EconS 503 - Microeconomic Theory II
Homework #6 - Answer key

1. **Cournot competition when all firms are uninformed - General setting**. Consider again the setting in Example 8.10 about two firms competing a la Cournot under incomplete information about their production costs. Let us now provide a more general analysis by considering marginal costs c_H and c_L for firm 2, where $c_H > c_L \geq 0$, occurring with probability p and $1-p$ respectively.

   (a) Find the best response function for every firm i, $q_i^k(q_j^L, q_j^H)$, where $k = \{L, H\}$ denotes firm i's marginal cost (high or low).

   - When firm i has low costs, it chooses $q_i^L \geq 0$ that solves the following expected profit maximization problem:

$$\begin{aligned}\max_{q_i^L \geq 0} \ \pi_i^L\left(q_i^L\right) &= \overbrace{(1-p)\left(1-q_i^L-q_j^L\right)q_i^L}^{\text{Profits if } j \text{ is low cost}} + \overbrace{p\left(1-q_i^L-q_j^H\right)q_i^L}^{\text{Profits if } j \text{ is high cost}} - c_L q_i^L \\ &= \left(1-c_L-q_i^L-(1-p)\,q_j^L-pq_j^H\right)q_i^L\end{aligned}$$

     Assuming interior solutions, that is, $q_i^L > 0$, the first order condition satisfies

$$\frac{\partial \pi_i^L\left(q_i^L\right)}{\partial q_i^L} = 1 - c_L - 2q_i^L - (1-p)\,q_j^L - pq_j^H = 0$$

     such that the best response function of firm i when its costs are low becomes

$$q_i^L\left(q_j^L, q_j^H\right) = \frac{1-c_L}{2} - \frac{(1-p)\,q_j^L + pq_j^H}{2}$$

     which originates at $\frac{1-c_L}{2}$, and decreases in its rival's production when the latter faces low costs, q_j^L, and when it faces high costs, q_j^H. In addition, the best response function becomes steeper when the "average output" that firm i assesses from firm j, $(1-p)\,q_j^L + pq_j^H$, increases. Intuitively, this occurs when output levels q_j^L or q_j^H increase, and when probability p increases as long as $q_j^L > q_j^H$.

   - When firm i has high costs, it chooses $q_i^H \geq 0$ that solves the following expected profit maximization problem:

$$\begin{aligned}\max_{q_i^H \geq 0} \ \pi_i^H\left(q_i^H\right) &= (1-p)\left(1-q_i^H-q_j^L\right)q_i^H + p\left(1-q_i^H-q_j^H\right)q_i^H - c_H q_i^H \\ &= \left(1-c_H-q_i^H-(1-p)\,q_j^L-pq_j^H\right)q_i^H\end{aligned}$$

     Assuming interior solutions, that is, $q_i^H > 0$, the first order condition satisfies

$$\frac{\partial \pi_i^H\left(q_i^H\right)}{\partial q_i^H} = \frac{1-c_H}{2} - 2q_i^H - (1-p)\,q_j^L - pq_j^H = 0$$

such that the best response function of firm i when its costs are high becomes

$$q_i^H\left(q_j^L, q_j^H\right) = \frac{1-c_H}{2} - \frac{(1-p)\,q_j^L + pq_j^H}{2}$$

which exhibits a similar intuition as firm i's best response function when facing low costs, $q_i^L\left(q_j^L, q_j^H\right)$; as discussed above.

(b) Use your results from part (a) to find the Bayesian Nash Equilibrium (BNE) of the game.

