ and $\theta_L = 1$, we obtain $e_3 = 1$.

1.4.2 Separating PBE – Applying the Intuitive Criterion

The most-costly separating PBE where the high-productivity worker chooses $e_H = e_4$ can only be sustained if the firm, upon observing any off-the-equilibrium education level in the interval $e_H \in [e_3, e_4)$, strictly below e_4, that it cannot be facing a high-productivity worker, i.e., $\mu(\theta_H | e_H) < 1$, and thus pays him strictly less than θ_H. But is these off-the-equilibrium beliefs sensible? No, which we can show using the Cho and Kreps' Intuitive Criterion, following our six-step approach.

1. *Step 1.* Consider a specific PBE, such as the most-costly separating PBE where $(e_L, e_H) = (0, e_4)$.

2. *Step 2.* Identify an off-the-equilibrium message for the worker, such as $e' \in [e_3, e_4)$, as depicted

in figure 8.27.[3]

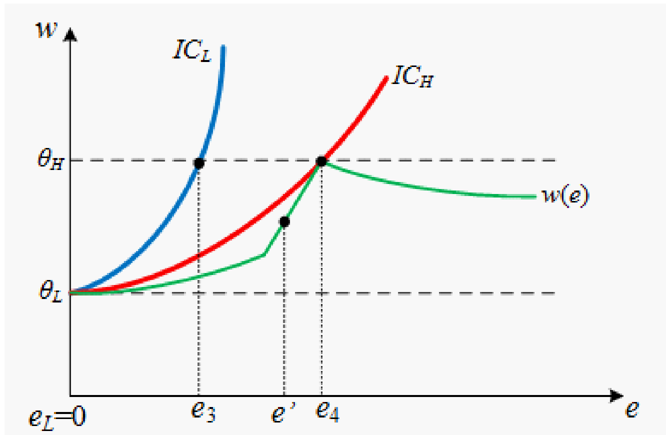


Figure 8.27. Applying the Intuitive Criterion to the separating PBEs.

3. *Step 3.* Find which types of worker can benefit by deviating to e'.

   - The low-productivity worker cannot benefit since, even if the firm responds paying him the highest salary, $w(e') = \theta_H$, the cost of effort is too large for this type of worker. Formally, $\theta_L - 0 \geq \theta_H - c(e', \theta_L)$, or

$$c(e', \theta_L) \geq \theta_H - \theta_L.$$

   - The high-productivity worker, however, can benefit from choosing e'. Intuitively, if the firm keeps paying him the highest salary, $w(e') = \theta_H$, but he incurs less education costs when acquiring e' rather than e_4, he will certainly deviate to e'. Formally, $\theta_H - c(e', \theta_H) \geq \theta_H - c(e_4, \theta_H)$ simplifies to $c(e_4, \theta_H) \geq c(e', \theta_H)$, which holds given that $e_4 > e'$. Overall, this means that education level e' can only originate from the high-productivity worker.

4. *Step 4.* We can now restrict the off-the-equilibrium beliefs of the firm. If education level e' is observed, it can only originate from the high-productivity worker, i.e., $\mu(\theta_H | e') = 1$.

5. *Step 5.* Let us find the optimal response given the restricted belief $\mu(\theta_H | e') = 1$. As the firm is convinced of dealing with a high-productivity worker, it optimally responds paying $w(e') = \theta_H$.

6. *Step 6.* Given the optimal response found in Step 5, we can see that the high-productivity worker has incentives to deviate from his equilibrium strategy of e_4 to education e'. Therefore, the most-costly separating PBE $(e_L, e_H) = (0, e_4)$ violates the Intuitive Criterion.

[3] This is an off-the-equilibrium education level since e' does not coincide with the education level of any type of worker in the above separating PBE, where $e_L = 0$ and $e_H = e_4$, that is, $e' \neq 0 \neq e_4$.

