EconS 503 - Microeconomic Theory II
Midterm Exam #1 - Answer key

1. **Gross substitutes.** Consider an economy with two individuals, Amelia and Bernardo, with utility functions $u^A(x^A, y^A) = \min\{x^A, 2y^A\}$ for Amelia and $u^B(x^B, y^B) = \min\{2x^B, y^B\}$ for Bernardo, and initial endowments given by $\mathbf{e}^A = (1, 0)$ and $\mathbf{e}^B = (0, 1)$.

   (a) Find the Walrasian demands of each individual.

   - *Amelia.* The UMP of Amelia is

   $$\max_{x^A, y^A \geq 0} \quad \min\{x^A, 2y^A\}$$
   $$\text{subject to} \quad p_x x^A + p_y y^A \leq p_x$$

   since she only owns one unit of good x, $\mathbf{e}^A = (1, 0)$ the market value of her resources (as captured in the right-hand side of the budget constraint) is p_x. As she would consume (x^A, y^A) pairs at the kink of her L-shaped indifference curves, optimal consumption bundles satisfy $x^A = 2y^A$. Plugging $x^A = 2y^A$ into her budget line, $p_x x^A + p_y y^A = p_x$, yields

   $$p_x \left(2y^A\right) + p_y y^A = p_x$$

   and solving for y^A, we obtain Amelia's Walrasian demand of good y

   $$y^A = \frac{p_x}{2p_x + p_y}$$

   while her demand for good x is

   $$x^A = 2y^A = \frac{2p_x}{2p_x + p_y}$$

   - *Bernardo.* Similarly, Bernardo's utility maximizing bundles (x^B, y^B) satisfy $2x^B = y^B$ (bundles at the kink of his indifference curve) and $p_x x^B + p_y y^B = p_y$ (budget line since he only owns one unit of good y). Simultaneously solving for x^B and y^B yields

   $$x^B = \frac{p_y}{p_x + 2p_y} \quad \text{and} \quad y^B = \frac{2p_y}{p_x + 2p_y}$$

   (b) Find the excess demand functions, $z_x(p_x, p_y)$ and $z_y(p_x, p_y)$.

   - The excess demand for good x is

   $$z_x(p_x, p_y) = \frac{2p_x}{2p_x + p_y} + \frac{p_y}{p_x + 2p_y} - 1 - 0 = \frac{p_x p_y - (p_y)^2}{(2p_x + p_y)(p_x + 2p_y)}$$

   while that of good y is

   $$z_y(p_x, p_y) = \frac{p_x}{2p_x + p_y} + \frac{2p_y}{p_x + 2p_y} - 0 - 1 = \frac{p_x p_y - (p_x)^2}{(2p_x + p_y)(p_x + 2p_y)}$$

(c) Check if goods are gross substitutes, i.e., for any two goods $k \neq j$ where $k, j = \{x, y\}$ their excess demand functions satisfy $\frac{\partial z_k(p_x, p_y)}{\partial p_j} > 0$.

- Using $z_x(p_x, p_y)$, we find

$$\frac{\partial z_x(p_x, p_y)}{\partial p_y} = \frac{2(p_x)^3 - 4(p_x)^2 p_y - 7p_x(p_y)^2}{(2(p_x)^2 + 2(p_y)^2 + 5p_x p_y)^2}$$

which is positive if

$$p_y < \frac{2p_x}{2 + 3\sqrt{2}} \simeq 0.32p_x$$

Similarly, using $z_y(p_x, p_y)$, we find that

$$\frac{\partial z_y(p_x, p_y)}{\partial p_x} = \frac{2(p_y)^3 - 4p_x(p_y)^2 - 7(p_x)^2 p_y}{(2(p_x)^2 + 2(p_y)^2 + 5p_x p_y)^2}$$

which is positive if

$$p_y > \frac{p_x}{2}\left(3\sqrt{2} + 2\right) \simeq 3.12p_x$$

Figure 1 depicts in the (p_x, p_y)-quadrant the two cutoffs we identified: (1) price pairs in area C entail that good x is a gross substitute of good y; whereas (2) price pairs in area A imply that good y is a gross substitute of good x. In other words, the conditions for goods to be gross substitutes are asymmetric, as there is no region where both good x is a gross substitute of y and vice versa. Last, note that price pairs in area B entail that good x is a gross complement of good y and, simultaneously, good y is a gross complement of good x.