- Since firms i and j are symmetric, we impose symmetry on the equilibrium output that

$$q^L = q_i^L = q_j^L$$
$$q^H = q_i^H = q_j^H$$

Substituting the above results into the best response functions in part (a),

$$q^L = \frac{1-c_L}{2} - \frac{(1-p)\,q^L + pq^H}{2}$$
$$q^H = \frac{1-c_H}{2} - \frac{(1-p)\,q^L + pq^H}{2}$$

Rearranging, we obtain the following simultaneous equations:

$$pq^H + (3-p)\,q^L = 1 - c_L$$
$$(2+p)\,q^H + (1-p)\,q^L = 1 - c_H$$

Applying Cramer's Rule, we find equilibrium output levels

$$q^{H*} = \frac{(3-p)(1-c_H) - (1-p)(1-c_L)}{6}$$
$$q^{L*} = \frac{(2+p)(1-c_L) - p(1-c_H)}{6}$$

(c) How do the equilibrium output levels you found in part (a) are affected by changes in c_H, c_L, and p? Interpret.

- Differentiating equilibrium output with respect to c_H, yields

$$\frac{\partial q^{H*}}{\partial c_H} = -\frac{3-p}{6} < 0$$
$$\frac{\partial q^{L*}}{\partial c_H} = p > 0$$

In words, as it becomes more costly for the high-cost type to produce the good, output of the high-cost (low-cost) firm decreases (increases, respectively).

- Differentiating equilibrium output with respect to c_L, we obtain

$$\frac{\partial q^{H*}}{\partial c_L} = \frac{1-p}{6} > 0$$
$$\frac{\partial q^{L*}}{\partial c_L} = -\frac{2+p}{6} < 0$$

which indicates that, as it becomes more costly for the low-cost type to produce the good, output of the low-cost (high-cost) firm decreases (increases, respectively).

- Differentiating equilibrium output with respect to probability p, we find

$$\frac{\partial q^{H*}}{\partial p} = \frac{c_H - c_L}{6} > 0$$
$$\frac{\partial q^{L*}}{\partial p} = \frac{c_H - c_L}{6} > 0$$

Intuitively, as firms are more likely to face high costs, output of both types of firms increases.

2. **Sequential version of a first-price auction.** Consider an auction with two bidders. Every bidder $i = \{1, 2\}$ privately observes his valuation v_i for the object, drawn from a uniform distribution $U[0, 1]$, which is common knowledge among players. Assume that all bidders are risk neutral. Unlike in the standard first-price auction, where players simultaneously and independently submit their bids, let us consider its sequential-move version with the following time structure: Bidder 1 submits his bid, b_1; bidder 2 observes b_1 and responds either buying the good at this price or steps out (which implies that bidder 1 purchases the object for the price he originally submitted b_1).

   (a) Find equilibrium bidding strategies for bidder 2 (follower) and bidder 1 (leader).

   - *Bidder 2.* Starting with bidder 2, responds buying the good if his valuation is higher than the bid submitted by bidder 1, that is, $v_2 \geq b_1$; and step out otherwise.
   - *Bidder 1.* Anticipating this response from bidder 2, bidder 1 understands that $v_2 \geq b_1$ occurs with probability b_1 since player 2's valuation is uniformly distributed in $U[0, 1]$. Therefore, bidder 1 submits a bid b_1 expecting to win the object with probability b_1. Therefore, bidder 1's expected utility maximization problem is

$$\max_{b_1 \geq 0} EU_1(v_1) = prob(win) \cdot (v_1 - b_1)$$
$$= b_1(v_1 - b_1)$$

Differentiating with respect to his bid b_1, yields $2b_1 - v_1 = 0$, and solving for b_1, we find that his equilibrium bidding function is

$$b_1(v_1) = \frac{1}{2}v_1$$

which indicates a bid shading of 1/2 of his valuation for the object. As a remark, note that this equilibrium bidding function then coincides with that in the simultaneous-move version of the first-price auction where two players compete for the object drawing their valuations from a uniform distribution.

(b) Is this auction efficient?

- No. The object could go to bidder 2 even if he has a lower valuation for the object. To see this point, consider valuations $v_1 = 0.7$ and $v_2 = 0.5$. In this setting, bidder 1 submit a bid of $b_1(0.7) = \frac{1}{2}(0.7) = 0.35$, and bidder 2 responds buying the good at this price, \$0.35, which provides bidder 2 with a positive surplus). Therefore, the object is not necessarily assigned to the bidder with the highest valuation.