A similar argument applies to any other separating PBE where $e_H \in (e_3, e_4]$, since only the high-productivity worker has incentives to deviate to a lower education level $e' < e_H$. His educational choice at the least-costly separating PBE, $e_H = e_3$, however, survives the Intuitive Criterion. To see this, note that both types of worker can benefit from deviating to education levels strictly below e_3, implying that the firm cannot restrict its beliefs upon observing off-the-equilibrium education e' satisfying $e' < e_3$, keeping its beliefs unaltered relative to those in the separating PBE. Therefore, only one separating PBE survives the Intuitive Criterion, namely, the least-costly separating PBE where $(e_L, e_H) = (0, e_3)$.

1.4.3 Pooling PBE

We next check if a pooling strategy profile where both types of worker acquires the same education level can be sustained as a PBE. We follow the same four steps as in the discrete version of the game.

First step. In a pooling strategy profile, both the high- and low-productivity worker choose education level e_P, where the subscript P denotes pooling equilibrium.

Second step. Upon observing education level e_P, firm posterior beliefs coincide with its prior, i.e., $\mu(\theta_H | e_P) = p$. Intuitively, the observation of education level e^P provides the firm with no additional information about the worker's productivity since all worker types choose to acquire the same education. Upon observing the off-the-equilibrium education level $e \neq e_P$, the firm cannot update its beliefs using Bayes' rule, leaving them unrestricted, that is, $\mu(\theta_H | e) \in [0, 1]$. (As with the separating PBEs, we will impose some conditions on off-the-equilibrium beliefs below.)

Third step. Given the above beliefs, upon observing the pooling education level e^P, the firm optimally responds with salary $w(e^P) = p\theta_H + (1-p)\theta_L$, which captures the worker's expected productivity and, for compactness, we denote as

$$E[\theta] \equiv p\theta_H + (1-p)\theta_L.$$

After observing any off-the-equilibrium education $e \neq e_P$, the firm responds with a salary $w(e) \in [\theta_L, \theta_H]$ since its beliefs in this case are $\mu(\theta_H | e) \in [0, 1]$; as described in the second step. Figure 8.28 depicts, as an example, a wage schedule that satisfies the above two properties: at the pooling education level e^P, the salary is $w(e^P) = E[\theta]$, and for all other education levels $e \neq e_P$ the firm pays salaries bounded between θ_L and θ_H.[4]

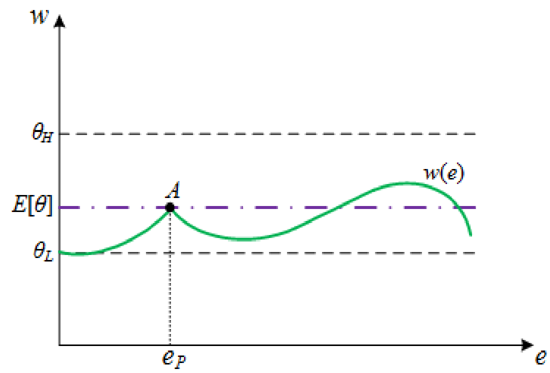


Figure 8.28. Pooling PBE - Example of wage schedule.

Fourth step. Given the above wage schedule from the firm, we next identify under which conditions both types of workers choose the same education level e^P. Let us start with the low-productivity worker. Figure 8.29 depicts an indifference curve of the low-productivity worker, IC_L, that originates at $(\theta_L, 0)$ on the vertical axis, where the worker acquires no education and receives the lowest salary θ_L, and passes through point A, which represents $(w, e) = (E[\theta], e^P)$. This type of worker is, then, indifferent between identifying himself as a low-productivity worker (i.e., acquiring no education and receiving salary θ_L) and acquiring the pooling education level e^P. The IC_L curve in figure 8.29 must satisfy

$$\theta_L - c(0, \theta_L) = E[\theta] - c(e^P, \theta_L)$$

or, given that $c(0, \theta_L) = 0$ by definition,

$$c(e^P, \theta_L) = E[\theta] - \theta_L,$$

where $E[\theta] > \theta_L$ since $p > 0$. For this to be the case, we need the wage schedule $w(e)$ to lie weakly below IC_L since that implies that, a deviation towards any education level $e \neq e_P$ produces an

[4] At first glance, one could expect wages to be increasing in the worker's education, so $w(e)$ increases in e. However, our above discussion about the optimal firm responses does not preclude the possibility of a decreasing portion in the wage schedule $w(e)$, as illustrated in figure 8.28.

overall utility lower than that at e^P for the low-productivity worker.