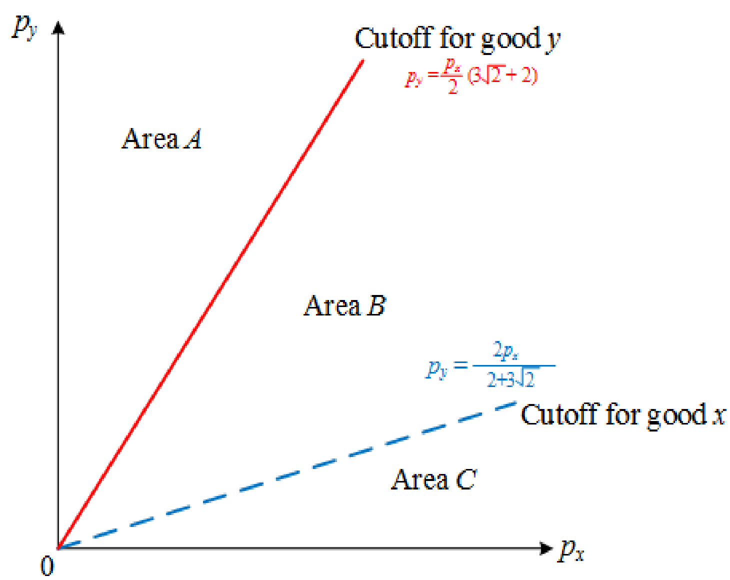


Figure 1. Areas for which goods x and y can be gross substitutes.

2. **Strict Nash equilibrium.** Consider the following definition: A strategy profile $s^* \equiv (s_1^*, ..., s_N^*)$ is a *strict Nash equilibrium* (SNE) if it satisfies

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*) \text{ for every player } i, \text{ and every } s_i \in S_i.$$

You probably noticed that this definition is almost identical to the definition of Nash equilibrium (NE), except for using a strict, rather than weak, inequality sign. In this exercise we connect both solution concepts, but first examine the relationship between a strict Nash equilibrium and IDSDS.

(a) Show that if a strategy s_i^* is a SNE it must also survive IDSDS. [*Hint*: Use contradiction.]

- Assume, by contradiction, that strategy profile s^* is a SNE, but s^* does not survive IDSDS. Then, there must be at least one player i who can find at least one strategy $s_i' \neq s_i^*$ such that $u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$ during some of the rounds of applying IDSDS. However, this would contradict that strategy s_i^* is a SNE for player i. Then, if strategy s_i^* is a SNE, it must also survive IDSDS.

(b) Show the opposite relationship to that in part (a), that is, if strategy s_i^* survives IDSDS then s_i^* must be a SNE.

- By the same argument as above, assume, by contradiction, that strategy profile s^* survives IDSDS, but s^* is not a SNE. Then, there must be at least one player i and at least one of his strategies $s_i' \neq s_i^*$ for which $u_i(s_i', s_{-i}^*) > u_i(s_i^*, s_{-i}^*)$, which contradicts strategy s_i^* surviving IDSDS.
- The above argument assumes that a SNE exists, because if it did not, strategy s_i^* could not survive IDSDS.

(c) Show that if strategy profile s^* is a NE, it doesn't need to be a SNE. An example suffices.

- Consider the following payoff matrix

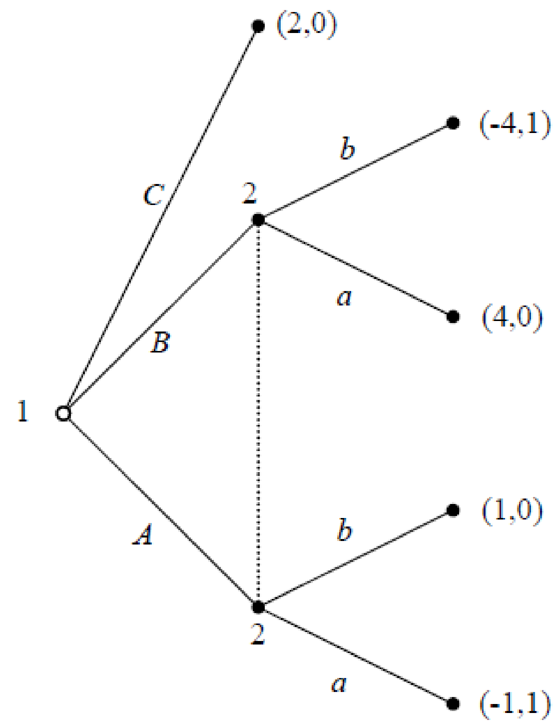
		Player 2		
		L	M	R
	U	$\underline{6},\underline{5}$	$7,3$	$\underline{8},4$
Player 1	C	$4,3$	$10,6$	$5,\underline{8}$
	D	$5,2$	$\underline{11},8$	$4,\underline{10}$

- *Nash equilibrium.* First, we underline best response payoffs of each player to find the Nash equilibria of the game. We identify only one Nash equilibrium (U, L), where every player chooses best responses to his opponent's equilibrium strategies.
- *Strict Nash equilibrium.* To see that this game has no SNE, note that there is no strictly dominant strategies, neither for player 1 nor for player 2. From our discussion in parts (a) and (b), this implies that there is no SNE.

(d) Show that if strategy profile s^* is a SNE, the game has no mixed strategy NE.

- If s^* is a SNE, then, it is the only strategy profile surviving IDSDS and, in turn, s^* is the unique pure strategy NE of the game. Both players play their strictly dominant strategies in this NE, and there is no belief that other player will play different strategies. Thus, we cannot sustain a mixed strategy NE.