3. **Bidders receiving independent signals**. Consider the following *second-price* sealed-bid auction with two bidders competing for the object. Players receive private and independent signals, t_1 and t_2, drawn from uniform distribution $U[0,1]$. Player i's valuation for the object is defined as

$$v_i = \alpha_i t_i + \alpha_j t_j,$$

where $i \neq j$ and parameters α_i and α_j satisfy $\alpha_i > \alpha_j \geq 0$. Intuitively, every player's valuation is a function of the signal he receives, t_i, and (to a lesser extent) the signal that his rival receives, t_j.

(a) Show that bidding function $\beta(t_i) = (\alpha_i + \alpha_j)t_i$, for every bidder $i = \{1, 2\}$ is a Bayesian Nash equilibrium of this game.

- *Expected payoff from not deviating.* The expected payoff of player 1 who has signal t_1 and bids $(\alpha_i + \alpha_j)t_1$ is given by

$$\begin{aligned} Prob[(\alpha_i + \alpha_j)t_1 &> (\alpha_i + \alpha_j)t_2] \\ &\times E\left[\alpha_i t_1 + \alpha_j t_2 - (\alpha_i + \alpha_j)t_2 | (\alpha_i + \alpha_j)t_1 > (\alpha_i + \alpha_j)t_2\right] \end{aligned}$$

which is equal to

$$\left(\alpha_i t_1 - \alpha_i \frac{t_1}{2}\right) t_1 = \alpha_i \frac{t_1^2}{2}$$

- *Expected payoff from deviating.* Now consider an arbitrary deviation to a bid b. Since player 2's bid cannot exceed $\alpha_i + \alpha_j$, the admissible range of deviation is given by $[0, (\alpha_i + \alpha_j)]$; or, equivalently, $(\alpha_i + \alpha_j)\, y$ for $y \in [0, 1]$, yielding an expected payoff from deviation of

$$\alpha_i \left(t_1 - \frac{y}{2}\right) y$$

- *Comparison.* Therefore, bidder i has no incentives to deviate if

$$\alpha_i \frac{t_1^2}{2} \geq \alpha_i \left(t_1 - \frac{y}{2}\right) y$$

or, after rearranging,

$$\frac{\alpha_i}{2}(t_1 - y)^2 \geq 0$$

entailing that there there exists no profitable deviation, because bid $(\alpha_i+\alpha_j)t_1$ is the best response to $(\alpha_i + \alpha_j)t_2$. Applying the symmetric argument to player 2, we can demonstrate that these strategies constitute a Bayesian Nash equilibrium.

(b) Characterize all symmetric equilibria in which player i bids $\beta(t_i)$, where β is strictly increasing and differentiable.

- Given a signal of t_1, the expected payoff of player 1 submitting a bid $b \geq 0$ is

$$\int_0^{\beta^{-1}(b)} (\alpha_i t_1 + \alpha_j t_2 - \alpha_j t_2)\, dt_2$$

Differentiating with respect to bid b, and assuming an interior solution, we obtain

$$\frac{1}{\beta'\left(\beta^{-1}(b)\right)}\left(\alpha t_1 + \alpha_j \beta^{-1}(b) - \beta\left(\beta^{-1}(b)\right)\right) = 0$$

and this must hold with equality for $b = \beta(t_1)$, entailing that

$$\frac{1}{\beta'(t_1)}\left(\alpha_i t_1 + \alpha_j t_1 - \beta(t_1)\right) = 0$$

and since the bidding function is monotonically increasing, $\beta' > 0$, we obtain that

$$\beta(t_1) = (\alpha_i + \alpha_j)t_1.$$

In part (a), we already show that $(\alpha_i + \alpha_j)\, t_i$ is a best response to $(\alpha_i + \alpha_j)\, t_j$, where $i \neq j$. Therefore, the unique symmetric Bayesian Nash equilibrium in strictly increasing and differentiable strategies is

$$\beta(t_i) = (\alpha_i + \alpha_j)\, t_i, \text{ for every bidder } i = \{1, 2\}.$$

4. **Third-price auction.** Consider an auction with $N \geq 3$ bidders, each of them privately observing his valuation of the good, $v_i \in [0, 1]$. Each bidder independently and simultaneously submit his own bid, b_i, and the bidder submitting the highest bid is selected as the winner of the auction and receives the good. In this case, however, let us assume that the winning bidder pays the *third* highest bid; thus justifying why this auction format is referred to as "third-price auction."