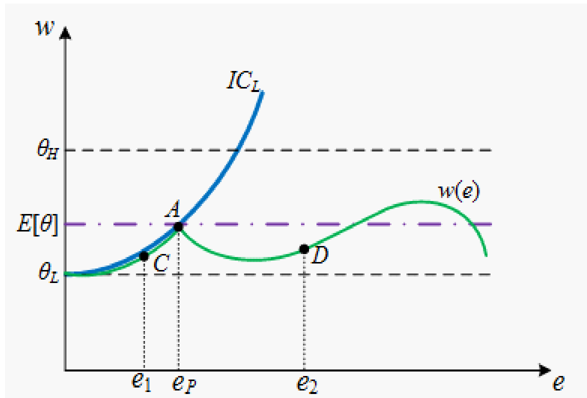


Figure 8.29. Pooling PBE - Low-productivity worker.

Example 8.9. Consider the parametric example of Example 8.7, where $\theta_H = 2$, $\theta_L = 1$, and the cost of education is $c(e, \theta_K) = \frac{e^2}{\theta_K}$. If we assume $p = 1/3$, we obtain an expected productivity $E[\theta] = \frac{4}{3}$. The above equation, $c(e^P, \theta_L) = E[\theta] - \theta_L$, becomes $\left(e^P\right)^2 = \frac{4}{3} - 1$, yielding a pooling education level of $e^P = 0.57$.

We now turn to the high-productivity worker, who must have incentives to choose the pooling education level e^P rather than deviating towards a different education $e \neq e_P$. Figure 8.30a illustrates a wage schedule $w(e)$ that does not provide incentives to deviate from e^P to this type of worker. Deviations to, for instance, education level e_1 yields a lower salary than $w(e^P) = E[\theta]$. While this worker's education cost is also lower, his overall utility is lower than at the pooling education e_P with associated salary $E[\theta]$. Graphically, the indifference curve passing through point C lies to the southeast of A, thus yielding a lower utility. Similarly, deviations to e_2 are not worth it since the worker needs to incur a larger education cost.

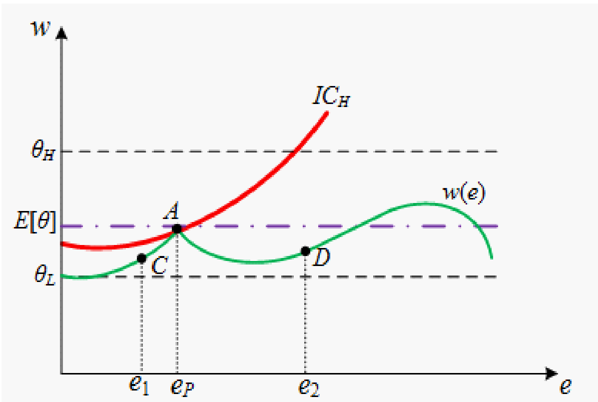


Figure 8.30a. Pooling PBE - High-productivity worker.