3. **Finding PSNE and SPNE.** Consider the game represented in the next figure. Player 1 is the first move (see left-hand side of the figure) choosing A, B or C. If player 1 chooses C the game is over. If player 1 chooses A or B, player 2 is called on to move not observing whether player 1 chose A or B; as depicted in player 2's information set. Player 2 then must respond with a or b. The first payoff in every payoff pair corresponds to player 1, and the second payoff to player 2.



(a) Find all pure strategy Nash equilibria (psNE).

- The only proper subgame of this game is the game itself. We can represent the normal form as follows

		2: a	2: b
1	A	$-1, 1$	$1, 0$
	B	$4, 0$	$-4, 1$
	C	$2, 0$	$2, 0$

Note that strategy A is strictly dominated by strategy C for player 1. Thus, we can delete this strategy and our reduced form game becomes

		2	
		a	b
1	B	$\underline{4},0$	$-4,\underline{1}$
	C	$2,\underline{0}$	$\underline{2},\underline{0}$

There is only one pure strategy Nash Equilibrium of this game, (C, b).

(b) Find all subgame perfect Nash equilibria (SPNE).

- The NE found in part (a), (C, b), must also be subgame perfect. That is, in this game the set of NE coincides with the set of SPNE.

4. **Contests with N players.** In this exercise we examine equilibrium investment in a contest, where $N \geq 2$ players compete to earn a prize of common value V, and the probability of winning the prize is a function of a player's own investment relative to the aggregate investment of all the players. Hence, contests are often used to model promotions within a firm (where every worker invests time and effort into being selected for a promotion), political campaigns (where candidates invest money and resources to capture a larger share of votes), and R&D races (where firms invest resources into discovering a new product, such as a drug). We consider player i's probability of winning the prize is given by

$$p_i = \frac{x_i^r}{\sum_{j=1}^{N} x_j^r},$$

where x_i denotes his investment and the parameter, $r \geq 1$, represents the effectiveness of his investment. For simplicity, we normalize the cost of every unit of investment to one dollar and assume that r satisfies $r \leq \frac{N}{N-1}$.

(a) Setup the utility maximization program for player i.

- Player i chooses his investment, x_i, to maximize the expected value of the prize, as follows

$$\max_{x_i \geq 0} \underbrace{\overbrace{p_i \times V}^{\text{Wins}} + \overbrace{(1 - p_i) \times 0}^{\text{Loses}}}_{\text{Expected value}} - \underbrace{x_i}_{\text{Cost}} = \frac{x_i^r}{\sum_{j=1}^{N} x_j^r} V - x_i$$

since, to contest for the prize, player i has to spend x_i.

(b) Find the every player's investment in equilibrium, x_i^*.

- Differentiating with respect to x_i, yields

$$\frac{r x_i^{r-1} \left(\sum_{j=1}^{N} x_j^r \right) - r x_i^{2r-1}}{\left(\sum_{j=1}^{N} x_j^r \right)^2} V - 1 = 0$$

and, rearranging, we obtain

$$\frac{r x_i^{r-1} \left(\sum_{j \neq i} x_j^r \right)}{\left(\sum_{j=1}^{N} x_j^r \right)^2} V = 1 \tag{1}$$

In a symmetric equilibrium, all players invest the same amount, $x_1^* = x_2^* = \cdots = x_N^* = x^*$, such that expression (1) becomes

$$\frac{rx^{r-1}(N-1)x^r}{(Nx^r)^2}V = 1$$

which, solving for x, yields an equilibrium investment of

$$x^* = rV\frac{N-1}{N^2}$$

- This investment is increasing in its effectiveness, r, and in the value of the prize, V, but decreasing in the number of players competing for the prize, N, given that

$$\frac{\partial x^*}{\partial N} = -rV\frac{N-2}{N^3} \leq 0.$$

Since $N \geq 2$, the more players competing for the prize, the less player i invests at equilibrium.

(c) Define Rent Dissipation as

$$D \equiv V - \sum_{i=1}^{N} x_i^*,$$

which can be understood as how many resources the society is left with after all players complete their investments. Find the Rent Dissipation D and show that it is positive for all the admissible values of r.

- Rent Dissipation is given by

$$\begin{aligned} D &= V - \sum_{i=1}^{N} x_i^* \\ &= V - Nx^* \\ &= V - N\left(rV\frac{N-1}{N^2}\right) \\ &= \left(1 - r\frac{N-1}{N}\right)V \end{aligned}$$

Since $r \leq \frac{N}{N-1}$ by assumption, we must have that $1 - r\frac{N-1}{N} \geq 1 - \frac{N}{N-1}\frac{N-1}{N}$, which helps us write D as follows

$$D = \left(1 - r\frac{N-1}{N}\right)V \geq \left(1 - \frac{N}{N-1}\frac{N-1}{N}\right)V = (1-1)V = 0$$

Therefore, rent dissipation must be positive, indicating that, as a whole, players invest less than the value of the prize, that is, $V \geq \sum_{i=1}^{N} x_i^*$.