(a) Assume a cumulative distribution function $F(v_i)$, with positive density for all valuations $f(v_i) > 0$ for all $v_i \in [0, 1]$. Find bidder i's equilibrium bidding function $b_i(v_i)$ in this auction.

- Similarly as in other auction formats, assume that there is a symmetric and increasing bidding function $b_i(v_i)$. In addition, consider that the payoff from the bidder with the lowest valuation, $v_i = 0$, is zero. We then have that, for every value v_i, the payoff (or payment) of bidder i to the seller is

$$m(v_i) = \int_0^{v_i} yg(y)dy$$

(recall that this is a direct result from the Revenue Equivalence Theorem). We now seek to write bidder i's payment $m(v_i)$ using an alternative expression, so we can ultimately set it equal to the right-hand side of the above equality. In particular, the seller's expected revenue is

$$\text{Prob}\,(\text{win}) \times E\left[b_i(X_2)|X_1 < v_i\right]$$

where X_1 (X_2) is the first-order (second-order) statistic. That is, X_1 (X_2) is the highest (second highest) valuation of all $N-1$ remaining bidders. We can use these statistics to describe that:

i) Bidder i wins the auction if his valuation is higher than that of all other $N-1$ bidders, that is, $X_1 < v_i$, which happens with probability $F_1^{(N-1)}(v_i)$; and

ii) The expected bid is

$$E\left[b_i(X_2)|X_1 < v_i\right] = \int_0^{v_i} b_i(y) f_2^{(N-1)}(y|X_1 < v_i)dy$$

where density $f_2^{(N-1)}(y|X_1 < v_i)$ is conditional on bidder i's valuation being larger than that of all $N-1$ remaining bidders, $X_1 < v_i$. Specifically, since the probability of $X_1 < v_i$ is $F_1^{(N-1)}(v_i)$, we can rewrite this conditional density as

$$f_2^{(N-1)}(y|X_1 < v_i) = \frac{1}{F_1^{(N-1)}(v_i)}(N-1)\left[F(v_i) - F(y)\right] f_1^{(N-2)}(y)$$

where $(N-1)\left[F(v_i) - F(y)\right]$ is the probability that $X_1 < v_i$ but $X_1 > y$, and $f_2^{(N-1)}(y)$ is the density of the highest of $N-2$ values.

- Using points (i) and (ii), we can write the seller's expected revenue in the third-price auction, Prob(win) $\times\, E\left[b_i(X_2)|X_1 < v_i\right]$, as follows

$$\begin{aligned}
& F_1^{(N-1)}(v_i) \int_0^{v_i} b_i(y) \overbrace{\frac{1}{F_1^{(N-1)}(v_i)}(N-1)\left[F(v_i) - F(y)\right] f_1^{(N-2)}(y)}^{f_2^{(N-1)}(y|X_1<v_i)}dy \\
= \;& (N-1)\frac{F_1^{(N-1)}(v_i)}{F_1^{(N-1)}(v_i)} \int_0^{v_i} b_i(y)\left[F(v_i) - F(y)\right] f_1^{(N-2)}(y)dy \\
= \;& (N-1)\int_0^{v_i} b_i(y)\left[F(v_i) - F(y)\right] f_1^{(N-2)}(y)dy
\end{aligned}$$

since $\frac{1}{F_1^{(N-1)}(v_i)}$ can be taken out of the integral. We are now ready to use this expression in the left-hand side of $m(v_i)$, to obtain

$$(N-1)\int_0^{v_i} b_i(y)\left[F(v_i) - F(y)\right] f_1^{(N-2)}(y)dy = \int_0^{v_i} yg(y)dy$$