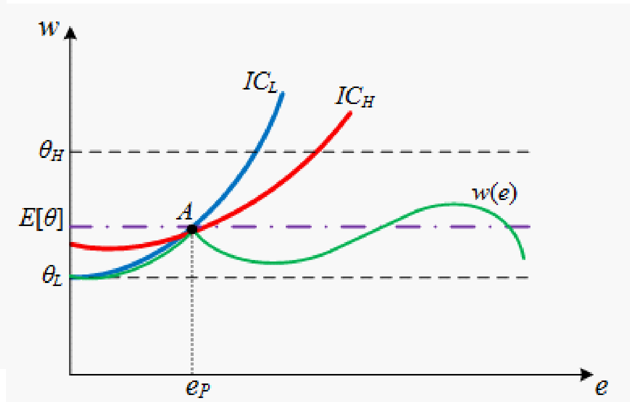


Figure 8.30b. Pooling PBE - Both types of worker.

Figure 8.30b superimposes figure 8.29, from our analysis of the low-productivity worker, and 8.30a, from our discussion of the high-producitivy worker. Note that the wage schedule $w(e)$ must lie weakly below both indifference curves IC_L and IC_H for both types of workers to have incentives to choose e^P rather than deviating. The pooling education level lies at the point where IC_L and IC_H cross each other.

Searching for more pooling PBEs. Are there any other pooling PBEs? Yes, if IC_L originates strictly above θ_L, the crossing point of IC_L and IC_H happens closer to the origin, as depicted in figure 8.31. A similar argument applies if we keep increasing the origin of IC_L until we reach a crossing point at $e^P = 0$. In this case, both types of workers acquire zero education and they receive a salary equal to their expected productivity $E[\theta]$; producing the same result as in an incomplete information game where workers cannot acquire education to signal their types.

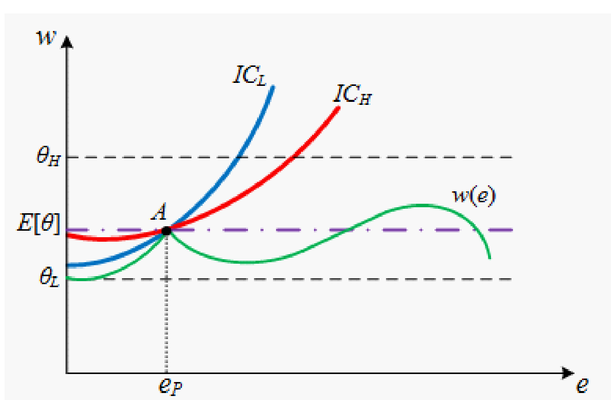


Figure 8.31. Pooling PBE - Other equilibria.

Therefore, we can summarize all pooling PBEs as (w, e)-pairs where both workers choose $e^P \in [0, e_A]$, where education e_A solves

$$\theta_L - c(0, \theta_L) = E[\theta] - c(e^P, \theta_L)$$

or, given that $c(0, \theta_L) = 0$ by definition,

$$c(e^P, \theta_L) = E[\theta] - \theta_L.$$

This equation just identifies the education level where the IC_L starting at θ_L crosses horizontal line $E[\theta]$ in figure 8.31. Intuitively, at this education level the low-productivity worker is indifferent between the pooling education e^P, earning a salary $w(e^P) = E[\theta]$, and a zero education level receiving a salary of θ_L. In all these pooling PBEs, the firm responds with a salary $w(e^P) = E[\theta]$ and a wage schedule $w(e)$ that lies below both IC_L and IC_H for all $e \neq e_P$.[5] Like in our above discussion of separating PBEs, our results indicate that we have a range of pooling PBEs: from the least-costly pooling PBE where both types of workers acquire zero education, $e^P = 0$, to the most-costly pooling PBE where both acquire the highest education $e^P = e_A$.

1.4.4 Pooling PBE – Applying the Intuitive Criterion

Our above discussion correctly identified under which conditions we can sustained pooling PBEs in this game, but do they survive the Intuitive Criterion? The answer is no, but before we formally show this, consider the above off-the-equilibrium beliefs for any education level $e \neq e_P$. Intuitively, the condition that $w(e)$ lies below both IC_L and IC_H for all $e \neq e_P$ means that, upon observing deviations from e^P, even to relatively high education levels, the firm infers that such deviation is not likely originating from the high-productivity worker, and thus pays a relatively low wage; as depicted by the height of $w(e)$ in the right-hand side of figure 8.31. This off-the-equilibrium beliefs are, of course, not sensible, since one could argue that deviations towards high education levels are more likely to stem from the high-type worker, as we show next following our six-steps approach to the Intuitive Criterion.