Differentiating with respect to v_i, yields

$$(N-1)f(v_i)\int_0^{v_i} b_i(y)f_1^{(N-2)}(y)dy = v_i g(v_i)$$

where $g(v_i)$ is the density of the cdf $G(v_i) = F(v_i)^{N-1}$, thus implying that $g(v_i) = (N-1)f(v_i)F(v_i)^{N-2}$. Using this term on the right-hand side, we obtain

$$(N-1)f(v_i)\int_0^{v_i} b_i(y)f_1^{(N-2)}(y)dy = v_i(N-1)f(v_i)F(v_i)^{N-2}$$

Cancelling $(N-1)f(v_i)$ on both sides, and rearranging, yields

$$\int_0^{v_i} b_i(y)f_1^{(N-2)}(y)dy = v_i F(v_i)^{N-2}$$

At this point, recall that our goal is to solve for bidding function $b_i(y)$. In order to eliminate the integral operator, let us differentiate again with respect to valuation v_i to obtain

$$b_i(v_i)f_1^{(N-2)}(v_i) = F(v_i)^{N-2} + v_i f_1(v_i)^{N-2}$$

Finally, solving for bidding function $b_i(v_i)$, yields

$$b_i(v_i) = \frac{F(v_i)^{N-2}}{f_1^{(N-2)}(v_i)} + \frac{v_i f_1(v_i)^{N-2}}{f_1^{(N-2)}(v_i)}$$

which simplifies to

$$b_i(v_i) = \frac{F(v_i)}{(N-2)f(v_i)} + v_i$$

(b) For the remainder of the exercise, consider that valuations are distributed according to a uniform distribution, so $F(v_i) = v_i$. Evaluate the equilibrium bidding function you found in part (a) using this distribution function. How does $b_i(v_i)$ change in the number of bidders N?

- Inserting $F(v_i) = v_i$ and $f(v_i) = 1$ in the equilibrium bidding function found in part (a), we obtain

$$\begin{aligned} b_i(v_i) &= v_i + \frac{v_i}{N-2} \\ &= \frac{N-1}{N-2} v_i \end{aligned}$$

which is increasing in bidder i's valuation, v_i; and decreases in the number of bidders since $\frac{\partial b_i(v_i)}{\partial N} = -\frac{v_i}{(N-2)^2} < 0$. Hence, unlike in first-price auctions (whereby an increase in the number of bidders raises bids, making them more aggressive), in a third-price auction bids decrease in N.

(c) Show that the equilibrium bidding function satisfies $b_i(v_i) > v_i$. Justify.

- Indeed, the difference

$$\begin{aligned} b_i(v_i) - v_i &= \frac{N-1}{N-2}v_i - v_i \\ &= \frac{1}{N-2}v_i \end{aligned}$$

which is positive since $N \geq 2$ by definition. Every bidder i submits a bid above his own valuation for the good, since he assigns a low probability to the third highest bid (the price that bidder i pays if winning) being higher than his own valuation v_i.

(d) Compare the equilibrium bidding functions in the first-, second-, and third-price auction. Interpret.

- Recall that, in a first-price auction with uniformly distributed valuations, the equilibrium bidding function is $b_i^{1st}(v_i) = \frac{N-1}{N}v_i$; whereas in the second-price auction it is $b_i^{2nd}(v_i) = v_i$.[1] Hence,

$$b_i^{3rd}(v_i) > b_i^{2nd}(v_i) \geq b_i^{1st}(v_i)$$

since

$$\frac{N-1}{N-2}v_i > v_i \geq \frac{N-1}{N}v_i$$

for all values of N. Moving from a first- to a second-price auction, bids increase as every bidder anticipates that, if winning, he will only pay the second-highest bid. A similar argument applies when we move from the second- to the third-price auction, as now every bidder knows that, upon winning, he will only pay the third highest bid.

[1] In the second-price auction, this equilibrium bidding function holds both when valuations are uniformly distributed and otherwise.