1. *Step 1.* Consider a specific PBE, such as the most-costly pooling PBE where $e^P = e_A$.

2. *Step 2.* Identify an off-the-equilibrium education, such as $e' > e_A$.

3. *Step 3.* Find which types of worker can benefit by deviating to e':

    - The low-productivity worker cannot benefit since, even if the firm responds paying him the highest salary, $w(e') = \theta_H$, the cost of effort is too large for this type of worker. Formally, $E[\theta] - c(e_A, \theta_L) \geq \theta_H - c(e', \theta_L)$, or

    $$c(e', \theta_L) - c(e_A, \theta_L) \geq \theta_H - E[\theta].$$

[5] Many wage schedules satisfy this property, such as $w(e) = \theta_L$ for all $e < e^P$ and $w(e) = E[\theta]$ otherwise.

Intuitively, the additional cost that the worker must incur offsets the wage increase he experiences. This is depicted in figure 8.32, where deviating towards education e', even if responded with the highest salary $w(e') = \theta_H$ at point B, yields a lower utility for the low-type worker than that he obtains in equilibrium. Graphically, the indifference curve crossing through point B lies to the southeast of IC_L.

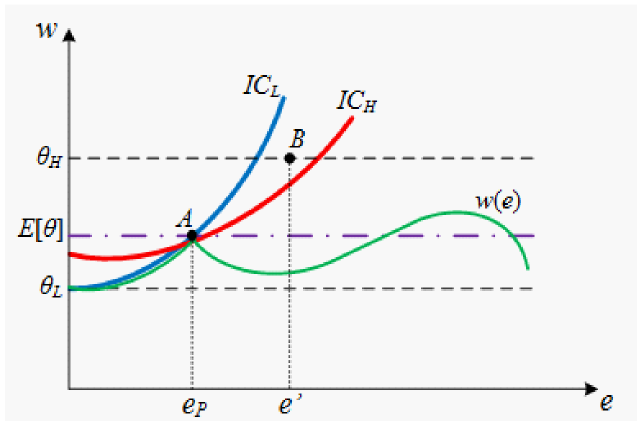


Figure 8.32. Applying the Intuitive Criterion to Pooling PBEs.

- The high-productivity worker, however, can benefit from choosing e', that is, $E[\theta] - c(e_A, \theta_H) \leq \theta_H - c(e', \theta_H)$, or

$$c(e', \theta_H) - c(e_A, \theta_H) \leq \theta_H - E[\theta].$$

  Intuitively, if the firm responds to the higher education level e' by paying him the highest salary $w(e') = \theta_H$, his wage increase would offset the additional education cost. At this point, we can combine the above two inequalities to obtain

$$c(e', \theta_H) - c(e_A, \theta_H) \leq \theta_H - E[\theta] \leq c(e', \theta_L) - c(e_A, \theta_L)$$

  which simplifies to

$$c(e', \theta_H) - c(e_A, \theta_H) \leq c(e', \theta_L) - c(e_A, \theta_L).$$

  This condition holds from our initial assumptions: the marginal cost of education (increasing e from e_A to e') is larger for the low- than the high-productivity worker. Overall, this means that education level e' can only originate from the high-productivity worker.

4. *Step 4.* We can now restrict the firm's off-the-equilibrium beliefs as follows: If education level e' is observed, it can only originate from the high-productivity worker, i.e., $\mu(\theta_H|e') = 1$.

5. *Step 5.* Let us find the optimal response given the restricted belief $\mu(\theta_H|e') = 1$. As the firm is convinced of dealing with a high-productivity worker, it optimally responds paying $w(e') = \theta_H$.

6. *Step 6.* Given the optimal response found in Step 5, we can see that the high-productivity worker has incentives to deviate from his equilibrium strategy of e_A to e'. Therefore, the most-costly pooling PBE $e^P = e_A$ violates the Intuitive Criterion.

A similar argument applies to all other pooling PBEs where $e^P < e_A$, including the least-costly pooling PBE where $e^P = 0$. Hence, no pooling PBE in the labor market signaling game survives the Intuitive Criterion, implying that the only PBE of the labor market signaling game, among those surviving the Intuitive Criterion, is the least-costly separating PBE where $(e_L, e_H) = (0, e_3)$.

1.5 Can signaling be welfare improving?

From our above results, a natural question is whether worker is better off when he uses education to signal his type (in the least-costly separating PBE found above) than when such signal is not available. When the worker cannot use education as a signal, the equilibrium outcome is a BNE, where the worker acquires no education regardless of his type, and the firm pays a salary equal to his expected productivity, $w = E[\theta]$. Figure 8.33 compares the indifference curve that each type of worker reaches in these two information settings.

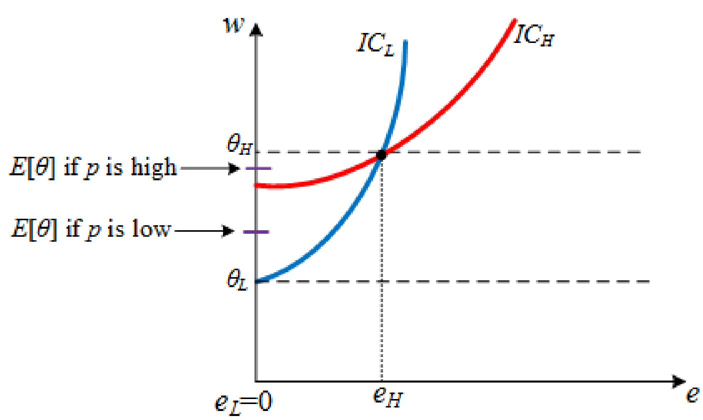


Figure 8.33. Utility comparison across information settings.

Low-productivity worker. The low-type worker is unambiguously worse off with signaling, where he acquires zero education but receives the lowest salary θ_L, than without signaling, where he still acquires no education but earns a higher salary $E[\theta]$. Graphically, the indifference curve passing through point $(0, E[\theta])$ reaches a higher utility level than IC_L. This result holds regardless of the specific probability that the worker is of high-productivity, p, which graphically means regardless of where $E[\theta]$ lies in the (θ_L, θ_H) interval.

High-productivity worker. In contrast, the high-type worker is better off with signaling, where he reaches IC_H, than without signaling, where he acquires no education and earns a salary $E[\theta]$, only if $E[\theta]$ is sufficiently low, which occurs when p is relatively low. Intuitively, this type of worker is better off acquiring education, despite its cost, and earning the highest wage θ_H, than not investing in education and receiving $E[\theta]$ when this wage is sufficiently low, which occurs when the firm believes that the high-productivity worker is relatively unlikely. If, instead, probability p (as then $E[\theta]$) is sufficiently high, the high-type worker is better off in the setting where education cannot be used as a signal to firms.

Example 8.10. Consider again the setting in Examples 8.7-8.9, where we found the least-costly separating PBE, $e_3 = 1$. In equilibrium, the high-productivity worker's utility is

$$u_H = w - \frac{e^2}{\theta_H} = \theta_H - \frac{1^2}{\theta_H} = 2 - \frac{1}{2} = \frac{3}{2}.$$

However, when signaling is not available, he earns a salary $E[\theta] = p2 + (1-p)1 = 1 + p$, yielding a utility of $1 + p$ since in this setting he acquires zero education. Therefore, this type of worker is better off in the environment where education cannot be used as signal to firms if $1 + p > \frac{3}{2}$ or, solving for p, when $p > 1/2$; as described in our above discussion.